Title
On the local center of Lienard-type systems (Dynamics of Functional Equations and Related Topics)

Author(s)
Hayashi, Makoto

Citation
数理解析研究所講究録 数学的解析学の研究 総合的な数学

Issue Date
2002-04

URL
http://hdl.handle.net/2433/41883

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
On the local center of Liénard-type systems

日本大学理工学部 林 誠 (MAKOTO HAYASHI)

1. Introduction

Our aim in this paper is to seek a necessary and sufficient condition in order that an analytic Liénard-type system has a local center. The equilibrium point is called a local center of the system if all the orbits in every neighborhood of it are closed. To decide the number of the non-trivial closed orbits of a Liénard-type system is important, and to see if an equilibrium point of the system is a center is a difficult problem. It has continued until today to draw attention of many mathematicians. For this purpose we assume the case where the corresponding linear system has a pair of pure imaginary eigenvalues (since otherwise the equilibrium point cannot be a center). Thus, we consider an analytic Liénard-type system of the following form:

\[
\begin{align*}
\dot{z} &= y \\
\dot{y} &= f_n(x)y^p - (x + g_q(x)),
\end{align*}
\]

where the dot (\(\dot{}\)) denotes differentiation, \(f_n(x)\) and \(g_q(x)\) are real analytic functions of the form (C) below.

\[f_n(x) = \sum_{k=n} a_k x^k \quad \text{and} \quad g_q(x) = \sum_{k=q} b_k x^k,\]

where \(n + p \geq 2^*\) and \(q \geq 2\).

Then the system (L) has an equilibrium point at the origin and the coefficient matrix of the linear system approximating the system at the origin has a pair of purely imaginary eigenvalues. In this case the equilibrium point is either a center or a focus.

In the old paper of T. Saito [Sa] he gave a necessary and sufficient condition on the case \(g_q(x) \equiv 0\). Recently, the author have treated on the special case \(n = p = 1\) and \(q = 2\) in [Ha]. Our results are an improvement of these papers and are stated as follows.

Theorem A. Suppose that \(g_q\) is an odd function. The system (L) with the form (C) has a local center at the origin if and only if one of the following conditions is satisfied:

1. \(p\) is an even number;
2. \(p\) is an odd number and \(f_n\) is an odd function.

*For the case \(n + p \leq 1\) see §3 Appendix
Theorem B. Suppose that \( f_n \) is an odd function and \( n + p \leq q \). The system (L) with the form (C) has a local center at the origin if and only if \( g_q \) is an odd function.

We shall apply our results to an analytic Liénard-type system of the form

\[
\begin{cases}
\dot{x} = y \\
\dot{y} = f_n(x)y^{2n-1} - \sin x.
\end{cases}
\]

with \( f_n(0) = 0 \) and \( n \geq 1 \). Using Theorem A for this system, it follows that the equilibrium point \((0, 0)\) is a local center if and only if \( f_n \) is an odd function.

2. Proof of Theorems

Now let us prove Theorem A. We suppose that \( g_q \) is an odd function. Let \((x(t), y(t))\) be a solution of the system (L). Then, if \( p \) is an odd number and \( f_n \) is an odd functions, \((-x(-t), y(-t))\) is also a solution of the system (L) with the form (C). Thus the orbits defined by the system (L) have mirror symmetry with respect to the \( y \)-axis. Hence the system (L) cannot have a focus at the origin. Similarly, if \( p \) is an even number, since \((x(-t), -y(-t))\) is also a solution of the system (L), the system cannot have a focus at the origin.

Conversely, we suppose that the origin is a local center. To prove the theorems we use the following fundament al tool which is well-known as Poincaré–Lyapunov' lemma (see [Ha], [P] or [Sch]).

Proposition. If the system (L) has a local center at the origin, then it has a nonconstant real analytic first integral \( M(x, y) = \text{const.} \) in a neighborhood of the origin. It can be written by a power series of the form

\[
M(x, y) = c(x^2 + y^2) + M_3(x, y) + M_4(x, y) + \cdots,
\]

(1)

where \( c \) is some real constant and \( M_m(x, y) \) is a homogeneous polynomial in \( x \) and \( y \) of degree \( m \geq 3 \).

Introducing the polar coordinates \( x = r \cos \theta \) and \( y = r \sin \theta \), the equality (1) is written as

\[
M(r \cos \theta, r \sin \theta) = r^2 \overline{M}_2(\theta) + r^3 \overline{M}_3(\theta) + \cdots,
\]

where \( r^m \overline{M}_m(\theta) = M_m(r \cos \theta, r \sin \theta) \) for \( m \geq 2 \) and \( \overline{M}_2(\theta) = c \).

Now let \((x(t), y(t))\) be a periodic solution of the system (L) with the form (C) and write \( x(t) = r(t) \cos \theta(t) \) and \( y(t) = r(t) \sin \theta(t) \). Then we have

\[
\dot{r} = \sum_{k=n} a_k r^{k+1} \cos^k \theta \sin^2 \theta - \sum_{k=q} b_k r^k \cos^k \theta \sin \theta
\]

(2)

and

\[
\dot{\theta} = -1 + \sum_{k=n} a_k r^k \cos^{k+1} \theta \sin \theta - \sum_{k=q} b_k r^{k-1} \cos^{k+1} \theta.
\]

(3)
Differentiating with respect to $t$ the relation

$$M(r(t) \cos \theta(t), r(t) \sin \theta(t)) = \sum_{m=2}^{\infty} r(t)^m \overline{M}_m(\theta(t)) \equiv \text{const},$$

we obtain

$$\sum_{m=2} mr^{m-1} \overline{M}_m(\theta) + \sum_{m=3} r^m \overline{M}_m(\theta) \dot{\theta} = 0, \quad (4)$$

where the prime (') denotes differentiation with respect to $\theta$. It follows from (2), (3) and (4) that

$$\sum_{m=3} r^m \overline{M}_m(\theta)' = \sum_{m=3} r^m \overline{M}_m(\theta)' \left\{ \sum_{k=n} a_k r^{k+p} \cos^k \theta \sin^{p+1} \theta - \sum_{k=q} b_k r^k \cos^k \theta \sin \theta \right\} + \sum_{m=2} mr^{m-1} \overline{M}_m(\theta) \left\{ \sum_{k=n} a_k r^{k+p-1} \cos^{k+1} \theta \sin^p \theta - \sum_{k=q} b_k r^k \cos^{k+1} \theta \sin \theta \right\}. \quad (5)$$

We give the proof by dividing all possible cases to the cases (I) $n + p + s = q$, $s \geq 0$ and (II) $n + p = q + t$, $t > 0$. Moreover, we need to divide these cases to the eight cases as is shown in the table below, where the sign e(resp. o) denotes an even(resp. odd) number.

<table>
<thead>
<tr>
<th></th>
<th>(i)</th>
<th>(ii)</th>
<th>(iii)</th>
<th>(iv)</th>
<th>(v)</th>
<th>(vi)</th>
<th>(vii)</th>
<th>(viii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>e</td>
<td>e</td>
<td>o</td>
<td>o</td>
<td>e</td>
<td>e</td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>$p$</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td>e</td>
<td>e</td>
<td>e</td>
<td>e</td>
</tr>
<tr>
<td>$q$</td>
<td>o</td>
<td>e</td>
<td>o</td>
<td>e</td>
<td>e</td>
<td>o</td>
<td>e</td>
<td>e</td>
</tr>
</tbody>
</table>

Case(I) : $n + p + s = q$, $s \geq 0$

First, we get the following lemma by comparing the terms of the same degree in $r$ on both sides of the equality (5).

**Lemma 1.** If $m \leq n + p$, then $\overline{M}_m(\theta) = 0$.

We shall consider the case (I)-(i).

**Lemma 2.** Suppose that $n + p < m \leq n + p + s = q$. Then $a_i = 0$ for even numbers $i \in [n, n + s - 1]$ and $\overline{M}_m(\theta)$ is a polynomial of $\sin \theta$ only.

The proof is given by the same discussion as in [Sa]. So we omit it.

**Lemma 3.** Suppose that $m > q$. Then $a_i = 0$ for even numbers $i \geq n + s$ and $\overline{M}_m(\theta)$ is a polynomial of $\sin \theta$ only.
Proof. From (5) we remark that the equality
\[
\widetilde{M}_{q+r}(\theta) = \sum_{k=0}^{s+r-1} (k + 2)\tilde{M}_{k+2}(\theta)a_{n+s+r-k-1}\cos^{n+s+r-k-1}\theta\sin^{p+1}\theta
\]
\[
- \sum_{k=0}^{r-1} (k + 2)\tilde{M}_{k+2}(\theta)b_{q+r-k-1}\cos^{q+r-k-1}\theta\sin\theta
\]
\[
+ \sum_{k=0}^{s+r-2} \tilde{M}_{k+3}(\theta)a_{n+s+r-k-2}\cos^{n+s+r-k-2}\theta\sin^{p}\theta
\]
\[
- \sum_{k=0}^{r-2} \tilde{M}_{k+3}(\theta)b_{q+r-k-2}\cos^{q+r-k-2}\theta.
\]
(6)
holds for \(1 \leq r\). When \(r = 1\), we have
\[
\widetilde{M}_{q+1}(\theta) = \sum_{k=0}^{s} (k + 2)\tilde{M}_{k+2}(\theta)a_{n+s-k}\cos^{n+s-k}\theta\sin^{p+1}\theta
\]
\[
- 2\tilde{M}_{2}(\theta)b_{q}\cos^{q}\theta\sin\theta
\]
\[
+ \sum_{k=0}^{s-1} \tilde{M}_{k+3}(\theta)a_{n+s-k-1}\cos^{n+s-k-1}\theta\sin^{p}\theta.
\]
By Lemma 1 and 2, since
\[
\widetilde{M}_{q+1}(2\pi) - \widetilde{M}_{q+1}(0) = 2\tilde{M}_{2}(\theta)a_{n+s}\int_{0}^{2\pi} \cos^{n+s}\theta\sin^{p+1}\theta d\theta = 0,
\]
we get \(a_{n+s} = 0\). Hence we see that \(\widetilde{M}_{q+1}(\theta)\) is a polynomial of \(\sin \theta\) only. Moreover, from (6) we have
\[
\widetilde{M}_{q+2}(\theta) = \sum_{k=0}^{s+1} (k + 2)\tilde{M}_{k+2}(\theta)a_{n+s-k+1}\cos^{n+s-k+1}\theta\sin^{p+1}\theta
\]
\[
- \sum_{k=0}^{1} (k + 2)\tilde{M}_{k+2}(\theta)b_{q-k+1}\cos^{q-k+1}\theta\sin\theta
\]
\[
+ \sum_{k=0}^{s} \tilde{M}_{k+3}(\theta)a_{n+s-k}\cos^{n+s-k}\theta\sin^{p}\theta
\]
\[
- \tilde{M}_{3}(\theta)b_{q}\cos^{q+1}\theta.
\]
By \(a_{n+s} = 0\) and the assumption that \(g_{q}\) is an odd function, we obtain that \(\widetilde{M}_{q+2}(\theta)\) is also a polynomial of \(\sin \theta\) only.

From now on, we suppose that for all \(l \geq 1\)
\[
a_{n+s} = a_{n+s+2} = \cdots = a_{n+s+2(l-1)} = 0
\]
and $\overline{M}_m(\theta)$ have been determined up to $m = q + 2l$ as polynomials of $\sin \theta$ only. Then, from (6), the equality determining $\overline{M}_{q+2l+1}(\theta)$ is given by

$$\overline{M}_{q+2l+1}(\theta) = \sum_{k=0}^{s+2l} (k + 2) \overline{M}_{k+2}(\theta) a_{n+s+2l-k} \cos^{n+s+2l-k} \theta \sin^{p+1} \theta$$

$$- \sum_{k=0}^{2l} (k + 2) \overline{M}_{k+2}(\theta) b_{q+2l-k} \cos^{q+2l-k} \theta \sin \theta$$

$$+ \sum_{k=0}^{s+2l-1} \overline{M}_{k+3}'(\theta) a_{n+s+2l-k} \cos^{n+s+2l-k} \theta \sin^p \theta$$

$$- \sum_{k=0}^{2l-1} \overline{M}_{k+3}^{l}(\theta) b_{q+2l-k} \cos^{q+2l-k} \theta$$

$$= 2 \overline{M}_2(\theta) a_{n+s+2l} \cos^{n+s+2l} \theta \sin^{p+1} \theta + \sum (\cdots). \quad (7)$$

From Lemma 2, the assumption of induction and that $g_q$ is an odd function, all the terms on the right-hand side of the equality (7), expect the first one, have the form (polynomial of $\sin \theta$) $\times$ (odd power of $\cos \theta$). Thus, since

$$\overline{M}_{q+2l+1}(2\pi) - \overline{M}_{q+2l+1}(0) = 2 \overline{M}_2(\theta) a_{n+s+2l} \int_0^{2\pi} \cos^{n+s+2l} \theta \sin^{p+1} \theta d\theta = 0,$$

we get $a_{n+s+2l} = 0$. Hence we see that $\overline{M}_{q+2l+1}(\theta)$ is a polynomial of $\sin \theta$ only.

Moreover we consider $\overline{M}_{q+2(l+1)}(\theta)$. By (6), $\overline{M}_{q+2(l+1)}(\theta)$ is determined from the equality

$$\overline{M}_{q+2(l+1)}(\theta) = \sum_{k=0}^{s+2l+1} (k + 2) \overline{M}_{k+2}(\theta) a_{n+s+2l-k-1} \cos^{n+s+2l-k-1} \theta \sin^{p+1} \theta$$

$$- \sum_{k=0}^{2l+1} (k + 2) \overline{M}_{k+2}(\theta) b_{q+2l-k-1} \cos^{q+2l-k-1} \theta \sin \theta$$

$$+ \sum_{k=0}^{s+2l-1} \overline{M}_{k+3}'(\theta) a_{n+s+2l-k-1} \cos^{n+s+2l-k-1} \theta \sin^p \theta$$

$$- \sum_{k=0}^{2l-1} \overline{M}_{k+3}^{l}(\theta) b_{q+2l-k-1} \cos^{q+2l-k} \theta$$

$$= 2 \overline{M}_2(\theta) a_{n+s+2l+1} \cos^{n+s+2l+1} \theta \sin^{p+1} \theta + \sum (\cdots). \quad (8)$$

From the above fact (i.e. $a_{n+s+2l} = 0$), the assumption of induction and that $g_q$ is an odd function, all the terms on the right-hand side of the equality (8) have the form (polynomial of $\sin \theta$) $\times$ (odd power of $\cos \theta$). Thus we conclude that $\overline{M}_{q+2(l+1)}(\theta)$ is a polynomial of $\sin \theta$ only.
Other seven cases are also proved by a similar method to the above one.

Case (II): \( n + p = q + t, \ t > 0 \)

First, we get the following lemma by comparing the terms of the same degree in \( r \) on both sides of the equality (5).

**Lemma 4.** If \( m \leq q \), then \( \overline{M}_m(\theta) = 0 \).

We shall consider the case (II)-(i). We get the following

**Lemma 5.** Suppose that \( q < m \leq q + t = n + p \). Then \( \overline{M}_m(\theta) \) is a polynomial of \( \cos \theta \) only.

**Proof.** From (5) we have

\[
\overline{M}_{q+1}(\theta) = -2\overline{M}_2(\theta)b_q \cos^q \theta \sin \theta.
\]

Thus \( \overline{M}_{q+1}(\theta) \) is a polynomial of \( \cos \theta \) only.

From now on, we suppose that \( \overline{M}_m(\theta) \) have been determined up to \( q + r - 1 \) \((2 \leq r \leq t)\) as polynomials of \( \cos \theta \) only. Then the equality determining \( \overline{M}_{q+r}(\theta) \) is given by

\[
\overline{M}_{q+r}(\theta) = -\sum_{k=0}^{r-1} (k+2)\overline{M}_{k+2}(\theta)b_{q+r-k-1} \cos^{q+r-k-1} \theta \sin \theta \\
- \sum_{k=0}^{r-2} \overline{M}'_{k+3}(\theta)b_{q+r-k-2} \cos^{q+r-k-1} \theta.
\]

Thus, we see from the assumption of induction and Lemma 4 that \( \overline{M}_{q+r}(\theta) \) is a polynomial of \( \cos \theta \) only.

**Lemma 6.** Suppose that \( m > q + t = n + p \). Then \( a_i = 0 \) for even numbers \( i \geq n \) and \( \overline{M}_m(\theta) \) is a polynomial of \( \cos \theta \) only.

**Proof.** From (5) we remark that the equality

\[
\overline{M}_{q+t+r}(\theta) = \sum_{k=0}^{r-1} (k+2)\overline{M}_{k+2}(\theta)a_{n+r-k-1} \cos^{n+r-k-1} \theta \sin^{p+1} \theta \\
- \sum_{k=0}^{r+t-1} (k+2)\overline{M}_{k+2}(\theta)b_{q+t+r-k-1} \cos^{q+t+r-k-1} \theta \sin \theta \\
+ \sum_{k=0}^{r-2} \overline{M}'_{k+3}(\theta)a_{n+r-k-2} \cos^{n+r-k-1} \theta \sin^p \theta \\
- \sum_{k=0}^{r+t-2} \overline{M}'_{k+3}(\theta)b_{q+t+r-k-2} \cos^{q+t+r-k-1} \theta
\]
holds for $1 \leq r$. When $r = 1$, we have

$$
\widetilde{M}_{q+t+1}''(\theta) = 2\overline{M}_2(\theta)a_n \cos^n \theta \sin^{p+1} \theta
$$

$$
- \sum_{k=0}^{t} (k+2) \overline{M}_{k+2}(\theta)b_{q+t-k} \cos^{q+t-k} \theta \sin \theta
$$

$$
- \sum_{k=0}^{t-1} \overline{M}_{k+3}'(\theta)b_{q+t-k-1} \cos^{q+t-k} \theta.
$$

By Lemma 4 and 5, since

$$
\overline{M}_{q+t+1}(2\pi) - \overline{M}_{q+t+1}(0) = 2\overline{M}_2(\theta)a_n \int_0^{2\pi} \cos^n \theta \sin^{p+1} \theta d\theta = 0,
$$

we get $a_n = 0$. Hence we see that $\widetilde{M}_{q+t+1}(\theta)$ is a polynomial of $\cos \theta$ only. As the result, we obtain from (10) that $\widetilde{M}_{q+t+2}(\theta)$ is also a polynomial of $\cos \theta$ only.

From now on, we suppose that for all $l \geq 1$

$$
a_n = a_{n+2} = \cdots = a_{n+2(l-1)} = 0
$$

(11)

and $\overline{M}_m(\theta)$ have been determined up to $m = q + t + 2l$ as polynomials of $\cos \theta$ only. Then, from (10), the equality determining $\widetilde{M}_{q+t+2l+1}(\theta)$ is given by

$$
\widetilde{M}_{q+t+2l+1}(\theta) = \sum_{k=0}^{2l} (k+2) \overline{M}_{k+2}(\theta)a_{n+2l-k} \cos^{n+2l-k} \theta \sin^{p+1} \theta
$$

$$
- \sum_{k=0}^{t+2l} (k+2) \overline{M}_{k+2}(\theta)b_{q+t+2l-k} \cos^{q+t+2l-k} \theta \sin \theta
$$

$$
+ \sum_{k=0}^{2l-1} \overline{M}_{k+3}'(\theta)a_{n+2l-k-1} \cos^{n+2l-k} \theta \sin^p \theta
$$

$$
- \sum_{k=0}^{t+2l-1} \overline{M}_{k+3}'(\theta)b_{q+t+2l-k-1} \cos^{q+t+2l-k} \theta
$$

$$
= 2\overline{M}_2(\theta)a_{n+2l} \cos^{n+2l} \theta \sin^{p+1} \theta + \sum(\cdots).
$$

(12)

From the assumption of induction and that $g_q$ is an odd function, all the terms on the right-hand side of the equality (12), except the first one, have the form (polynomial of $\sin \theta$) $\times$ (odd power of $\cos \theta$). Thus, since

$$
\widetilde{M}_{q+t+2l+1}(2\pi) - \widetilde{M}_{q+t+2l+1}(0) = 2\overline{M}_2(\theta)a_{n+2l} \int_0^{2\pi} \cos^{n+2l} \theta \sin^{p+1} \theta d\theta = 0,
$$

we get $a_{n+2l} = 0$. Hence we see that $\widetilde{M}_{q+t+2l+1}(\theta)$ is a polynomial of $\cos \theta$ only.
Moreover we consider $\overline{M}_{q+t+2(l+1)}(\theta)$. By (10), $\overline{M}_{q+t+2(l+1)}(\theta)$ is determined from the equality

$$
\overline{M}_{q+t+2(l+1)}(\theta) = \sum_{k=0}^{2l+1} (k+2)\overline{M}_{k+2}^{l}(\theta)a_{n+2l-k+1}\cos^{n+2l-k+1}\theta \sin^{p+1}\theta
$$

$$
- \sum_{k=0}^{t+2l+1} (k+2)\overline{M}_{k+2}^{l}(\theta)b_{q+t+2l-k+1}\cos^{q+t+2l-k+1}\theta \sin\theta
$$

$$
+ \sum_{k=0}^{2l} \overline{M}_{k+3}^{l}(\theta)a_{n+2l-k}\cos^{n+2l-k+1}\theta \sin^{p}\theta
$$

$$
- \sum_{k=0}^{t+2l} \overline{M}_{k+3}^{l}(\theta)b_{q+t+2l-k}\cos^{q+t+2l-k+1}\theta
$$

$$
= 2\overline{M}_{2}(\theta)a_{n+2l+1}\cos^{n+s+2l+1}\theta \sin^{p+1}\theta + \sum(\cdots). \quad (13)
$$

From the above fact (i.e. $a_{n+2l} = 0$), the assumption of induction and that $g_{q}$ is an odd function, all the terms on the right-hand side of the equality (13) have the form (polynomial of $\sin\theta$) $\times$ (odd power of $\cos\theta$). Thus we conclude that $\overline{M}_{q+t+2(l+1)}(\theta)$ is a polynomial of $\cos\theta$ only.

Other seven cases are also proved by a similar method to the above one. Therefore the proof of Theorem A is now completed. \(\square\)

The following fact is a key in the proof of Theorem B.

**Lemma 7.** Suppose that $n + p < m \leq n + p + s = q$. If $m$ is an odd (resp. even) number, then $\overline{M}_{m}(\theta)$ is a polynomial of $\cos\theta$ of odd(resp. even) degree only.

We omit the details for the proofs of Lemma 7 and Theorem B.

### 3. Appendix

[1] We consider the case $n + p \leq 1$ in the form (C). If $(n, p) = (1, 0)$ and $a_{1} > 1$, then there exists the first integral $(1/2)y^{2} + \int_{0}^{x} (f_{1}(\xi) - \xi - g_{q}(\xi))d\xi = \text{const.}$ of the system (L). Since $x\{f_{1}(x) - x - g_{q}(x)\} > 0$ ($x \neq 0$) in the neighborhood of the origin, the equilibrium point is a center.

If $(n, p) = (0, 1)$ and $a_{1} > 1$, then we can apply Theorem A and B to this system.

We set $P(x) = f_{0}(x) - x - g_{q}(x)$. Let a solution of the equation $P(x) = 0$ be $x = \alpha$. If $(n, p) = (0, 0)$ and $P'(\alpha) > 0$, then we also can apply Theorem A and B to this system.

[2] By combining the mentioned facts above and the result in [Su], we have the following result on a global center of the system (L).
Corollary. Consider the system (L) with $p = 1$ of the form (C). Suppose that
(C1) $g_q$ is an odd function with $g_q(0) = 0$ and $x \left\{ x + g_q(x) \right\} > 0$ ($x \neq 0$),
(C2) there exists $0 \leq \lambda < \sqrt{8}$ such that
\[
\left| \int_0^x f_n(\xi) d\xi \right| \leq \lambda \sqrt{\int_0^x g_q(\xi) d\xi} \quad \text{for sufficiently large } x.
\]
Then the equilibrium point $(0, 0)$ of the system (L) is a global center if and only if $\int_0^\infty g(x) dx = \infty$.

References


