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On the local center of Liénard-type systems

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1. Introduction

Our aim in this paper is to seek a necessary and sufficient condition in order that an analytic Liénard-type system has a local center. The equilibrium point is called a local center of the system if all the orbits in every neighborhood of it are closed. To decide the number of the non-trivial closed orbits of a Liénard-type system is important, and to see if an equilibrium point of the system is a center is a difficult problem. It has continued until today to draw attention of many mathematicians. For this purpose we assume the case where the corresponding linear system has a pair of pure imaginary eigenvalues (since otherwise the equilibrium point cannot be a center). Thus, we consider an analytic Liénard-type system of the following form:

\[
\begin{align*}
\dot{z} &= y \\
\dot{y} &= f_n(x)y^p - (x + g_q(x)),
\end{align*}
\]

where the dot (\(\dot{\cdot}\)) denotes differentiation, \(f_n(x)\) and \(g_q(x)\) are real analytic functions of the form (C) below.

\[
f_n(x) = \sum_{k=n} a_k x^k \quad \text{and} \quad g_q(x) = \sum_{k=q} b_k x^k,
\]

where \(n + p \geq 2^*\) and \(q \geq 2\).

Then the system (L) has an equilibrium point at the origin and the coefficient matrix of the linear system approximating the system at the origin has a pair of purely imaginary eigenvalues. In this case the equilibrium point is either a center or a focus.

In the old paper of T. Saito\([Sa]\) he gave a necessary and sufficient condition on the case \(g_q(x) \equiv 0\). Recently, the author have treated on the special case \(n = p = 1\) and \(q = 2\) in [Ha]. Our results are an improvement of these papers and are stated as follows.

Theorem A. Suppose that \(g_q\) is an odd function. The system (L) with the form (C) has a local center at the origin if and only if one of the following conditions is satisfied:

1. \(p\) is an even number;
2. \(p\) is an odd number and \(f_n\) is an odd function.

*For the case \(n + p \leq 1\) see §3 Appendix
Theorem B. Suppose that $f_n$ is an odd function and $n + p \leq q$. The system (L) with the form (C) has a local center at the origin if and only if $g_q$ is an odd function.

We shall apply our results to an analytic Liénard-type system of the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = f_n(x)y^{2n-1} - \sin x. \end{cases}$$

with $f_n(0) = 0$ and $n \geq 1$. Using Theorem A for this system, it follows that the equilibrium point $(0, 0)$ is a local center if and only if $f_n$ is an odd function.

2. Proof of Theorems

Now let us prove Theorem A. We suppose that $g_q$ is an odd function. Let $(x(t), y(t))$ be a solution of the system (L). Then, if $p$ is an odd number and $f_n$ is an odd functions, $(-x(-t), y(-t))$ is also a solution of the system (L) with the form (C). Thus the orbits defined by the system (L) have mirror symmetry with respect to the $y$-axis. Hence the system (L) cannot have a focus at the origin. Similarly, if $p$ is an even number, since $(x(-t), -y(-t))$ is also a solution of the system (L), the system cannot have a focus at the origin.

Conversely, we suppose that the origin is a local center. To prove the theorems we use the following fundamental tool which is well-known as Poincaré–Lyapunov’ lemma (see [Ha], [P] or [Sch]).

**Proposition.** If the system (L) has a local center at the origin, then it has a nonconstant real analytic first integral $M(x, y) = \text{const.}$ in a neighborhood of the origin. It can be written by a power series of the form

$$M(x, y) = c(x^2 + y^2) + M_3(x, y) + M_4(x, y) + \cdots,$$

where $c$ is some real constant and $M_m(x, y)$ is a homogeneous polynomial in $x$ and $y$ of degree $m \geq 3$.

Introducing the polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$, the equality (1) is written as

$$M(r \cos \theta, r \sin \theta) = r^2 \overline{M}_2(\theta) + r^3 \overline{M}_3(\theta) + \cdots,$$

where $r^m \overline{M}_m(\theta) = M_m(r \cos \theta, r \sin \theta)$ for $m \geq 2$ and $\overline{M}_2(\theta) = c$.

Now let $(x(t), y(t))$ be a periodic solution of the system (L) with the form (C) and write $x(t) = r(t) \cos \theta(t)$ and $y(t) = r(t) \sin \theta(t)$. Then we have

$$\dot{r} = \sum_{k=n}^q a_k r^{k+1} \cos^k \theta \sin^2 \theta - \sum_{k=q} b_k r^k \cos^k \theta \sin \theta$$

(2)

and

$$\dot{\theta} = -1 + \sum_{k=n}^q a_k r^k \cos^{k+1} \theta \sin \theta - \sum_{k=q} b_k r^{k-1} \cos^{k+1} \theta.$$  (3)
Differentiating with respect to $t$ the relation

$$M(r(t) \cos \theta(t), r(t) \sin \theta(t)) = \sum_{m=2}^{\infty} r(t)^{m} \overline{M}_{m}(\theta(t)) \equiv \text{const.},$$

we obtain

$$\sum_{m=2} \dot{r} r^{m-1} \overline{M}_{m}(\theta) + \sum_{m=3} r^{m} \overline{M}_{m}(\theta) \dot{\theta} = 0,$$

(4)

where the prime (') denotes differentiation with respect to $\theta$. It follows from (2), (3) and (4) that

$$\sum_{m=3} r^{m} \overline{M}_{m}'(\theta) = \sum_{m=3} r^{m} \overline{M}_{m}'(\theta) \left[ \sum_{k=n} a_{k} r^{k+p-1} \cos^{k+1} \theta \sin^{p} \theta - \sum_{k=q} b_{k} r^{k-1} \cos^{k+1} \theta \right]$$

$$+ \sum_{m=2} \dot{r} r^{m-1} \overline{M}_{m}(\theta) \left[ \sum_{k=n} a_{k} r^{k+p} \cos^{k} \theta \sin^{p+1} \theta - \sum_{k=q} b_{k} r^{k} \cos^{k} \theta \sin \theta \right].$$

We give the proof by dividing all possible cases to the cases (I) $n+p+s = q$, $s \geq 0$ and (II) $n+p = q+t$, $t > 0$. Moreover, we need to divide these cases to the eight cases as is shown in the table below, where the sign e(resp. o) denotes an even(resp. odd) number.

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Case(I) : $n+p+s = q$, $s \geq 0$

First, we get the following lemma by comparing the terms of the same degree in $r$ on both sides of the equality (5).

**Lemma 1.** If $m \leq n+p$, then $\overline{M}_{m}(\theta) = 0$.

We shall consider the case (I)-(i).

**Lemma 2.** Suppose that $n+p < m \leq n+p+s = q$. Then $a_i = 0$ for even numbers $i \in [n,n+s-1]$ and $\overline{M}_{m}(\theta)$ is a polynomial of $\sin \theta$ only.

The proof is given by the same discussion as in [Sa]. So we omit it.

**Lemma 3.** Suppose that $m > q$. Then $a_i = 0$ for even numbers $i \geq n+s$ and $\overline{M}_{m}(\theta)$ is a polynomial of $\sin \theta$ only.
Proof. From (5) we remark that the equality
\[
\overline{M}_{q+r}^{l}(\theta) = \sum_{k=0}^{s+r-1} (k + 2)\overline{M}_{k+2}^{l}(\theta)a_{n+s+r-k-1}\cos^{n+s+r-k-1}\theta \sin^{p+1}\theta
\]
\[
- \sum_{k=0}^{r-1} (k + 2)\overline{M}_{k+2}^{l}(\theta)b_{q+r-k-1}\cos^{q+r-k-1}\theta \sin\theta
\]
\[
+ \sum_{k=0}^{s+r-2} \overline{M}_{k+3}^{l}(\theta)a_{n+s+r-k-2}\cos^{n+s+r-k-2}\theta \sin^{p}\theta
\]
\[
- \sum_{k=0}^{r-2} \overline{M}_{k+3}^{l}(\theta)b_{q+r-k-2}\cos^{q+r-k-2}\theta.
\]
holds for \(1 \leq r\). When \(r = 1\), we have
\[
\overline{M}_{q+1}^{l}(\theta) = \sum_{k=0}^{s} (k + 2)\overline{M}_{k+2}^{l}(\theta)a_{n+s-k}\cos^{n+s-k}\theta \sin^{p+1}\theta
\]
\[
- 2\overline{M}_{2}^{l}(\theta)b_{q}\cos^{q}\theta \sin\theta
\]
\[
+ \sum_{k=0}^{s-1} \overline{M}_{k+3}^{l}(\theta)a_{n+s-k-1}\cos^{n+s-k-1}\theta \sin^{p}\theta.
\]
By Lemma 1 and 2, since
\[
\overline{M}_{q+1}(2\pi) - \overline{M}_{q+1}(0) = 2\overline{M}_{2}(\theta)a_{n+s}\int_{0}^{2\pi} \cos^{n+s}\theta \sin^{p+1}\theta d\theta = 0,
\]
we get \(a_{n+s} = 0\). Hence we see that \(\overline{M}_{q+1}(\theta)\) is a polynomial of \(\sin \theta\) only. Moreover, from (6) we have
\[
\overline{M}_{q+2}^{l}(\theta) = \sum_{k=0}^{s+1} (k + 2)\overline{M}_{k+2}^{l}(\theta)a_{n+s-k+1}\cos^{n+s-k+1}\theta \sin^{p+1}\theta
\]
\[
- \sum_{k=0}^{1} (k + 2)\overline{M}_{k+2}^{l}(\theta)b_{q-k+1}\cos^{q-k+1}\theta \sin\theta
\]
\[
+ \sum_{k=0}^{s} \overline{M}_{k+3}^{l}(\theta)a_{n+s-k}\cos^{n+s-k+1}\theta \sin^{p}\theta
\]
\[
- \overline{M}_{3}^{l}(\theta)b_{q}\cos^{q+1}\theta.
\]
By \(a_{n+s} = 0\) and the assumption that \(g_{q}\) is an odd function, we obtain that \(\overline{M}_{q+2}(\theta)\) is also a polynomial of \(\sin \theta\) only.
From now on, we suppose that for all \(l \geq 1\)
\[
a_{n+s} = a_{n+s+2} = \cdots = a_{n+s+2(l-1)} = 0
\]
and \( \overline{M}_m(\theta) \) have been determined up to \( m = q + 2l \) as polynomials of \( \sin \theta \) only. Then, from (6), the equality determining \( \overline{M}_{q+2l+1}(\theta) \) is given by

\[
\overline{M}_{q+2l+1}(\theta) = \sum_{k=0}^{s+2l} (k + 2) \overline{M}_{k+2}(\theta) a_{n+s+2l-k} \cos^{n+s+2l-k} \theta \sin^{p+1} \theta \\
- \sum_{k=0}^{2l} (k + 2) \overline{M}_{k+2}(\theta) b_{q+2l-k} \cos^{q+2l-k} \theta \sin \theta \\
+ \sum_{k=0}^{s+2l-1} \overline{M}'_{k+3}(\theta) a_{n+s+2l-k} \cos^{n+s+2l-k} \theta \sin^{p} \theta \\
- \sum_{k=0}^{2l-1} \overline{M}'_{k+3}(\theta) b_{q+2l-k-1} \cos^{q+2l-k} \theta \\
= 2\overline{M}_2(\theta) a_{n+s+2l} \cos^{n+s+2l} \theta \sin^{p+1} \theta + \sum(\cdots). \tag{7}
\]

From Lemma 2, the assumption of induction and that \( g_q \) is an odd function, all the terms on the right-hand side of the equality (7), except the first one, have the form \((\text{polynomial of } \sin \theta) \times \text{(odd power of } \cos \theta)\). Thus, since

\[
\overline{M}_{q+2l+1}(2\pi) - \overline{M}_{q+2l+1}(0) = 2\overline{M}_2(\theta) a_{n+s+2l} \int_0^{2\pi} \cos^{n+s+2l} \theta \sin^{p+1} \theta \, d\theta = 0,
\]

we get \( a_{n+s+2l} = 0 \). Hence we see that \( \overline{M}_{q+2l+1}(\theta) \) is a polynomial of \( \sin \theta \) only.

Moreover we consider \( \overline{M}_{q+2(l+1)}(\theta) \). By (6), \( \overline{M}_{q+2(l+1)}(\theta) \) is determined from the equality

\[
\overline{M}_{q+2(l+1)}(\theta) = \sum_{k=0}^{s+2l+1} (k + 2) \overline{M}_{k+2}(\theta) a_{n+s+2l-k+1} \cos^{n+s+2l-k+1} \theta \sin^{p+1} \theta \\
- \sum_{k=0}^{2l+1} (k + 2) \overline{M}_{k+2}(\theta) b_{q+2l-k+1} \cos^{q+2l-k+1} \theta \sin \theta \\
+ \sum_{k=0}^{s+2l} \overline{M}'_{k+3}(\theta) a_{n+s+2l-k+1} \cos^{n+s+2l-k+1} \theta \sin^{p} \theta \\
- \sum_{k=0}^{2l} \overline{M}'_{k+3}(\theta) b_{q+2l-k} \cos^{q+2l-k} \theta \\
= 2\overline{M}_2(\theta) a_{n+s+2l+1} \cos^{n+s+2l+1} \theta \sin^{p+1} \theta + \sum(\cdots). \tag{8}
\]

From the above fact (i.e. \( a_{n+s+2l} = 0 \)), the assumption of induction and that \( g_q \) is an odd function, all the terms on the right-hand side of the equality (8) have the form \((\text{polynomial of } \sin \theta) \times \text{(odd power of } \cos \theta)\). Thus we conclude that \( \overline{M}_{q+2(l+1)}(\theta) \) is a polynomial of \( \sin \theta \) only.
Other seven cases are also proved by a similar method to the above one.

Case(II) : \( n + p = q + t, \ t > 0 \)

First, we get the following lemma by comparing the terms of the same degree in \( r \) on both sides of the equality (5).

**Lemma 4.** If \( m \leq q \), then \( \overline{M}_m(\theta) = 0 \).

We shall consider the case (II)-(i). We get the following

**Lemma 5.** Suppose that \( q < m \leq q + t = n + p \). Then \( \overline{M}_m(\theta) \) is a polynomial of \( \cos \theta \) only.

*Proof.* From (5) we have

\[
\overline{M}_{q+1}(\theta) = -2\overline{M}_2(\theta)b_q \cos^q \theta \sin \theta.
\]

Thus \( \overline{M}_{q+1}(\theta) \) is a polynomial of \( \cos \theta \) only.

From now on, we suppose that \( \overline{M}_m(\theta) \) have been determined up to \( q + r - 1 \) (\( 2 \leq r \leq t \)) as polynomials of \( \cos \theta \) only. Then the equality determining \( \overline{M}_{q+r}(\theta) \) is given by

\[
\overline{M}_{q+r}(\theta) = -\sum_{k=0}^{r-1} (k+2)\overline{M}_{k+2}(\theta)b_{q+r-k-1} \cos^{q+r-k-1} \theta \sin \theta
\]

\[
-\sum_{k=0}^{r-2} \overline{M}_{k+3}(\theta)b_{q+r-k-2} \cos^{q+r-k-1} \theta.
\]

(9)

Thus, we see from the assumption of induction and Lemma 4 that \( \overline{M}_{q+r}(\theta) \) is a polynomial of \( \cos \theta \) only. \( \square \)

**Lemma 6.** Suppose that \( m > q + t = n + p \). Then \( a_i = 0 \) for even numbers \( i \geq n \) and \( \overline{M}_m(\theta) \) is a polynomial of \( \cos \theta \) only.

*Proof.* From (5) we remark that the equality

\[
\overline{M}_{q+t+r}(\theta) = \sum_{k=0}^{r-1} (k+2)\overline{M}_{k+2}(\theta)a_{n+r-k-1} \cos^{n+r-k-1} \theta \sin^{p+1} \theta
\]

\[
-\sum_{k=0}^{r+t-1} (k+2)\overline{M}_{k+2}(\theta)b_{q+t+r-k-1} \cos^{q+t+r-k-1} \theta \sin \theta
\]

\[
+\sum_{k=0}^{r-2} \overline{M}_{k+3}(\theta)a_{n+r-k-2} \cos^{n+r-k-1} \theta \sin^p \theta
\]

\[
-\sum_{k=0}^{r+t-2} \overline{M}_{k+3}(\theta)b_{q+t+r-k-2} \cos^{q+t+r-k-1} \theta.
\]

(10)
holds for $1 \leq r$. When $r = 1$, we have

\[ \overline{M}_{q+t+1}^{'}(\theta) = 2\overline{M}_{2}(\theta) a_{n} \cos^{n} \theta \sin^{p+1} \theta \]

\[ - \sum_{k=0}^{t} (k+2) \overline{M}_{k+2}(\theta) b_{q+t-k} \cos^{q+t-k} \theta \sin \theta \]

\[ - \sum_{k=0}^{t-1} \overline{M}_{k+3}(\theta) b_{q+t-k-1} \cos^{q+t-k} \theta. \]

By Lemma 4 and 5, since

\[ \overline{M}_{q+t+1}(2\pi) - \overline{M}_{q+t+1}(0) = 2\overline{M}_{2}(\theta) a_{n} \int_{0}^{2\pi} \cos^{n} \theta \sin^{p+1} \theta d\theta = 0, \]

we get $a_{n} = 0$. Hence we see that $\overline{M}_{q+t+1}(\theta)$ is a polynomial of $\cos \theta$ only. As the result, we obtain from (10) that $\overline{M}_{q+t+2}(\theta)$ is also a polynomial of $\cos \theta$ only.

From now on, we suppose that for all $l \geq 1$

\[ a_{n} = a_{n+2} = \cdots = a_{n+2(l-1)} = 0 \] (11)

and $\overline{M}_{m}(\theta)$ have been determined up to $m = q + t + 2l$ as polynomials of $\cos \theta$ only. Then, from (10), the equality determining $\overline{M}_{q+t+2l+1}(\theta)$ is given by

\[ \overline{M}_{q+t+2l+1}(\theta) = \sum_{k=0}^{2l} (k+2) \overline{M}_{k+2}(\theta) a_{n+2l-k} \cos^{n+2l-k} \theta \sin^{p+1} \theta \]

\[ - \sum_{k=0}^{t+2l} (k+2) \overline{M}_{k+2}(\theta) b_{q+t+2l-k} \cos^{q+t+2l-k} \theta \sin \theta \]

\[ + \sum_{k=0}^{2l-1} \overline{M}_{k+3}(\theta) a_{n+2l-k-1} \cos^{n+2l-k} \theta \sin^{p} \theta \]

\[ - \sum_{k=0}^{t+2l-1} \overline{M}_{k+3}(\theta) b_{q+t+2l-k-1} \cos^{q+t+2l-k} \theta \]

\[ = 2\overline{M}_{2}(\theta) a_{n+2l} \cos^{n+2l} \theta \sin^{p+1} \theta \sin \theta + \sum (\cdots). \] (12)

From the assumption of induction and that $g_{q}$ is an odd function, all the terms on the right-hand side of the equality (12), except the first one, have the form (polynomial of $\sin \theta$) $\times$ (odd power of $\cos \theta$). Thus, since

\[ \overline{M}_{q+t+2l+1}(2\pi) - \overline{M}_{q+t+2l+1}(0) = 2\overline{M}_{2}(\theta) a_{n+2l} \int_{0}^{2\pi} \cos^{n+2l} \theta \sin^{p+1} \theta d\theta = 0, \]

we get $a_{n+2l} = 0$. Hence we see that $\overline{M}_{q+t+2l+1}(\theta)$ is a polynomial of $\cos \theta$ only.
Moreover we consider $\overline{M}_{q+t+2(l+1)}(\theta)$. By (10), $\overline{M}_{q+t+2(l+1)}(\theta)$ is determined from the equality

$$
\overline{M}_{q+t+2(l+1)}(\theta) = \sum_{k=0}^{2l+1} (k+2)\overline{M}_{k+2}^{l}(\theta)a_{n+2l-k+1}\cos^{n+2l-k+1}\theta\sin^{p+1}\theta - \sum_{k=0}^{t+2l+1} (k+2)\overline{M}_{k+2}^{l}(\theta)b_{q+t+2l-k+1}\cos^{q+t+2l-k+1}\theta\sin\theta \nonumber 
+ \sum_{k=0}^{2l} \overline{M}_{k+3}^{l}(\theta)a_{n+2l-k}\cos^{n+2l-k+1}\theta\sin^{p}\theta 
- \sum_{k=0}^{t+2l} \overline{M}_{k+3}^{l}(\theta)b_{q+t+2l-k}\cos^{q+t+2l-k+1}\theta 
= 2\overline{M}_{2}(\theta)a_{n+2l+1}\cos^{n+s+2l+1}\theta\sin^{p+1}\theta + \sum(\cdots). 
$$

(13)

From the above fact (i.e. $a_{n+2l} = 0$), the assumption of induction and that $g_q$ is an odd function, all the terms on the right-hand side of the equality (13) have the form (polynomial of $\sin\theta$) $\times$ (odd power of $\cos\theta$). Thus we conclude that $\overline{M}_{q+t+2(l+1)}(\theta)$ is a polynomial of $\cos\theta$ only.

Other seven cases are also proved by a similar method to the above one. Therefore the proof of Theorem A is now completed. □

The following fact is a key in the proof of Theorem B.

**Lemma 7.** Suppose that $n + p < m \leq n + p + s = q$. If $m$ is an odd (resp. even) number, then $\overline{M}_{m}(\theta)$ is a polynomial of $\cos\theta$ of odd (resp. even) degree only.

We omit the details for the proofs of Lemma 7 and Theorem B.

### 3. Appendix

[1] We consider the case $n + p \leq 1$ in the form (C). If $(n, p) = (1, 0)$ and $a_1 > 1$, then there exists the first integral $(1/2)y^2 + \int_{0}^{x} \{f_1(\xi) - \xi - g_q(\xi)\}d\xi = \text{const.}$ of the system (L). Since $x\{f_1(x) - x - g_q(x)\} > 0$ ($x \neq 0$) in the neighborhood of the origin, the equilibrium point is a center.

If $(n, p) = (0, 1)$ and $a_1 > 1$, then we can apply Theorem A and B to this system.

We set $P(x) = f_0(x) - x - g_q(x)$. Let a solution of the equation $P(x) = 0$ be $x = \alpha$. If $(n, p) = (0, 0)$ and $P'(-\alpha) > 0$, then we also can apply Theorem A and B to this system.

[2] By combining the mentioned facts above and the result in [Su], we have the following result on a global center of the system (L).
Corollary. Consider the system (L) with \( p = 1 \) of the form (C). Suppose that

1. \( g_q \) is an odd function with \( g_q(0) = 0 \) and \( x\{x + g_q(x)\} > 0 \) \((x \neq 0)\),

2. there exists \( 0 \leq \lambda < \sqrt{8} \) such that

\[
|\int_0^x f_n(\xi)d\xi| \leq \lambda \sqrt{\int_0^x g_q(\xi)d\xi} \quad \text{for sufficiently large } x.
\]

Then the equilibrium point \((0, 0)\) of the system (L) is a global center if and only if \( \int_0^\infty g(x)dx = \infty \).

References


