数理解析研究所講究録 1252 巻 2002 年 211-217

Competitive Facility Location Problem with Fuzzy Relative Distance

Shigehiro OSUMI (大角 盛広)* and Shogo SHIODE (塩出 省吾)**
* Faculty of Design, Kobe Design University (神戸芸術工科大学 芸術工学部)
8-1-1 Gakuennishi-machi, Nishi-ku, Kobe 6512196, Japan
E-mail: osumi@kobe-du.ac.jp
** Faculty of Economics, Kobe Gakuin University (神戸学院大学 経済学部)
518 Arise, Ikawadani-cho, Nishi-ku, Kobe 6512180, Japan
E-mail: shiode@eb.kobegakuin.ac.jp

Abstract. This paper investigate a competitive facility location problem where there are two facilities on a linear market. Customers at a demand point utilize the facility which seems to be the nearest one from them. We assume that they do not distinguish the small difference between two distances, i.e., they do not always utilize the nearest facility by strictly measuring physical distance. We formulate this preference by introducing fuzzy difference of the actual distance between two points.

In our model, two companies, the leader and the follower, establish their facilities in this market to get as much buying power as possible. This paper considers the problems to find the optimal location for the follower and for the leader. We formulate these problems as a medianoid problem and a centroid problem respectively, and show the domains which contain the solutions for these problems.
Keywords. continuous location, noncooperative games, fuzzy programming

1 Introduction

Competitive location problems were introduced by H.Hotelling [1], who studied the Nash equilibrium problem of two sellers on a linear market. S.L.Hakimi considered the Stackelberg equilibrium problem on a network [2], that is, two companies "leader" and "follower" establish their facilities on nodes in order to capture as much buying power as possible. He showed that the problem is NP-hard. Z.Drezner studied the same kind of a competitive problem on a plane [3]. Although a large number of studies have been made on Nash equilibrium, but there is few on Stackelberg equilibrium.

Many of these kind of models were based on a hypothesis that the customers utilize strictly the nearest facility. But actually, customers at a demand point measure the distance to the facilities by some mental way and choose one which is relatively near. We assume that they do not distinguish the small difference between two distances, i.e., they do not always feel 1.01km is nearer than 1.05km. So we introduce relative distance and fuzzy set to represent the concept of nearness, which is determined by actual distances between facilities and a customer.

In out model, leader X and follower Y establish their facilities on the market in order to capture as much buying power as possible. X locate his facility first, and Y locates his facility, knowing the decision of the company X. So, the company X must determine his optimal location by considering that the competitor locate his facility myopically afterward. There are two types of problems, i.e., to find the optimal location for Y and to find that for X. We formulate these problems as a medianoid problem and a centroid problem, and show the domains which contain the solutions for these problems.

2 Our Model and Formulation

2.1 Relative Distance

Let $d(p, x)$ denote the distance between a demand point P and the facility X. We introduce the following function f_Y which represent the relative distance between P and Y, when the location of X is given.

$$f_Y(p, y|x) = \frac{d(p, y) - d(p, x)}{\frac{d(p,y)+d(p,x)}{2}}$$

Function f_X can be defined by the same way. By introducing a coefficient $\alpha > 0$ which satisfies $d(p, y) = \alpha\, d(p, x)$, we can redefine f_Y as follows.

$$f_Y(\alpha) = \frac{2(\alpha - 1)}{\alpha + 1}, \quad \alpha = \frac{d(p, y)}{d(p, x)}$$

provided that when $d(p, x) = 0$ and $d(p, y) \neq 0$ then $f_Y(\alpha) = 2$, when $d(p, x) = 0$ and $d(p, y) = 0$ then $f_Y(\alpha) = 0$.

2.2 Evaluating the Relative Distance

We introduce the following function g which represent the degree of feeling "there is no difference".

$$g(f_Y(\alpha)) = \begin{cases} 0, & f_Y(\alpha) \leq -f_1 \\ \frac{1}{f_1 - f_0}(f_Y(\alpha) + f_1), & -f_1 < f_Y(\alpha) < -f_0 \\ 1, & 0 \leq f_Y(\alpha) \leq f_0 \\ \frac{1}{f_0 - f_1}(f_Y(\alpha) - f_1), & f_0 < f_Y(\alpha) < f_1 \\ 0, & f_Y(\alpha) \geq f_1 \end{cases}$$

f_0, f_1 are constants which satisfy $0 < f_0 < f_1 < 2$.
If $g = 1$ then customers feel X and Y are at the same distance. If $g = 0$ then they feel X or Y is obviously near.

2.3 Preference

Customers utilize facility Y only when they feel Y is nearer than X. We use the fuzzy set $\tilde{Y}$ for relative nearness with the following membership function.

$$\mu_Y(\alpha) = \begin{cases} 1, & 0 \leq \alpha \leq \alpha_0 \\ 1 - \frac{0.5}{f_1 - f_0}\left(\frac{2(\alpha-1)}{\alpha+1} + f_1\right), & \alpha_0 < \alpha < \alpha_1 \\ 0.5, & \alpha_1 \leq \alpha \leq \frac{1}{\alpha_1} \\ \frac{0.5}{f_0 - f_1}\left(\frac{2(\alpha-1)}{\alpha+1} - f_1\right), & \frac{1}{\alpha_1} < \alpha < \frac{1}{\alpha_0} \\ 0, & \alpha \geq \frac{1}{\alpha_0} \end{cases}$$

α_0, α_1 are constants which satisfies $\frac{2(\alpha_0-1)}{\alpha_0+1} = -f_1, \frac{2(\alpha_1-1)}{\alpha_1+1} = -f_0$ and $0 < \alpha_0 < \alpha_1 < 1$. Figure 1 shows the shape of the preference function $\mu_Y(\alpha)$. Note that the curves are hyperbolas.

We assume that one's profit is in proportion to the amount of captured buying power, which is shared in proportion to $\mu_Y(\alpha)$. The sum of all buying power is always assumed to

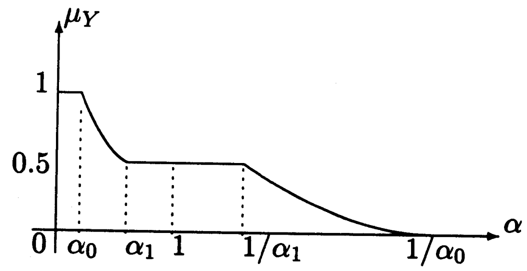

Figure 1: Preference function $\mu_Y(\alpha)$

2.4 Medianoid Problem and Centroid Problem

Using a distribution function $F(p)$, the profit of Y is denoted by

$$M_Y(x,y) = \int_{-\infty}^{\infty} \mu_Y(\alpha)dF(p) = \int_{-\infty}^{\infty} \mu_Y(p,x,y)dF(p) \ .$$

With given x, medianoid problem is the problem to find y which maximizes $M_Y(x,y)$. Let $y^*(x)$ denote the solution for the problem, then centroid problem is to find x which satisfies $\max_x M_X(x, y^*(x))$. Since this game is zero sum game, this problem is equivalent to $\min_x M_Y(x, y^*(x))$.

3 Uniform Distribution on a Line

We investigate a linear market on the interval $[0,1]$. Function M_Y becomes

$$\begin{aligned}
M_Y(x,y) &= \int_{\frac{y+\alpha_0 x}{1+\alpha_0}}^{\min\{1,\frac{y-\alpha_0 x}{1-\alpha_0}\}} dF(p) \\
&+ \int_{\frac{y+\alpha_1 x}{1+\alpha_1}}^{\frac{y+\alpha_0 x}{1+\alpha_0}} g(p)dF(p) + \int_{\min\{1,\frac{y-\alpha_0 x}{1-\alpha_0}\}}^{\min\{1,\frac{y-\alpha_1 x}{1-\alpha_1}\}} g(p)dF(p) \\
&+ \frac{1}{2}\int_0^{\max\{0,\frac{-x+\alpha_1 y}{-1+\alpha_1}\}} dF(p) + \frac{1}{2}\int_{\frac{x+\alpha_1 y}{1+\alpha_1}}^{\frac{y+\alpha_1 x}{1+\alpha_1}} dF(p) + \frac{1}{2}\int_{\min\{1,\frac{y-\alpha_1 x}{1-\alpha_1}\}}^{1} dF(p) \\
&+ \int_{\max\{0,\frac{-x+\alpha_1 y}{-1+\alpha_1}\}}^{\max\{0,\frac{-x+\alpha_0 y}{-1+\alpha_0}\}} h(p)dF(p) + \int_{\frac{x+\alpha_0 y}{1+\alpha_0}}^{\frac{x+\alpha_1 y}{1+\alpha_1}} h(p)dF(p) \ .
\end{aligned}$$

Obviously, if $x > \frac{1}{2}$ then Y gets no advantage by locating at $y > x$. So, we consider the case where $x \le \frac{1}{2}$ and $y > x$.

Now we investigate the change of M_Y with given x. Let M_{1Y} denote the first term of M_Y. It becomes

$$M_{1Y}(x,y) = \int_{\frac{y+\alpha_0 x}{1+\alpha_0}}^{\min\{1,\frac{y-\alpha_0 x}{1-\alpha_0}\}} 1\,dp = \begin{cases} \frac{2\alpha_0(x-y)}{\alpha_0{}^2-1}, & \frac{y-\alpha_0 x}{1-\alpha_0} \le 1 \\ 1 - \frac{\alpha_0 x + y}{1+\alpha_0}, & \text{otherwise} \end{cases} .$$

Calculating $\frac{\partial M_{1Y}(x,y)}{\partial y}$, M_{1Y} is linear on each domain.

The second term becomes

$$M_{2Y}(x,y) = \int_{\frac{y+\alpha_1 x}{1+\alpha_1}}^{\frac{y+\alpha_0 x}{1+\alpha_0}} \left(1 - \frac{\frac{2(1-\alpha_0)}{1+\alpha_0} + \frac{2\left(\frac{y-p}{p-x}-1\right)}{\frac{y-p}{p-x}+1}}{2\left(\frac{2(1-\alpha_0)}{1+\alpha_0} - \frac{2(1-\alpha_1)}{1+\alpha_1}\right)} \right) dp$$

$$= \frac{3\,(\alpha_1-\alpha_0)}{4\,(1+\alpha_0)\,(1+\alpha_1)}\,(y-x)\ .$$

Since the coefficient of y is positive, M_{2Y} is linear and increasing on the interval $(x,1]$.

$$M_{3Y}(x,y)=\int_{\min\{1,\frac{y-\alpha_0 x}{1-\alpha_0}\}}^{\min\{1,\frac{y-\alpha_1 x}{1-\alpha_1}\}}\left(1-\frac{\frac{2(1-\alpha_0)}{1+\alpha_0}+\frac{2\left(\frac{y-p}{p-x}-1\right)}{\frac{y-p}{p-x}+1}}{2\left(\frac{2(1-\alpha_0)}{1+\alpha_0}-\frac{2(1-\alpha_1)}{1+\alpha_1}\right)}\right)dp\ .$$

When $\frac{y-\alpha_0 x}{1-\alpha_0}<1$, $\frac{y-\alpha_1 x}{1-\alpha_1}<1$, i.e., $x<y<1-\alpha_1(1-x)$ then the coefficient of y is positive, so M_{3Y} is linear and increasing function on this interval.

When $\frac{y-\alpha_0 x}{1-\alpha_0}<1, 1<\frac{y-\alpha_1 x}{1-\alpha_1}$, i.e., $1-\alpha_0(1-x)<y<1-\alpha_1(1-x)$ then the third term becomes more complicated and the second partial derivative of M_{3Y} with respect to y becomes

$$\frac{\partial^2 M_{3Y}(x,y)}{\partial y^2}=\frac{-(1+\alpha_0)\,(1+\alpha_1)\,(-1+x)^2}{2\,(\alpha_0-\alpha_1)\,(x-y)\,(-2+x+y)^2}<0\ .$$

So M_{3Y} is concave downward on this interval.

When $1<\frac{y-\alpha_0 x}{1-\alpha_0}, 1<\frac{y-\alpha_1 x}{1-\alpha_1}$, i.e., $y>1-\alpha_0(1-x)$ then the third term becomes $M_{3Y}(x,y)=0$.

The fourth term becomes

$$M_{4Y}(x,y)=\frac{1}{2}\int_0^{\max\{0,\frac{-x+\alpha_1 y}{-1+\alpha_1}\}}1\,dp=\begin{cases}0, & \frac{\alpha_1 y-x}{\alpha_1-1}\le 0\\ \frac{\alpha_1 y-x}{2(\alpha_1-1)}, & \text{otherwise}\end{cases}\ .$$

So M_{4Y} is linear and strictly decreasing function where $y<\frac{x}{\alpha_1}$.

The fifth term becomes

$$M_{5Y}(x,y)=\frac{1}{2}\int_{\frac{x+\alpha_1 y}{1+\alpha_1}}^{\frac{y+\alpha_1 x}{1+\alpha_1}}1\,dp=\frac{(1-\alpha_1)}{2\,(1+\alpha_1)}\,(y-x)\ .$$

Since the coefficient of y is positive, $M_{5Y}$is linear and increasing on the interval $(x,1)$.

The sixth term becomes

$$M_{6Y}=\frac{1}{2}\int_{\min\{\frac{y-\alpha_1 x}{1-\alpha_1}\}}^{1}1\,dp=\begin{cases}\frac{-1\alpha_1-\alpha_1 x+y}{2(\alpha_1-1)}, & \frac{y-\alpha_1 x}{1-\alpha_1}<1\\ 0, & \text{otherwise}\end{cases}\ .$$

So M_{6Y} is linear and increasing function where $y<1-\alpha_1(1-x)$.

When $\frac{\alpha_1 y-x}{\alpha_1-1}>0, \frac{\alpha_0 y-x}{\alpha_0-1}>0$, i.e,. $x<y<\frac{x}{\alpha_1}$ then the seventh term is linear and strictly increasing.

When $\frac{\alpha_1 y-x}{\alpha_1-1}<0, \frac{\alpha_0 y-x}{\alpha_0-1}>0$, i.e., $\frac{x}{\alpha_1}<y<\frac{x}{\alpha_0}$ then the seventh term is not linear and the second derivative of M_{7Y} with respect to y becomes

$$\frac{\partial^2 M_{7Y}(x,y)}{\partial y^2}=\frac{(1+\alpha_0)\,(1+\alpha_1)\,x^2}{2\,(\alpha_0-\alpha_1)\,(x-y)\,(x+y)^2}>0$$

Therefore M_{7Y} is concave upward on the interval $(\frac{x}{\alpha_1},\frac{x}{\alpha_0})$.

When $\frac{\alpha_1 y-x}{\alpha_1-1}<0, \frac{\alpha_0 y-x}{\alpha_0-1}<0$, i.e., $y>\frac{x}{\alpha_0}$ then $M_{7Y}(x,y)=0$.

The eighth term M_{8Y} becomes

$$M_{8Y}(x,y)=\int_{\frac{x+\alpha_0 y}{1+\alpha_0}}^{\frac{x+\alpha_1 y}{1+\alpha_1}}\left(\frac{\frac{2(\alpha_0-1)}{1+\alpha_0}+\frac{2\left(\frac{y-p}{p-x}-1\right)}{1+\frac{y-p}{p-x}}}{2\left(\frac{2(\alpha_0-1)}{1+\alpha_0}-\frac{2(\alpha_1-1)}{1+\alpha_1}\right)}\right)dp=\frac{(\alpha_1-\alpha_0)}{4\,(1+\alpha_0)\,(1+\alpha_1)}\,(y-x)\ .$$

Since the coefficient of y is positive, M_{8Y} is linear and increasing function on the interval $(x, 1]$.

Comparing the slopes of $M_{1Y} \cdots M_{8Y}$ on each interval, the maximal value of M_Y exists on the interval $[1-\alpha_1(1-x), 1-\alpha_0(1-x)]$. In this interval, $M_{1Y}, M_{2Y}, M_{5Y}, M_{8Y}$ are linear functions, $M_{4Y}=0, M_{6Y}=0$. So we check $M_{3Y}+M_{7Y}$ as follows.

$$\frac{\partial^2(M_{3Y}+M_{7Y})}{\partial y^2} = \frac{-(1+\alpha_0)\,(1+\alpha_1)\,(x\,(2\,x+2\,y-3)-y)}{2\,(\alpha_0-\alpha_1)\,(x+y-2)^2\,(x+y)^2} < 0$$

Therefore $M_{3Y}+M_{7Y}$ is concave downward.

Considering $M_{4Y}=0$ and $M_{6Y}=0$, M_Y becomes

$$M_Y(x,y) = \frac{1}{2} + \frac{(1+\alpha_0)\,(1-\alpha_1)\,(1-x-y)}{4\,(\alpha_0-\alpha_1)} - \frac{(1+\alpha_0)\,(1+\alpha_1)\,(x-y)\,\log(\frac{x+y}{2-x-y})}{8\,(\alpha_0-\alpha_1)}\,.$$

So, the location of y which maximizes M_Y, i.e., the solution for the medianoid problem exists on the extreme points of the interval $[1-\alpha_1(1-x), 1-\alpha_0(1-x)]$ or on the point which makes the derivative zero as follows.

$$\frac{\partial M_Y}{\partial y} = \frac{(1+\alpha_0)\,\left(2\,(\alpha_1-1)+\frac{2\,(1+\alpha_1)\,(x-y)}{(x+y-2)\,(x+y)}+(1+\alpha_1)\,\log(\frac{x+y}{2-x-y})\right)}{8\,(\alpha_0-\alpha_1)} = 0$$

This equation cannot be solved by algebraic way, but we can solve actual concrete problems by some numerical methods. Let $\bar{y}$ denote the solution, then the solution for the medianoid problem with given x is

$$y^*(x) = \max_y M_Y(x,y)\,,\; y = \{1-\alpha_1(1-x)\,,\,\bar{y}\,,\,1-\alpha_0(1-x)\}\,.$$

Then the solution for the centroid problem is

$$x^* = \min_x M_Y(x, y^*(x))\,.$$

On the above part, we treated x as a given number, since X locate his facility first. Let $R_Y = \{(x,y)\}$ denote the set of Y's optimal reaction strategy against X, then $M_R(x) = M_Y(x,y)$, $(x,y) \in R_Y$ is the set of peak points of the functions M_Y which shapes are fixed by the location of x.

Then the solution for the centroid problem is rewrote as follows.

$$x^* = \min_x M_R(x)$$

We can examine the concavity of the function $M_R(x)$ by using the symmetricity between x and y.

4 Separated Market on a Line

In this section, we investigate a liner market with a gap where no customer exists. We assume the width of gap is denoted by S and the market is symmetrical about the middle point. In this case, following properties hold.

Property 1 *The solution for the medianoid problem and the centroid problem are on the interval* $[0,1]$. *More generally, they are in the convex hull of demand points.*

Brief Proof If $x<0$ then Y can get 1 (=all buying power) locating at $y=0$. Similar logic holds in other cases.

Figure 2 shows the numerical example of uniform distribution with $\alpha_0 = 0.45, \alpha_1 = 0.8$. The horizontal axis expresses the location of X and depth expresses the location of Y on the same line. If X locate at $x = \frac{1}{2}$, then M_Y does not exceed $\frac{1}{2}$ wherever Y may locate. So, in this case, the solution for the centroid problem is $x = \frac{1}{2}$.

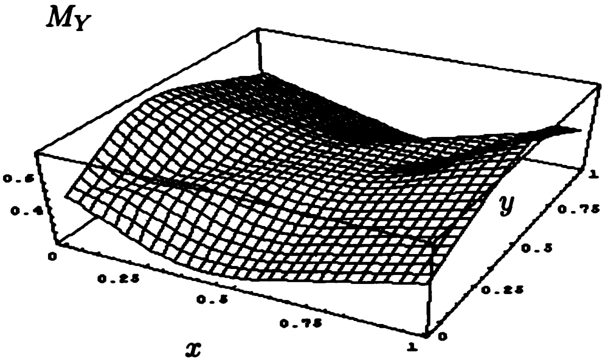

Figure 2: Uniform Distribution

Property 2 *The solution for the centroid problem does not exist on the gap. This property holds even if there are more than one gap.*

Brief Proof If $x \geq \frac{1}{2}$ is on the gap, then there exist some points on the interval $[\frac{1-S}{2}, x]$ where Y can get more than $\frac{1}{2}$ by making $\frac{y-\alpha_1 x}{1-\alpha_1}$ greater than $\frac{1+S}{2}$.

At first, we investigate the case where X is on the extreme point, i.e., $x = \frac{1-S}{2}$. In this case, if the gap is narrow then Y can get more than $\frac{1}{2}$ by moving y from $\frac{1+S}{2}$ to 1, making $\frac{y+\alpha_0 x}{1+\alpha_0}$ greater than $\frac{1+S}{2}$. However, if the gap is wide, Y cannot increase his profit by moving from $\frac{1+S}{2}$ to any direction. So we check the condition that Y cannot get more than $\frac{1}{2}$. The critical point is where $\frac{\alpha_0 y - x}{\alpha_0 - 1}$ becomes less than 0, so the condition is $S \geq \frac{1-\alpha_0}{1+\alpha_0}$.

Therefore if the gap is wider than or equal to $\frac{1-\alpha_0}{1+\alpha_0}$, the solutions for the centroid problem and the medianoid problem become

$$\begin{cases} x^* = \frac{1-S}{2} \\ y^* = \frac{1+S}{2} \end{cases} \quad \text{or} \quad \begin{cases} x^* = \frac{1+S}{2} \\ y^* = \frac{1-S}{2} \end{cases} .$$

In this case, X, Y share the buying power half and half.

When $S < \frac{1-\alpha_0}{1+\alpha_0}$, if x, y are on the extreme points, i.e., $x = \frac{1-S}{2}, y = \frac{1+S}{2}$ then Y can get more buying power by moving to the right while the inequality $\frac{1+S}{2} \leq \frac{y+\alpha_0 x}{1+\alpha_0}$ holds. Y begin to lose some buying power when the inequality $\frac{y+\alpha_1 x}{1+\alpha_1} \leq \frac{1+S}{2}$ or $\frac{y-\alpha_0 x}{1-\alpha_0} \leq 1$ holds. So, if X locate at $x = \frac{1-S}{2}$ then the solution y^* for the medianoid problem satisfies the following inequality.

$$\frac{(1+S)(1+\alpha_0)}{2} - \alpha_0 x \leq y^* \leq \min\{\alpha_0 x + 1 - \alpha_0, \frac{(1+\alpha_1)(1+S)}{2} - \alpha_1 x\}$$

So, when $S < \frac{1-\alpha_0}{1+\alpha_0}$ and $x = \frac{1-S}{2}$, Y can get more than $\frac{1}{2}$ by locating at y^*. Conversely thinking, X may have better solution in this case, i.e., at the beginning X locate at the point symmetrical with y^*, not $\frac{1-S}{2}$. But in this case Y can get more buying power by moving from y^* to 1, since $M_{1Y} + +M_{2Y} + M_{3Y} + M_{7Y} + M_{8Y}$ is concave downward (same as uniform distribution).

Figure 3 and 4 shows the results of numerical experiment with $\alpha_0 = 0.45$, $\alpha_1 = 0.8$. Figure 3 shows a market with wide gap ($S = \frac{1}{2}$) and Figure 4 shows the case of narrow gap ($S = \frac{1}{8}$). On each figure, the seven curves show the value of M_Y in $x = 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5$ from the left to the right, respectively. The graphs are drawn for y on $[x, 0.9]$.

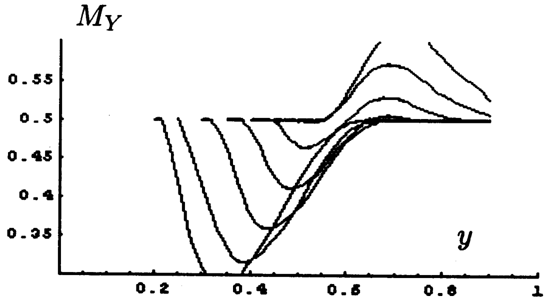

Figure 3: $S = \frac{1}{2}$

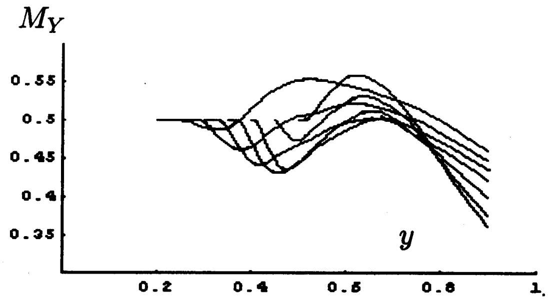

Figure 4: $S = \frac{1}{8}$

When $S = \frac{1}{2}$, the solution for centroid problem is $x = 0.25$ which makes M_Y less than or equal to $\frac{1}{2}$. When $S = \frac{1}{8}$, it can read that $x = 0.3$ is a good approximation solution for centroid problem.

5 Conclusion

- We use the preference based on fuzzy relative difference of distance.
- Competitive facility location problems with "leader" and "follower" are formulated as the centroid problem and the medianoid medianoid problem.
- The interval is shown which contains the solution for the medianoid problem on a linear market with uniform distribution.
- In the divided market, the conditions are shown for which the solution for the centroid problem does not come to the extreme points of a market.
- In the divided market, the domain is shown in which the solution for the medianoid problem exists.

Our further research is finding a solution with other distributions, and extending the market on a plane.

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