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フィードバックをもつ混合型ガウス型通信路の容量について, II

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1　はじめに

前回の講究録 (No.1186) においては混合型ガウス型通信路の容量に関してその性質を明らかにしたが、今回はその続きである. まず第 2 章では未解決問題 1 として Cover の conjecture をあげる. 次に第 3 章では未解決問題 2 として $C_{n,FB,\cdot}(P)$ の凸性を示す. また第 4 章においては未解決問題 3 として $R_{\tilde{Z}}^{(n)}=\alpha R_{Z_1}^{(n)}+\beta R_{Z_2}^{(n)}$ で定義される雑音 $\tilde{Z}$ をもつときの容量 $C_{n,FB,\tilde{Z}}(\alpha P_1+\beta P_2)$ と $C_{n,FB,Z_1}(P_1)$ と $C_{n,FB,Z_2}(P_2)$ との間に成り立つであろう関係式を扱う.

今まで何度もフィードバックをもつガウス型通信路の容量について報告しているのでその詳細な定義は省略する. もし厳密な定義を必要とする場合は他の報告書を参照していただきたい. フィードバックをもつ有限ブロック長容量は次のように定義される.

$$C_{n,FB,Z}(P)=\max\frac{1}{2n}\log\frac{|R_X^{(n)}+R_Z^{(n)}|}{|R_Z^{(n)}|},$$

ただし $|\cdot|$ は行列式を表し、最大値は

$$Tr[(I+B)R_X^{(n)}(I+B)^t+BR_Z^{(n)}B^t]\leq nP$$

を満たす狭義下三角行列 B と非負対称行列 $R_X^{(n)}$ についてとる. 同様にフィードバックがないときには容量 $C_{n,Z}(P)$ は $B=0$ としたときの最大値である. これらの条件の下で Cover and Pombra [5] は次を得た.

Proposition 1 (Cover and Pombra [5]) 任意の $\epsilon > 0$ に対して各 $n = 1, 2, \ldots$ でブロック長 n で $2^{n(C_{n,FB,Z}(P)-\epsilon)}$ 個の符号語が存在して $n \to \infty$ のとき $Pe^{(n)} \to 0$ とできる. 逆に任意の $\epsilon > 0$ とブロック長 n で $2^{n(C_{n,FB,Z}(P)+\epsilon)}$ 個の符号語からなる任意の符号の列に対しても $Pe^{(n)} \to 0$ $(n \to \infty)$ が成り立たない. これはフィードバックをもたない場合も成り立つ.

$C_{n,Z}(P)$ は正確に得られている.

Proposition 2 (Gallager [9])

$$C_{n,Z}(P) = \frac{1}{2n} \sum_{i=1}^{k} \log \frac{nP + r_1 + \cdots + r_k}{k r_i},$$

ただし $0 < r_1 \le r_2 \le \cdots \le r_n$ は $R_Z^{(n)}$ の固有値、$k(\le n)$ は $nP + r_1 + r_2 + \cdots + r_k > k r_k$ を満たす最大整数である.

ところで $C_{n,FB,Z}(P)$ は正確には得られていないので、今まで多くの人々によって様々な形の上界が得られている ([1],[2],[3], [5],[7],[8],[11], [12],[14],[15],[16]). 以下計算の都合上、対数は自然対数を用いることにする.

2 未解決問題 1

未解決問題 1

$$C_{n,FB,Z}(P) \le C_{n,Z}(2P)\ ?$$

今まで次の結果が得られている.

Theorem 1 (Cover-Pombra [5])

$$C_{n,FB,Z}(P) \le \min\{2C_{n,Z}(P), C_{n,Z}(P) + \frac{1}{2}\log 2\}.$$

Theorem 2 (Chen-Yanagi [1])

$$C_{n,Z}(2P) \leq \min\{2C_{n,Z}(P), C_{n,Z}(P) + \frac{1}{2}\log 2\}.$$

Theorem 3 (Chen-Yanagi [1])

$$C_{2,FB,Z}(P) \leq C_{2,Z}(2P).$$

3 未解決問題 2

Definition 1 任意の $\alpha, \beta \geq 0 (\alpha + \beta = 1)$ と任意のガウス雑音 Z_1, Z_2 に対して $R_{\tilde{Z}} = \alpha R_{Z_1} + \beta R_{Z_2}$ とおく. このときガウス雑音 $\tilde{Z}$ をもつ通信路を混合型ガウス型通信路という.

未解決問題 2

$$C_{n,FB,\tilde{Z}}(P) \leq \alpha C_{n,FB,Z_1}(P) + \beta C_{n,FB,Z_2}(P)\ ?$$

今までは次の結果が得られている.

Theorem 4 (Yanagi-Chen-Yu [16])

$$C_{n,\tilde{Z}}(P) \leq \alpha C_{n,Z_1}(P) + \beta C_{n,Z_2}(P).$$

Theorem 5 (Yanagi-Chen-Yu [16]) $P = \alpha P_1 + \beta P_2$ を満たす $P_1, P_2 \geq 0$ が存在して

$$C_{n,FB,\tilde{Z}}(P) \leq \alpha C_{n,FB,Z_1}(P_1) + \beta C_{n,FB,Z_2}(P_2).$$

が成り立つ.

Theorem 6 (Yanagi-Chen-Yu [16]) 次の (a) 又は (b) の条件があれば未解決問題 *2* が成り立つ.

(a) R_{Z_1} の n 行 n 列を除いた部分行列と R_{Z_2} のそれが一致する.

(b) $\tilde{Z}$ がホワイト型である. 即ち $R_{\tilde{Z}}$ が対角行列である.

4 未解決問題 3

未解決問題 3 任意の $P_1, P_2 \geq 0$ と任意の $\alpha, \beta \geq 0(\alpha+\beta=1)$ に対して

$$\alpha C_{n,FB,Z_1}(P_1) + \beta C_{n,FB,Z_2}(P_2)$$
$$\leq C_{n,FB,\tilde{Z}}(\alpha P_1 + \beta P_2) + \frac{1}{2n}\log\frac{|R_{\tilde{Z}}|}{|R_{Z_1}|^{\alpha}|R_{Z_2}|^{\beta}} \ ?$$

今まで次のような結果が得られている.

Theorem 7 (Chen-Yanagi [3]) $Z_1 = Z_2$ のとき成り立つ. 即ち $C_{n,FB,Z}(\cdot)$ の凹性が成り立つ.

$$\alpha C_{n,FB,Z}(P_1) + \beta C_{n,FB,Z}(P_2) \leq C_{n,FB,Z}(\alpha P_1 + \beta P_2).$$

Theorem 8 (Yanagi-Yu-Chao [17]) $P_1 = P_2$ のとき成り立つ. 即ち

$$\alpha C_{n,FB,Z_1}(P) + \beta C_{n,FB,Z_2}(P) \leq C_{n,FB,\tilde{Z}}(P) + \frac{1}{2n}\log\frac{|R_{\tilde{Z}}|}{|R_{Z_1}|^{\alpha}|R_{Z_2}|^{\beta}}.$$

Theorem 9 (Yanagi-Yu-Chao [17])

$$\alpha C_{n,FB,Z_1}(P_1) + \beta C_{n,Z_2}(P_2) \leq C_{n,FB,\tilde{Z}}(\alpha P_1 + \beta P_2) + \frac{1}{2n}\log\frac{|R_{\tilde{Z}}|}{|R_{Z_1}|^{\alpha}|R_{Z_2}|^{\beta}}.$$

Theorem 10 (Yanagi-Yu-Chao [17])

$$\alpha C_{n,FB,Z_1}(P_1) + \beta C_{n,FB,Z_2}(P_2) \leq C_{n,\tilde{Z}}(\alpha P_1 + \beta P_2) + \frac{1}{2}\log 2 + \frac{1}{2n}\log\frac{|R_{\tilde{Z}}|}{|R_{Z_1}|^{\alpha}|R_{Z_2}|^{\beta}}.$$

Theorem 11 (Yanagi-Yu-Chao [17])

$$\alpha C_{n,FB,Z_1}(P_1) + \beta C_{n,FB,Z_2}(P_2) \leq 2C_{n,FB,\tilde{Z}}(\alpha P_1 + \beta P_2) + \frac{1}{2n}\log\frac{|R_{\tilde{Z}}|}{|R_{Z_1}|^{\alpha}|R_{Z_2}|^{\beta}}.$$

5 証明

Proof of Theorem 10. Since $R_{S_i+Z_i} \leq 2(R_{S_i} + R_{Z_i})$ $(i = 1, 2)$, we have the following.

$$\begin{aligned} \alpha R_{S_1+Z_1} + \beta R_{S_2+Z_2} &\leq 2\alpha(R_{S_1} + R_{Z_1}) + 2\beta(R_{S_2} + R_{Z_2}) \\ &= 2(\alpha R_{S_1} + \beta R_{S_2} + \alpha R_{Z_1} + \beta R_{Z_2}). \end{aligned}$$

Then

$$|R_{S_1+Z_1}|^\alpha |R_{S_2+Z_2}|^\beta \leq |2(\alpha R_{S_1} + \beta R_{S_2} + \alpha R_{Z_1} + \beta R_{Z_2})|.$$

And we have

$$\frac{|R_{S_1+Z_1}|^\alpha}{|R_{Z_1}|^\alpha} \cdot \frac{|R_{S_2+Z_2}|^\beta}{|R_{Z_2}|^\beta} \leq \frac{|2(R_{\tilde{S}} + R_{\tilde{Z}})|}{|2R_{\tilde{Z}}|} \cdot \frac{|2R_{\tilde{Z}}|}{|R_{Z_1}|^\alpha |R_{Z_2}|^\beta}.$$

Then

$$\begin{aligned} &\alpha \frac{1}{2n} \log \frac{|R_{S_1+Z_1}|}{|R_{Z_1}|} + \beta \frac{1}{2n} \log \frac{|R_{S_2+Z_2}|}{|R_{Z_2}|} \\ \leq\ & \frac{1}{2n} \log \frac{|R_{\tilde{S}} + R_{\tilde{Z}}|}{|R_{\tilde{Z}}|} + \frac{1}{2} \log 2 + \frac{1}{2n} \log \frac{|R_{\tilde{Z}}|}{|R_{Z_1}|^\alpha |R_{Z_2}|^\beta}. \end{aligned}$$

Therefore

$$\begin{aligned} &\alpha C_{n,FB,Z_1}(P_1) + \beta C_{n,FB,Z_2}(P_2) \\ \leq\ & C_{n,\tilde{Z}}(\alpha P_1 + \beta P_2) + \frac{1}{2} \log 2 + \frac{1}{2n} \log \frac{|R_{\tilde{Z}}|}{|R_{Z_1}|^\alpha |R_{Z_2}|^\beta}. \end{aligned}$$

□

Proof of Theorem 11. Since

$$\begin{aligned} R_{S_1 Z_1} &= R_{S_1}^{1/2} V R_{Z_1}^{1/2} \\ R_{S_2 Z_2} &= R_{S_2}^{1/2} W R_{Z_2}^{1/2}, \end{aligned}$$

we have the following.

$$\begin{aligned} &\alpha R_{S_1+Z_1} + \beta R_{S_2+Z_2} \\ =\ & \alpha R_{S_1} + \beta R_{S_2} + \alpha R_{Z_1} + \beta R_{Z_2} + \alpha R_{S_1 Z_1} + \beta R_{S_2 Z_2} + \alpha R_{Z_1 S_1} + \beta R_{Z_2 S_2} \\ =\ & R_{\tilde{S}} + R_{\tilde{Z}} + \alpha R_{S_1}^{1/2} V R_{Z_1}^{1/2} + \beta R_{S_2}^{1/2} W R_{Z_2}^{1/2} + \alpha R_{Z_1}^{1/2} V^t R_{S_1}^{1/2} + \beta R_{Z_2}^{1/2} W^t R_{S_2}^{1/2} \\ =\ & R_{\tilde{S}} + R_{\tilde{Z}} + (\alpha R_{S_1})^{1/2} V (\alpha R_{Z_1})^{1/2} + (\beta R_{S_2})^{1/2} W (\beta R_{Z_2})^{1/2} \\ & + (\alpha R_{Z_1})^{1/2} V^t (\alpha R_{S_1})^{1/2} + (\beta R_{Z_2})^{1/2} W^t (\beta R_{S_2})^{1/2}. \end{aligned}$$

It follows from $\alpha R_{S_1} \leq R_{\tilde{S}}$ that

$$(\alpha R_{S_1})^{1/2} = R_{\tilde{S}}^{1/2} L, \quad \|L\| \leq 1.$$

Similarly,

$$(\beta R_{S_2})^{1/2} = R_{\tilde{S}}^{1/2} M, \quad \|M\| \leq 1,$$
$$(\alpha R_{Z_1})^{1/2} = R_{\tilde{Z}}^{1/2} T, \quad \|T\| \leq 1,$$
$$(\beta R_{Z_2})^{1/2} = R_{\tilde{Z}}^{1/2} S, \quad \|S\| \leq 1.$$

We put

$$K = \frac{LVT^t + MWS^t}{2}.$$

Then

$$\begin{aligned} & \alpha R_{S_1+Z_1} + \beta R_{S_2+Z_2} \\ = \; & R_{\tilde{S}} + R_{\tilde{Z}} + R_{\tilde{S}}^{1/2} LVT^t R_{\tilde{Z}}^{1/2} + R_{\tilde{S}}^{1/2} MWS^t R_{\tilde{Z}}^{1/2} + R_{\tilde{Z}}^{1/2} TV^t L^t R_{\tilde{S}}^{1/2} + R_{\tilde{Z}}^{1/2} SW^t M^t R_{\tilde{S}}^{1/2} \\ = \; & R_{\tilde{S}} + R_{\tilde{Z}} + R_{\tilde{S}}^{1/2} (LVT^t + MWS^t) R_{\tilde{Z}}^{1/2} + R_{\tilde{Z}}^{1/2} (TV^t L^t + SW^t M^t) R_{\tilde{S}}^{1/2} \\ = \; & R_{\tilde{S}} + R_{\tilde{Z}} + (R_{\sqrt{2}\tilde{S}})^{1/2} K (R_{\sqrt{2}\tilde{Z}})^{1/2} + (R_{\sqrt{2}\tilde{Z}})^{1/2} K^t (R_{\sqrt{2}\tilde{S}})^{1/2} \\ = \; & R_{\sqrt{2}\tilde{S}} + R_{\sqrt{2}\tilde{Z}} + (R_{\sqrt{2}\tilde{S}})^{1/2} K (R_{\sqrt{2}\tilde{Z}})^{1/2} + (R_{\sqrt{2}\tilde{Z}})^{1/2} K^t (R_{\sqrt{2}\tilde{S}})^{1/2} - R_{\tilde{S}} - R_{\tilde{Z}}. \end{aligned}$$

Then

$$\begin{aligned} & \alpha R_{S_1+Z_1} + \beta R_{S_2+Z_2} + R_{\tilde{S}} + R_{\tilde{Z}} \\ = \; & R_{\sqrt{2}\tilde{S}} + R_{\sqrt{2}\tilde{Z}} + (R_{\sqrt{2}\tilde{S}})^{1/2} K (R_{\sqrt{2}\tilde{Z}})^{1/2} + (R_{\sqrt{2}\tilde{Z}})^{1/2} K^t (R_{\sqrt{2}\tilde{S}})^{1/2}. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{\alpha}{2} R_{S_1+Z_1} + \frac{\beta}{2} R_{S_2+Z_2} + \frac{1}{2}(R_{\tilde{S}} + R_{\tilde{Z}}) \\ = \; & R_{\tilde{S}} + R_{\tilde{Z}} + (R_{\tilde{S}})^{1/2} K (R_{\tilde{Z}})^{1/2} + (R_{\tilde{Z}})^{1/2} K^t (R_{\tilde{S}})^{1/2}. \end{aligned}$$

Since $\frac{\alpha}{2} + \frac{\beta}{2} + \frac{1}{2} = 1$, we have the following.

$$\begin{aligned} & |R_{S_1+Z_1}|^{\alpha/2} |R_{S_2+Z_2}|^{\beta/2} |R_{\tilde{S}} + R_{\tilde{Z}}|^{1/2} \\ \leq \; & |R_{\tilde{S}} + R_{\tilde{Z}} + (R_{\tilde{S}})^{1/2} K (R_{\tilde{Z}})^{1/2} + (R_{\tilde{S}})^{1/2} K^t (R_{\tilde{Z}})^{1/2}|. \end{aligned}$$

Thus

$$\begin{aligned} & \frac{\alpha}{2} \frac{1}{2n} \log \frac{|R_{S_1+Z_1}|}{|R_{Z_1}|} + \frac{\beta}{2} \frac{1}{2n} \log \frac{|R_{S_2+Z_2}|}{|R_{Z_2}|} + \frac{1}{2} \frac{1}{2n} \log \frac{|R_{\tilde{S}} + R_{\tilde{Z}}|}{|R_{\tilde{Z}}|} \\ \leq \; & \frac{1}{2n} \log \frac{|R_{\tilde{S}} + R_{\tilde{Z}} + (R_{\tilde{S}})^{1/2} K (R_{\tilde{Z}})^{1/2} + (R_{\tilde{S}})^{1/2} K^t (R_{\tilde{Z}})^{1/2}|}{|R_{\tilde{Z}}|} + \frac{1}{2} \frac{1}{2n} \log \frac{|R_{\tilde{Z}}|}{|R_{Z_1}|^{\alpha} |R_{Z_2}|^{\beta}}. \end{aligned}$$

Then we have

$$
\begin{aligned}
& \alpha C_{n,FB,Z_1}(P_1) + \beta C_{n,FB,Z_2}(P_2) \\
\leq\ & 2C_{n,FB,\tilde{Z}}(\alpha P_1 + \beta P_2) + \frac{1}{2n}\log\frac{|R_{\tilde{Z}}|}{|R_{Z_1}|^{\alpha}|R_{Z_2}|^{\beta}}.
\end{aligned}
$$

□

したがって未解決問題 3 に関連して次の問題も提起される.

未解決問題 4 任意の $P_1, P_2 \geq 0$ と任意の $\alpha, \beta \geq 0 (\alpha + \beta = 1)$ に対して

$$
\alpha C_{n,FB,Z_1}(P_1) + \beta C_{n,FB,Z_2}(P_2)
$$

$$
\leq 2C_{n,\tilde{Z}}(\alpha P_1 + \beta P_2) + \frac{1}{2n}\log\frac{|R_{\tilde{Z}}|}{|R_{Z_1}|^{\alpha}|R_{Z_2}|^{\beta}} ?
$$

もちろん未解決問題 3 が解決されれば未解決問題4 は当然解決されることに注意する.

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