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Optimization of threshold memberships over fuzzy decision process

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1 Introduction

Since Bellman and Zadeh has proposed three – *deterministic, stochastic* and *fuzzy* – systems on multistage decision processes in fuzzy environment [4, §4,5], an intensive study on fuzzy decision making under uncertainty has been developed both in theory and in its wide applications ([1, 5, 15] and others). In Markov decision processes [6, 17], it has been tacitly known that there exists an optimal policy which is *Markov* for the *additive* criterion, where Markov policy takes decision on the basis of only today's state. Recently, from a stochastic control theory, Iwamoto has developed Bellman and Zadeh's fuzzy decision-making on the stochastic system for a *non-additive (minimum)* criterion. He has shown that an optimal policy does not exist in Markov class for minimum criterion but does exist in *general* class, where general policy depends on state-sequence up until today [7, 8, 9, 10, 11, 12, 13, 14, 18]. His tool is identical twins – both dynamic programming [2] and invariant imbedding [3, 16] – for non-additive criterion in stochastic system [8, 9].

In this paper, we consider a "threshold probability" decision-making in fuzzy environment. On the multi-stage stochastic control process, we evaluate the threshold probability that the minimum criterion exceeds a lower membership-degree. The minimum criterion denotes a *total* membership function of the multistage fuzzy decison process with *stage-wise* membership functions and a *goal* membership function. It is the membership function of intersection of the underlying fuzzy sets [4, p.144, §4,5]. Under the controlled Markov chain we optimize the threshold probability not in general class but in Markov class. We show that this choice will be successful; there exists an optimal policy in Markov class. We also derive the recursive relation for the threshold probability. We use the notations and terminology in [4, 8, 9].

2 Decision Process with Threshold Probability

Let us consider an N-stage ($N \geq 2$) stochastic decision process $\{(X_n, U_n)\}_0^N$ on a finite state space X and decision space U, which is governed by a Markov transition law $p = \{p(\cdot|\cdot,\cdot)\}$:

$$p(y|x,u) \geq 0, \qquad \sum_{y \in X} p(y|x,u) = 1.$$

Thus $p(y|x,u)$ is a conditional probability that the next state X_{n+1} will be y when the current state X_n is x and current decision U_n is u :

$$P(X_{n+1} = y \,|\, X_n = x, U_n = u) = p(y|x,u).$$

This transition is expressed as $X_{n+1} \sim p(\,\cdot\,|x,u)$.

We begin to introduce a large class of policies, which depend not only on today's state but also on state-to-date. Let $X^n := X \times X \times \cdots \times X$ be direct product of n state spaces X. A mapping $\sigma_n : X^{n+1} \to U$ is called *n-th general decision function*, whose sequence $\sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_{N-1}\}$ constitutes a *general policy*. The set of all general policies Π_g is called *general class*. When each general decision function σ_n depends only on the last (= current) state, the general policy reduces to a *Markov policy* $\pi = \{\pi_0, \pi_1, \ldots, \pi_{N-1}\}$. Let *Markov class* Π denote the set of all Markov policies. Thus we have an inclusion relation : $\Pi \subset \Pi_g$.

Further, given an *n-th membership function* $\mu_n : X \times U \to [0,\ 1]$ $(0 \le n \le N-1)$ and a *goal membership function* $\mu_G : X \to [0,\ 1]$, the random variables $\mu_n = \mu_n(X_n, U_n)$, $\mu_G = \mu_G(X_N)$ denote the resulting grade of membership [4].

Now we consider the problem of maximizing a threshold probability that *total membership* is greater than or equal to a given *lower grade* $\alpha \in [0,\ 1]$:

$$\mathrm{P}_0(x_0) \qquad \begin{array}{ll} \text{Maximize } P^{\pi}_{x_0}(\,\mu_0 \wedge \mu_1 \wedge \cdots \wedge \mu_{N-1} \wedge \mu_G \ge \alpha\,) & \\ \text{subject to (i)}_\mathrm{n}\ X_{n+1} \sim p(\,\cdot\,|x_n, u_n) & \\ \qquad\qquad\ \text{(ii)}_\mathrm{n}\ u_n \in U & 0 \le n \le N-1 \end{array}$$

where $P^{\pi}_{x_0}$ is the (discrete) probability measure on history space

$$H_N := X \times U \times X \times U \times \cdots \times U \times X \quad (2N+1)\text{-factors}$$

induced through an initial state x_0, the Markov transition law p and a Markov policy $\pi(\in \Pi)$.

We dare to maximize the threshold probability over Markov class Π. We do not optimize it over general class Π_g. This choice will be turned to generate a valid recursive equation. Any Markov policy $\pi(\in \Pi)$ determines the threshold probability in $\mathrm{P}_0(x_0)$, which is a "partial" multiple sum :

$$P^{\pi}_{x_0}(\,\mu_0 \wedge \mu_1 \wedge \cdots \wedge \mu_{N-1} \wedge \mu_G \ge \alpha\,) \;=\; \sum\sum\cdots\sum_{(x_1,x_2,\ldots,x_N)\in(*)} p_0 p_1 \cdots p_{N-1} \qquad (1)$$

$$(\ p_n = p(x_{n+1}|x_n, u_n)\ )$$

where the domain $(*)$ is the set of all $(x_1, x_2, \ldots, x_N) \in X^N$ satisfying

$$\mu_0(x_0, u_0) \wedge \mu_1(x_1, u_1) \wedge \cdots \wedge \mu_{N-1}(x_{N-1}, u_{N-1}) \wedge \mu_G(x_N) \;\ge\; \alpha. \qquad (2)$$

Here the sequence of decisions $\{u_0,\ u_1,\ \ldots,\ u_{N-1}\}$ in (1),(2) is uniquely determined through Markov policy $\pi = \{\pi_0,\ \ldots,\ \pi_{N-1}\}$:

$$u_0 = \pi_0(x_0),\ u_1 = \pi_1(x_1),\ \ldots,\ u_{N-1} = \pi_{N-1}(x_{N-1}). \qquad (3)$$

As for controlling threshold probability on the Markov chain $\{(X_n, U_n)\}$ with reward functions $\{\{r_n\}_0^{N-1},\ r_N\}$ and a lower level value c, Markov class Π is not enough for *additive* criteria :

$$\mathrm{Q}_0(x_0) \qquad \begin{array}{l} \text{Maximize } P^{\sigma}_{x_0}(\,r_0 + r_1 + \cdots + r_{N-1} + r_N \ge c\,) \\ \text{subject to (i)}_\mathrm{n},\ \text{(ii)}_\mathrm{n} \quad 0 \le n \le N-1 \end{array}$$

but general class Π_g is enough [8, 18]. However, in this paper, we dare to maximize the threshold probability for *minimum* criteria over *Markov class*.

Thus our problem $\mathrm{P}_0(x_0)$ is to find the *maximum value function* $v_0 = v_0(x_0)$ and an *optimal policy* $\pi^*(\in \Pi)$ which attains the maximum :

$$\begin{aligned} v_0(x_0) &= P^{\pi^*}_{x_0}(\,\mu_0 \wedge \cdots \wedge \mu_{N-1} \wedge \mu_G \ge \alpha\,) \\ &= \operatorname*{Max}_{\pi\in\Pi} P^{\pi}_{x_0}(\,\mu_0 \wedge \cdots \wedge \mu_{N-1} \wedge \mu_G \ge \alpha\,). \end{aligned} \qquad x_0 \in X \qquad (4)$$

3 Recursive formula

We consider the subproblem starting at state $x_n(\in X)$ on n-th stage and terminating on the final N-th stage $(0 \le n \le N-1)$:

$$\mathrm{P}_n(x_n) \qquad \text{Max } P^{\pi}_{x_n}(\mu_n \wedge \cdots \wedge \mu_{N-1} \wedge \mu_G \ge \alpha) \quad \text{s.t. (i)}_{\mathrm{m}},\ \text{(ii)}_{\mathrm{m}} \quad n \le m \le N-1$$

where $\pi = \{\pi_n, \ldots, \pi_{N-1}\}$ is taken over *Markov class from n-th stage on* $\Pi(n)$.

Let $v_n(x_n)$ be the maximum value, where

$$v_N(x_N) \triangleq P(\mu_G \ge \alpha \mid X_N = x_N) = \begin{cases} 1 & x_N \ge \alpha \\ 0 & \text{otherwise} \end{cases} \quad x_N \in X. \tag{5}$$

Lemma 3.1 *We have for any* $0 \le n \le N-1$, $x_n \in X$ *and* $\pi = \{\pi_n, \ldots, \pi_{N-1}\} \in \Pi(n)$

$$P^{\pi}_{x_n}(\mu_n \wedge \cdots \wedge \mu_G \ge c) = \begin{cases} \sum_{x_{n+1} \in X} P^{\pi'}_{x_{n+1}}(\mu_{n+1} \wedge \cdots \wedge \mu_G \ge c) p(x_{n+1}|x_n, u_n) \\ \qquad\qquad\qquad \textit{if } \mu_n(x_n, u_n) \ge \alpha \\ 0 \qquad\qquad\qquad \textit{otherwise} \end{cases}$$

where $u_n = \pi_n(x_n)$, $\pi' = \{\pi_{n+1}, \ldots, \pi_{N-1}\}$ *and* $P^{\pi'}_{x_N} := P$ *in (5) for* $\pi = \{\pi_{N-1}\}$.
Equivalently, in terms of multiple sum, we get

$$\sum\sum\cdots\sum_{(x_{n+1}, x_{n+2}, \ldots, x_N) \in (*)} p_{n+1} p_{n+2} \cdots p_N = \begin{cases} \sum_{x_{n+1} \in X} \left[\sum\sum\cdots\sum_{(x_{n+2}, \ldots, x_N) \in (\star)} p_{n+2} \cdots p_N \right] p(x_{n+1}|x_n, u_n) \\ \qquad\qquad\qquad \textit{if } \mu_n(x_n, u_n) \ge \alpha \\ 0 \qquad\qquad\qquad \textit{otherwise} \end{cases}$$

where $p_m = p(x_m|x_{m-1}, u_{m-1})$, $u_m = \pi_m(x_m)$, $(*)$ *denotes the partial multiple sum over* $(x_{n+1}, \ldots, x_N) \in X \times \cdots \times X$ *satisfying* $\mu_n(x_n, u_n) \wedge \cdots \wedge \mu_G(x_N) \ge \alpha$, *and* $(\star)$ *denotes* $(x_{n+2}, \ldots, x_N)$ *satisfying* $\mu_{n+1}(x_{n+1}, u_{n+1}) \wedge \cdots \wedge \mu_G(x_N) \ge \alpha$.

Thus we have the backward recursive relation :

Theorem 3.1 *(Recursive Equation)*

$$v_N(x) = \begin{cases} 1 & \textit{if } \mu_G(x) \ge \alpha \\ 0 & \textit{otherwise} \end{cases} \qquad x \in X$$

$$v_n(x) = \begin{cases} \underset{u;\mu_n(x,u) \ge c}{\text{Max}} \sum_{y \in X} v_{n+1}(y) p(y|x,u) & \textit{if } \exists\, u\ ;\ \mu_n(x,u) \ge \alpha \\ 0 & \textit{otherwise} \end{cases} \tag{6}$$

$$x \in X, \quad 0 \le n \le N-1.$$

Now let us take any pair (n, x). If it satisfies $\mu_n(x, u) \ge \alpha$, then let $\pi^*_n(x)$ denote a $u^* \in U$ which attains the maximum in (6). Otherwise, let $\pi^*_n(x)$ denote any $u \in U$. Then we have an optimal n-th decision function $\pi^*_n : X \to U$. Thus we construct an optimal policy $\pi^* = \{\pi^*_0, \ldots, \pi^*_{N-1}\}$ in Markov class Π.

4 Bellman and Zadeh's Model

Let us consider maximizing the threshold probability with lower membership-degree $\alpha = 0.7$ on Bellman and Zadeh's model [4, pp.B154] :

$$\begin{array}{lll} \text{Max} & P^{\pi}_{x_0}(\,\mu_0(U_0) \wedge \mu_1(U_1) \wedge \mu_G(X_2) \geq 0.7\,) & \\ \text{s.t.} & (\text{i})_\text{n} \;\; X_{n+1} \sim p(\,\cdot\,|x_n, u_n) & n = 0, 1 \\ & (\text{ii})_\text{n} \;\; u_n \in U & \end{array}$$

where the numerical data is theirs :

$$\mu_0(a_1) = 0.7 \qquad \mu_0(a_2) = 1.0\;;$$
$$\mu_1(a_1) = 1.0 \qquad \mu_1(a_2) = 0.6\;;$$
$$\mu_G(s_1) = 0.3 \qquad \mu_G(s_2) = 1.0 \qquad \mu_G(s_3) = 0.8$$

	$p(x_{n+1}\|x_n, a_1)$			$p(x_{n+1}\|x_n, a_2)$		
$x_n \backslash x_{n+1}$	s_1	s_2	s_3	s_1	s_2	s_3
s_1	0.8	0.1	0.1	0.1	0.9	0.0
s_2	0.0	0.1	0.9	0.8	0.1	0.1
s_3	0.8	0.1	0.1	0.1	0.0	0.9

The backward recursion (6) yields optimal solution in Markov class Π — a pair of a sequence of optimal value fuctions

$$v_0 = v_0(x_0),\;\; v_1 = v_1(x_1),\;\; v_2 = v_2(x_2)$$

and an optimal policy

$$\pi^* = \{\pi^*_0(x_0),\;\; \pi^*_1(x_1)\}.$$

The optimal solution is tabulated as

x_n	$v_2(x_2)$	$v_1(x_1)$	$\pi^*_1(x_1)$	$v_0(x_0)$	$\pi^*_0(x_0)$
s_1	0	0.2	a_1	0.92	a_2
s_2	1	1.0	a_1	0.28	a_1, a_2
s_3	1	0.2	a_1	0.28	a_1

Table 1: Optimal Solution

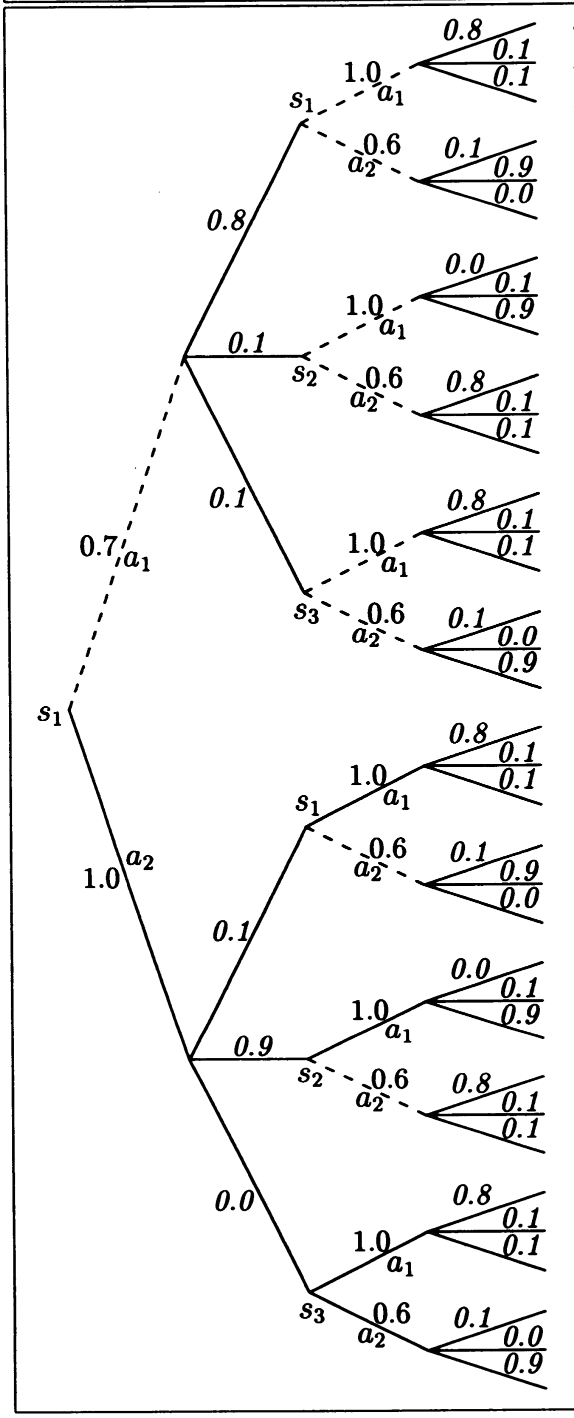
Furthermore, we have another two methods. One is stochastic decision tree-table methods(Figure 1). The other is total enumalation of all Markov policy and related threshold probability vector(Table 2).

So we have three approaches. Through these three approaches, we have obtained optimal solution ; optimal value (0.92, 0.28, 0.28) and optimal policy π^*.

$$w_0(s_1) = \underset{\pi\in\Pi}{\mathrm{Max}}\, P^{\pi}_{x_0}(\,\mu_0(U_0) \wedge \mu_1(U_1) \wedge \mu_G(X_2) \geq 0.7\,)$$

Figure 1 : Two-stage stochastic decision tree-table from state s_1

history: x_0 μ_0/u_0 p_0 x_1 μ_1/u_1 p_1



x_2	μ_G	mini mum	*path prob.*	thre-prob.	sub-prob.	total-prob.
s_1	0.3	0.3	*0.64*	0		
s_2	1.0	0.7	*0.08*	0.08	**0.16**	
s_3	0.8	0.7	*0.08*	0.08		
s_1	0.3	0.3	*0.08*	0		
s_2	1.0	0.6	*0.72*	0	0	
s_3	0.8	0.6	*0.0*	0		
s_1	0.3	0.3	*0.0*	0		
s_2	1.0	0.7	*0.01*	0.01	**0.1**	
s_3	0.8	0.7	*0.09*	0.09		0.28
s_1	0.3	0.3	*0.08*	0		
s_2	1.0	0.6	*0.01*	0	0	
s_3	0.8	0.6	*0.01*	0		
s_1	0.3	0.3	*0.08*	0		
s_2	1.0	0.7	*0.01*	0.01	**0.02**	
s_3	0.8	0.7	*0.01*	0.01		
s_1	0.3	0.3	*0.01*	0		
s_2	1.0	0.6	*0.0*	0	0	
s_3	0.8	0.6	*0.09*	0		
s_1	0.3	0.3	*0.08*	0		
s_2	1.0	1.0	*0.01*	0.01	**0.02**	
s_3	0.8	0.8	*0.01*	0.01		
s_1	0.3	0.3	*0.01*	0		
s_2	1.0	0.6	*0.09*	0	0	
s_3	0.8	0.6	*0.0*	0		
s_1	0.3	0.3	*0.0*	0		
s_2	1.0	1.0	*0.09*	0.09	**0.9**	
s_3	0.8	0.8	*0.81*	0.81		**0.92**
s_1	0.3	0.3	*0.72*	0		
s_2	1.0	0.6	*0.09*	0	0	
s_3	0.8	0.6	*0.09*	0		
s_1	0.3	0.3	*0.0*	0		
s_2	1.0	1.0	*0.0*	0.0	**0**	
s_3	0.8	0.8	*0.0*	0.0		
s_1	0.3	0.3	*0.0*	0		
s_2	1.0	0.6	*0.0*	0.0	**0**	
s_3	0.8	0.6	*0.0*	0		

$$J(x_0;\pi) \quad = \quad P^{\pi}_{x_0}(\mu_0 \wedge \mu_1 \wedge \mu_G \geq 0.7)$$

le 2 : all threshold-probability vectors $J(\pi) = \begin{pmatrix} J(s_1;\pi) \\ J(s_2;\pi) \\ J(s_3;\pi) \end{pmatrix}$, where $\pi = \{\pi_0,\, \pi_1\}$ is Ma

$\pi_0 \setminus \pi_1$	$\begin{pmatrix} a_1 \\ a_1 \\ a_1 \end{pmatrix}$	$\begin{pmatrix} a_1 \\ a_1 \\ a_2 \end{pmatrix}$	$\begin{pmatrix} a_1 \\ a_2 \\ a_1 \end{pmatrix}$	$\begin{pmatrix} a_1 \\ a_2 \\ a_2 \end{pmatrix}$	$\begin{pmatrix} a_2 \\ a_1 \\ a_1 \end{pmatrix}$	$\begin{pmatrix} a_2 \\ a_1 \\ a_2 \end{pmatrix}$	$\begin{pmatrix} a_2 \\ a_2 \\ a_1 \end{pmatrix}$	$\begin{pmatrix} a_2 \\ a_2 \\ a_2 \end{pmatrix}$
$\begin{pmatrix} a_1 \\ a_1 \\ a_1 \end{pmatrix}$	$\begin{pmatrix} 0.28 \\ 0.28 \\ 0.28 \end{pmatrix}$	$\begin{pmatrix} 0.26 \\ 0.1 \\ 0.26 \end{pmatrix}$	$\begin{pmatrix} 0.18 \\ 0.18 \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.16 \\ 0.0 \\ 0.16 \end{pmatrix}$	$\begin{pmatrix} 0.12 \\ 0.28 \\ 0.12 \end{pmatrix}$	$\begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix}$	$\begin{pmatrix} 0.02 \\ 0.18 \\ 0.02 \end{pmatrix}$	$\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$
$\begin{pmatrix} a_1 \\ a_1 \\ a_2 \end{pmatrix}$	$\begin{pmatrix} 0.28 \\ 0.28 \\ 0.2 \end{pmatrix}$	$\begin{pmatrix} 0.26 \\ 0.1 \\ 0.02 \end{pmatrix}$	$\begin{pmatrix} 0.18 \\ 0.18 \\ 0.2 \end{pmatrix}$	$\begin{pmatrix} 0.16 \\ 0.0 \\ 0.02 \end{pmatrix}$	$\begin{pmatrix} 0.12 \\ 0.28 \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.1 \\ 0.1 \\ 0.0 \end{pmatrix}$	$\begin{pmatrix} 0.02 \\ 0.18 \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$
$\begin{pmatrix} a_1 \\ a_2 \\ a_1 \end{pmatrix}$	$\begin{pmatrix} 0.28 \\ 0.28 \\ 0.28 \end{pmatrix}$	$\begin{pmatrix} 0.26 \\ 0.26 \\ 0.26 \end{pmatrix}$	$\begin{pmatrix} 0.18 \\ 0.18 \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.16 \\ 0.16 \\ 0.16 \end{pmatrix}$	$\begin{pmatrix} 0.12 \\ 0.12 \\ 0.12 \end{pmatrix}$	$\begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix}$	$\begin{pmatrix} 0.02 \\ 0.02 \\ 0.02 \end{pmatrix}$	$\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$
$\begin{pmatrix} a_1 \\ a_2 \\ a_2 \end{pmatrix}$	$\begin{pmatrix} 0.28 \\ 0.28 \\ 0.2 \end{pmatrix}$	$\begin{pmatrix} 0.26 \\ 0.26 \\ 0.02 \end{pmatrix}$	$\begin{pmatrix} 0.18 \\ 0.18 \\ 0.2 \end{pmatrix}$	$\begin{pmatrix} 0.16 \\ 0.16 \\ 0.02 \end{pmatrix}$	$\begin{pmatrix} 0.12 \\ 0.12 \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.1 \\ 0.1 \\ 0.0 \end{pmatrix}$	$\begin{pmatrix} 0.02 \\ 0.02 \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$
$\begin{pmatrix} a_2 \\ a_1 \\ a_1 \end{pmatrix}$	$\begin{pmatrix} \mathbf{0.92} \\ \mathbf{0.28} \\ \mathbf{0.28} \end{pmatrix}$	$\begin{pmatrix} 0.92 \\ 0.1 \\ 0.26 \end{pmatrix}$	$\begin{pmatrix} 0.02 \\ 0.18 \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.02 \\ 0.0 \\ 0.16 \end{pmatrix}$	$\begin{pmatrix} 0.9 \\ 0.28 \\ 0.12 \end{pmatrix}$	$\begin{pmatrix} 0.9 \\ 0.1 \\ 0.1 \end{pmatrix}$	$\begin{pmatrix} 0.0 \\ 0.18 \\ 0.02 \end{pmatrix}$	$\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$
$\begin{pmatrix} a_2 \\ a_1 \\ a_2 \end{pmatrix}$	$\begin{pmatrix} 0.92 \\ 0.28 \\ 0.2 \end{pmatrix}$	$\begin{pmatrix} 0.92 \\ 0.1 \\ 0.26 \end{pmatrix}$	$\begin{pmatrix} 0.02 \\ 0.18 \\ 0.2 \end{pmatrix}$	$\begin{pmatrix} 0.02 \\ 0.0 \\ 0.02 \end{pmatrix}$	$\begin{pmatrix} 0.9 \\ 0.28 \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.9 \\ 0.1 \\ 0.0 \end{pmatrix}$	$\begin{pmatrix} 0.0 \\ 0.18 \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$
$\begin{pmatrix} a_2 \\ a_2 \\ a_1 \end{pmatrix}$	$\begin{pmatrix} \mathbf{0.92} \\ \mathbf{0.28} \\ \mathbf{0.28} \end{pmatrix}$	$\begin{pmatrix} 0.92 \\ 0.26 \\ 0.26 \end{pmatrix}$	$\begin{pmatrix} 0.02 \\ 0.18 \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.02 \\ 0.16 \\ 0.16 \end{pmatrix}$	$\begin{pmatrix} 0.9 \\ 0.12 \\ 0.12 \end{pmatrix}$	$\begin{pmatrix} 0.9 \\ 0.1 \\ 0.1 \end{pmatrix}$	$\begin{pmatrix} 0.0 \\ 0.02 \\ 0.02 \end{pmatrix}$	$\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$
$\begin{pmatrix} a_2 \\ a_2 \\ a_2 \end{pmatrix}$	$\begin{pmatrix} 0.92 \\ 0.28 \\ 0.2 \end{pmatrix}$	$\begin{pmatrix} 0.92 \\ 0.26 \\ 0.26 \end{pmatrix}$	$\begin{pmatrix} 0.02 \\ 0.18 \\ 0.2 \end{pmatrix}$	$\begin{pmatrix} 0.02 \\ 0.16 \\ 0.02 \end{pmatrix}$	$\begin{pmatrix} 0.9 \\ 0.12 \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.9 \\ 0.1 \\ 0.0 \end{pmatrix}$	$\begin{pmatrix} 0.0 \\ 0.02 \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$

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