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Kyoto University
ON DUCK SOLUTIONS IN $R^4$

KIYOYUKI TCHIZAWA (知沢 清之)

Dept of Maths, Musashi Institute of Technology

Abstract. In this paper, we will prove the existence of duck solutions with winding in the coupled Fitzhugh-Nagumo equation. As the system is described by the slow-fast one in $R^4$, we will find the ducks in $R^4$.

Let consider the following slow-fast system:

\begin{align*}
\epsilon dx_1/dt &= h_1(x_1, x_2, y_1, y_2, u), \\
\epsilon dx_2/dt &= h_2(x_1, x_2, y_1, y_2, u), \\
y_1/dt &= f(x_1, x_2, y_1, y_2, u), \\
y_2/dt &= g(x_1, x_2, y_1, y_2, u),
\end{align*}

where $\epsilon$ is infinitesimally small and $u$ is a parameter. We assume that $H = (h_1, h_2)$ has \textit{rank} $H = 2$ at almost every where. In this paper, we put

\begin{align*}
h_1 &= y_1 + x_1 - x_1^3/3 + \gamma(x_1 - x_2), \\
h_2 &= y_2 + x_2 - x_2^3/3 + \gamma(x_2 - x_1), \\
f &= -(x_1 - a + by_1)/c, \\
g &= -(x_2 - a + by_2)/c,
\end{align*}

and for the simplicity, we put $a = 0$, $\gamma = -1$. So, the parameters are $b$ and $c$ but only $b$ is essential. This slow-fast system is reduced from the coupled Fitzhugh-Nagumo equation proposed by S.A.Campbell[1], 2000. When $\epsilon = 0$, the fast system gives a 2-dim differentiable manifold as a constrained surface. Because of satisfying \textit{rank} $H = 2$ regarding especially $x_1, x_2$, the system(1) can be reduced to the slow-fast system projected in $R^2$:

\begin{align*}
y_1/dt &= -(x_1 + by_1)/c, \\
y_2/dt &= -(x_1^3/3 - y_1 + by_2)/c, \\
\epsilon dx_1/dt &= y_2 - (x_1^3/3 - y_1)^3/3 + x_1,
\end{align*}

under the condition, which $|dx_1/dt - dx_2/dt|$ is limited. On the constrained surface in the system(3), we can get the time scaled reduced system:

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Then, the pseudo singular point, that is, the singular point of the system (4) is determined by

$$\begin{align*}
\left(x_1^3/3 - y_1\right)^2(x_1 + by_1) + x_1^3/3 - y_1 + by_2 &= 0, \\
1 - \left(x_1^3/3 - y_1\right)^2x_1^2 &= 0.
\end{align*}$$

(5)

Note that the second equation in (5) can be expressed as $$x_1^3/3 - y_1 = \pm1/x_1$$. In the case $$(-)$$, there are 2 pseudo singular points:

$$(1, 4/3, -1, -4/3), \text{ and } (-1, -4/3, 1, 4/3).$$

These points do not depend on the parameter $$b$$, therefore they are structurally stable.

As the characteristic equation of the linearized system is

$$\lambda(\lambda - (2 + 8b/3))(-\lambda + 8b/3) = 0,$$

we can conclude that these will be node if $$-3/4 < b < 0$$. Then there are duck solutions at the pseudo-singular node. This fact implies they are winding. See[3], [4], [6].

In the case $$(+)$$, there are 4 pseudo singular points which depend on the parameter $$b$$. The characteristic equation in this case is

$$\lambda(A\lambda^2 + B\lambda + C)/(3 + D)^3 = 0,$$

(7)

where

$$\begin{align*}
A &= -D^3 - 27D + 36b^2 - 108 \\
B &= 2[(4b^2 - 9)D^3 + (16b^4 - 90b^2 + 243)]D - 162b^2 + 486]/3b \\
C &= -4[405D^3 + (64b^6 - 720b^4 + 291b^2 - 3645)D + 576b^6 - 3024b^4 + 3888b^2]/9b^2 \\
D &= \sqrt{(3 - 2b)(3 + 2b)}.
\end{align*}$$

If $$0 < b < 3/2$$, there exist the pseudo singular points, as

$$x_1 = \pm(-)\sqrt{3/2b}, \text{ and } \pm(\pm)\sqrt{(9/b^2 - 4)/2}.$$  

(8)

The eigenvalues of all four singular points are the same due to the symmetry. They arise in some sort of pitchfork bifurcation from the singular points in the $$(-)$$equation at $$b = 3/2$$. If $$0.388 < b < 1.4489$$, there will be the ducks at the pseudo-singular node and spirals if $$0 < b < 0.388$$ or $$1.4489 < b < 3/2$$. 
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REFERENCES