数理解析研究所講究録 1254 巻 2002 年 163-171

常微分方程式の定性解析によるファジィ境界値問題

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Keywords : Fuzzy Numbers; Fuzzy Differential Equation; Fuzzy Boudary Condition; Couple Parametric Representation;

1 Introduction

There are many fruitful results on representations of fuzzy numbers, differentials and integrals of fuzzy functions (see, e.g., in Aumann [1], Goetschel-Voxman [8, 9], Dubois-Prade [3, 4, 5, 6], Puri-Ralescue [13], Furukawa [7], Kaleva [10, 11] etc). They establish fundamental results concerning differentials, integrals and fuzzy differential equations of fuzzy functions which map **R**, where **R** is the set of real numbers, to a set of fuzzy numbers. By using the results it seems to be difficult to apply all the practical and significant problems. In this study we introduce the couple parametric representation(see [14]) corresponding to the representation of fuzzy numbers due to Goetschel-Voxman so that it is easy to solve fuzzy differential equations.

In Buckley [2], Kaleva [10, 11], Park [12] and Song [17], various types of conditions for the existence and uniqueness of solutions to fuzzy differential equations. By the couple representation some kinds of differential and integral of fuzzy functions can be easily treated in an analogous way with the real analysis as well as some type of fuzzy differential equations can be solved without difficulty. In Section 2 we denote a fuzzy number x by (x_1, x_2), where x_1, x_2 are endpoints of $\alpha-$cut set of the membership function μ_x, respectively. We consider some kind of metric space which includes the set of fuzzy numbers as well as prove the continuity of x_1, x_2. In Section 3 we give definitions of differential and integral of fuzzy functions and sufficient conditions for fuzzy functions to be differentiable or integrable. In Section 4 we show the existence and uniqueness of solutions for initial value problems of fuzzy differential equations $x' = F(t, x), x(t_0) = x_0$, where $t \in \mathbf{R}$ and x is a fuzzy number. Moreover we discuss global behaviours of solutions for $x' = p(t)x$, where p is a continuous fuzzy function on **R**. In Section 5 we treat a fuzzy differential equation $x'' = f(t, x, x')$ with fuzzy boundary conditions $x(a) = A, x(b) = B$,where f is a fuzzy-valued function defined on $J = [a, b]$ in the set of real numbers **R**, and A, B are fuzzy numbers.

2 Parametric Representation of Fuzzy Numbers

In order to introduce a metric space which includes the set of fuzzy numbers, we define the following set.

$$X = \{x = (x_1, x_2) \in C(I) \times C(I)\}$$

where $I = [0,1] \subset \mathbf{R}$ and $C(I)$ is the set of continuous functions from I to $\mathbf{R}$. Denote a metric by $d(x,y) = \sup_{\alpha \in I}(|x_1(\alpha) - y_1(\alpha)| + |y_2(\alpha) - y_2(\alpha)|)$ for $x = (x_1, x_2), y = (y_1, y_1) \in X$. Then the metric space (X, d) is complete. The following definition means that fuzzy numbers are identified with membership functions.

Definition 1 *Consider a set of fuzzy numbers with bounded supports as follows:*

$\mathcal{F}_{\mathbf{b}}^{st} = \{\mu : \mathbf{R} \to I$ *satisfying* (i)-(iv) *below*$\}$.

(i) *There exists a unique* $m \in \mathbf{R}$ *such that* $\mu(m) = 1$.

(ii) *The set* $supp(\mu) = cl(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$ *is bounded in* $\mathbf{R}$.

(iii) *One of the following conditions holds:*
(a) μ *is strictly fuzzy convex, i.e.,*

$$\mu(c\xi_1 + (1-c)\xi_2) > \min[\mu(\xi_1), \mu(\xi_2)]$$

for $\xi_1, \xi_2 \in \mathbf{R}, 0 < c < 1$;
(b) $\mu(m) = 1$ *and* $\mu(\xi) = 0$ *for* $\xi \neq m$.

(iv) μ *is upper semi-continuous on* $\mathbf{R}$.

Remark 1 *The above condition* (iiia) *is stronger than one in the usual case where* μ *is fuzzy convex. From* (iiia) *it follows that* $\mu(\xi)$ *is strictly increasing in* $\xi \in (-\infty, m)$ *and strictly decreasing in* $\xi \in (m, \infty)$. *This condition plays an important role in the proof of Theorem* 1.

We introduce the following parametric representation of $\mu \in \mathcal{F}_{\mathbf{b}}^{st}$,

$$\begin{aligned} x_1(\alpha) &= \min L_\alpha(\mu), \\ x_2(\alpha) &= \max L_\alpha(\mu) \end{aligned}$$

for $0 < \alpha \leq 1$ and

$$\begin{aligned} L_\alpha(\mu) &= \{\xi \in \mathbf{R} : \mu(\xi) \geq \alpha\}, \\ x_1(0) &= \min cl(supp(\mu)), \\ x_2(0) &= \max cl(supp(\mu)). \end{aligned}$$

Remark 2 *From the extension principle of Zadeh, it follows that*

$$\begin{aligned} & \mu_{x+y}(\xi) \\ = & \max_{\xi=\xi_1+\xi_2} \min_{i=1,2} (\mu_i(\xi_i))) \\ = & \max\{\alpha \in I : \xi = \xi_1 + \xi_2, \xi_i \in L_\alpha(\mu_i)\} \\ = & \max\{\alpha \in I : \\ & \xi \in [x_1(\alpha) + y_1(\alpha), x_2(\alpha) + y_2(\alpha)]\}, \end{aligned}$$

where μ_1, μ_2 *are membership functions of* x, y, *respectively. Thus we get* $x + y = (x_1 + y_1, x_2 + y_2)$.

The following theorem is a basic result.

Theorem 1 *Denote* $\mu = (x_1, x_2) \in \mathcal{F}_{\mathbf{b}}^{st}$, *where* $x_1, x_2 : I \to \mathbf{R}$. *The following properties* (i)-(iii) *hold.*

(i) x_1, x_2 *are continuous on* I.

(ii) $\max x_1(\alpha) = x_1(1) = m$ *and* $\min x_2(\alpha) = x_2(1) = m$.

(iii) *One of the following statements holds:*
(a) x_1 *is strictly increasing and* x_2 *is strictly decreasing with* $x_1(\alpha) < x_2(\alpha)$;
(b) $x_1(\alpha) = x_2(\alpha) = m$ *for* $0 < \alpha \leq 1$.

Conversely, under the above conditions (i)-(iii), *if we denote*

$$\mu(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leq \xi \leq x_2(\alpha)\}$$

then $\mu \in \mathcal{F}_{\mathbf{b}}^{st}$. *Moreover it follows that* $\mathbf{R} \subset \mathcal{F}_{\mathbf{b}}^{st}$ *and that* $\mathcal{F}_{\mathbf{b}}^{st}$ *is a complete metric spcae in* X.

In the following example we illustrate typical three types of fuzzy numbers.

Example 1 *Consider the following* $L-R$ *fuzzy number* $x \in \mathcal{F}_{\mathbf{b}}^{st}$ *with a membership function as follows:*

$$\mu_x(\xi) = \begin{cases} L(\frac{m-\xi}{l})_+ & \text{for } \xi \leq m \\ R(\frac{\xi-m}{r})_+ & \text{for } \xi > m \end{cases}$$

where $m \in \mathbf{R}, l > 0, r > 0$. L, R *are into mappings defined on* $\mathbf{R}_+ = [0, \infty)$. *Let* $L(\xi)_+ = \max(L(\xi), 0)$ *etc. We identify* μ_x *with* $x = (x_1, x_2)$ *Then we have* $x_1(\alpha) = m - L^{-1}(\alpha)l$ *and* $x_2(\alpha) = m + R^{-1}(\alpha)r$ *provided that* L^{-1} *and* R^{-1} *exist.*

Let $L(\xi) = -c_1\xi + 1$, *where* $c_1 > 0$. *We illustrate the following cases* (i)-(iii).

(i) *Let* $R(\xi) = -c_2\xi + 1$, *where* $c_2 > 0$. *Then* $c_2 l(x_2 - m) = c_1 r(m - x_1)$.

(ii) *Let* $R(\xi) = -c_2\sqrt{\xi} + 1$, *where* $c_2 > 0$. *Then* $c_2 l(x_2 - m)^2 = c_1 r^2(m - x_1)$.

(iii) *Let* $R(\xi) = -c_2\xi^2 + 1$, *where* $c_2 > 0$. *Then* $c_2^2 l^2(x_2 - m) = c_1^2 r(x_1 - m)^2$.

3 Differential and Integral of Fuzzy-valued Functions

Let an interval $J \subset \mathbf{R}$. Denote an $\mathcal{F}_{\mathbf{b}}^{st}$–valud function by

$$\begin{aligned} x(t) &= (x_1(t), x_2(t)) \\ &= \{(x_1(t,\alpha), x_2(t,\alpha))^T \in \mathbf{R}^2 : \alpha \in I\}. \end{aligned}$$

We define the continuiety and differentiability of fuzzy-valued function as follows:

Definition 2 *A fuzzy-valued function* $x : J \to \mathcal{F}_{\mathbf{b}}^{st}$ *is continuous at* $t \in J$ *if*

$$\lim_{h \to 0} d(x(t+h), x(t)) = 0.$$

Let $x : J \to \mathcal{F}_{\mathbf{b}}^{st}$ *be*

$$\begin{aligned} x(t) &= \{(x_1(t,\alpha), x_2(t,\alpha))^T \in \mathbf{R}^2 : \alpha \in I\} \\ &= (x_1(t,\cdot), x_2(t,\cdot)) = x(t,\cdot) \end{aligned}$$

for $t \in J$. *The function* x *is said to be differentiable at* $t \in J$ *if for any* $\alpha \in I$ *there exist* $\dfrac{\partial x_1}{\partial t}(t,\alpha), \dfrac{\partial x_2}{\partial t}(t,\alpha)$ *such that* $\dfrac{\partial x_2}{\partial t}(t,\alpha) \leq \dfrac{\partial x_2}{\partial t}(t,\alpha)$ *and* $\mu_{\partial x}(t,\cdot) \in \mathcal{F}_{\mathbf{b}}^{st}$, *where* $\mu_{\partial x}(t,\xi) = \sup\{\alpha \in I : \frac{\partial x_1}{\partial t}(t,\alpha) \leq \xi \leq \frac{\partial x_2}{\partial t}(t,\alpha)\}$. *The function* x *is said to be differentiable on* J *if* x *is differentiable at any* $t \in J$. *Denote* $\dfrac{dx}{dt}(t) = x'(t) = (\dfrac{\partial x_1}{\partial t}(t,\cdot), \dfrac{\partial x_2}{\partial t}(t,\cdot))$ *and it is said to be the derivative of* $x(t)$.

We consider the following definition of the integral of $\mathcal{F}_{\mathbf{b}}^{st}$– valued functions.

Definition 3 *Let* $x : J \to \mathcal{F}_{\mathbf{b}}^{st}$ *be* $x(t,\cdot) = (x_1(t,\cdot), x_2(t,\cdot))$ *for* $t \in J$. *The function* x *is* *said to be integrable over* $[t_1, t_2]$, *if* x_1, x_2 *are Riemann integrable over* $[t_1, t_2]$. *Then we define the integral as follows:*

$$\int_{t_1}^{t_2} x(s,\cdot)ds = \{(\int_{t_1}^{t_2} x_1(s,\alpha)ds, \int_{t_1}^{t_2} x_2(s,\alpha)ds)^T \in \mathbf{R}^2 : \alpha \in I\}.$$

Remark 3 *Let* $x(t) = (x_1(t,\cdot), x_2(t,\cdot)) \in \mathcal{F}_{\mathbf{b}}^{st}$ *for* $t \in J$.

(i) *If* x *is differentiable at* t, *we get the integral over* $[t_1, t_2] \subset J$ *as follows:*

$$\int_{t_1}^{t_2} x'(s,\cdot)ds + x(t_1,\cdot) = x(t_2,\cdot).$$

(ii) *If* $x(t) \in \mathcal{F}_{\mathbf{b}}^{st}$ *is integrable over* $[t_1, t_2]$, *then we have* $\int_{t_1}^{t_2} x(s,\cdot)ds \in \mathcal{F}_{\mathbf{b}}^{st}$. *We have*

$$d(\int_{t_1}^{t_2} x(s,\cdot)ds, 0) \leq \int_{t_1}^{t_2} d(x(s,\cdot),0)ds.$$

4 Initial Value Problems of Fuzzy Differential Equations

Consider the following initial value problem of a differential equation

$$x'(t) = f(t,x), \quad x(t_0) = x_0 \qquad (N)$$

where $t_0 \in \mathbf{R}, x_0 \in \mathcal{F}_{\mathbf{b}}^{st}$. Let $f : J_c \times \mathcal{B}(x_0, r) \to \mathcal{F}_{\mathbf{b}}^{st}$, where $J_c = [t_0, t_0 + c], c > 0, \mathcal{B}(x_0, r) = \{x \in \mathcal{F}_{\mathbf{b}}^{st} : d(x_0, 0) \leq r\}$.

By applying the contraction principle we get the following theorem.

Theorem 2 (*cf.* [17]) *Suppose that the following conditions* (i) *and* (ii) *are satisfied.*

(i) f *is bounded, i.e., there exists an* $M > 0$ *such that* $d(f(t,x), 0) \leq M$ *for* $(t,x) \in J_c \times \mathcal{B}(x_0, r)$;

(ii) f *is Lipschitzian in* x *,i.e., there exists an* $L > 0$ *such that* $d(f(t,x), f(t,y)) \leq Ld(x,y)$ *for* $(t,x),(t,y) \in J_c \times \mathcal{B}(x_0, r)$.

Then there exists a unique solution x *for* (N) *such that* $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s,\cdot))ds$ *for* $t \in J_\rho = [t_0, t_0 + \rho]$, *where* $\rho = \min(c, r/M)$.

In the following example we obtain an initial valu problem of ordinary differential equations which are arising from fuzzy problems.

Example 2 *Consider the following problem of fuzzy differential equation*

$$x' = p(t)x + q(t), \quad x(t_0) = x_0 \qquad (E)$$

$t \in \mathbf{R}, x_0, x(t) \in \mathcal{F}_{\mathbf{b}}^{st}$. *Functions* $p, q : \mathbf{R} \to \mathbf{R}$ *are continuous, respectively.*

Let $p : \mathbf{R} \to (-\infty, 0]$ and $x(t) = (x_1(t), x_2(t))$. Then we have $x_1'(t) = p(t)x_2(t) + q(t), x_2' = p(t)x_1(t) + q(t)$, by denoting $x_0 = (a_0, b_0)$, so $x_1(t,\alpha)$ and $x_2(t,\alpha)$ satisfy

$$\begin{pmatrix} x_1(t,\alpha) \\ x_2(t,\alpha) \end{pmatrix} = \Phi(t,\alpha) \begin{pmatrix} a_0(t,\alpha) \\ b_0(t,\alpha) \end{pmatrix} + \Phi(t,\alpha) \int_{t_0}^{t} \Phi^{-1}(s,\alpha) \begin{pmatrix} q(s,\alpha) \\ q(s,\alpha) \end{pmatrix} ds,$$

where $\Phi(\cdot,\cdot)$ is a fundamental matrix of

$$\frac{d}{dt}(x_1(t,\alpha), x_2(t,\alpha))^T = (p(t,\alpha)x_2(t,\alpha), p(t,\alpha)x_1(t,\alpha))^T$$

,i.e.,

$$\Phi(t,\alpha) = \begin{pmatrix} \phi_{11}(t,\alpha) & \phi_{12}(t,\alpha) \\ \phi_{21}(t,\alpha) & \phi_{22}(t,\alpha) \end{pmatrix},$$

where

$$\phi_{11}(t,\alpha) = \frac{e^{\int_{t_0}^t p(s,\alpha)ds} + e^{-\int_{t_0}^t p(s,\alpha)ds}}{2}$$

$$\phi_{12}(t,\alpha) = \frac{e^{\int_{t_0}^t p(s,\alpha)ds} - e^{-\int_{t_0}^t p(s,\alpha)ds}}{2}$$

$$\phi_{21}(t,\alpha) = \frac{e^{\int_{t_0}^t p(s,\alpha)ds} - e^{-\int_{t_0}^t p(s,\alpha)ds}}{2}$$

$$\phi_{22}(t,\alpha) = \frac{e^{\int_{t_0}^t p(s,\alpha)ds} + e^{-\int_{t_0}^t p(s,\alpha)ds}}{2}$$

for $t \geq t_0, \alpha \in I$. Then we get the following theorem in which solutions of fuzzy differential equation mean unstability in case that the initial value $x_0 \in \mathcal{F}_{\mathbf{b}}^{st} \backslash \mathbf{R}$.

Theorem 3 *Let $q(t) \equiv 0$. Then solutions of (E) satisfy following statements* (i) - (iii).

(i) *Any solutions x such that $x_0 \in \mathbf{R}$ satisfy* $\lim_{t\to\infty} d(x(t), 0) = 0$;

(ii) *Any solutions x such that $x_0 \in \mathcal{F}_{\mathbf{b}}^{st} \backslash \mathbf{R}$ satisfy* $\lim_{t\to\infty} d(x(t), 0) = \infty$ *and* $\lim_{t\to\infty} |x_1(t,\alpha) + x_2(t,\alpha)| = 0$ *for $\alpha \in I$.*

Seikkala [16] calculates the solution in case that $p(t) \equiv -1$. In what follows we consider the equation (E) with $q(t) \equiv 0$.

Example 3 *Consider behaviors of solutions of the following problem of a fuzzy differential equation*

$$x' = p(t)x, \quad x(t_0) = x_0 \qquad (E_0)$$

where $t \in \mathbf{R}, x_0$ and $x(t) \in \mathcal{F}_{\mathbf{b}}^{st}$. Here $p(t) = (p_1(t,\cdot), p_2(t,\cdot)) : \mathbf{R} \to \mathcal{F}_{\mathbf{b}}^{st}$ is continuous.

Remark 4 *Let $T(x) = p(t)x$. It follows that T is non-linear.*

In analyzing the ordinary differential equation $x' = a(t)x$, where $a : \mathbf{R} \to \mathbf{R}$ are continuous, the condition that $\lim_{t\to\infty} \int^t a(s)ds = -\infty$ plays an important role in showing the property that $\lim_{t\to\infty} x(t) = 0$. Concerning fuzzy differential equation (E_0), we get an extension result of asymptotic behaviors of ordinary linear differential equations as well as we observe a little different result as follows. When $p = (p_1, p_2)$ is a fuzzy function, we have the following theorem.

Theorem 4 *Consider Problem (E_0). Let $p_2(t,\alpha) \leq 0$ on $\mathbf{R} \times I$ and*

$$\lim_{t\to\infty} \int_{t_0}^t p_2(s,\cdot)ds = -\infty$$

for $t_0 \in \mathbf{R}$. Then solutions of (E_0) satisfy following statements (i) - (iii).

(i) *Any solutions x such that $x_0 \in \mathbf{R}$ satisfy* $\lim_{t\to\infty} d(x(t), 0) = 0$;

(ii) *Any solutions x such that $x_0 \in \mathcal{F}_{\mathbf{b}}^{st} \backslash \mathbf{R}$ satisfy* $\lim_{t\to\infty} d(x(t), 0) = \infty$.

(iii) *Let the solution $x(t) = \{(x_1(t,\alpha), x_2(t,\alpha))^T \in \mathbf{R}^2 : \alpha \in I\}$ satisfy $|x_1(t,\alpha)| \leq x_2(t,\alpha)$ for $J_1 = [\tau, \sigma]$. Then it follows that $0 \leq x_1(t,\alpha) + x_2(t,\alpha) \leq e^{\int_\tau^t p_1(s,\alpha)ds}$ for $\tau, t \in J_1, \alpha \in I$.*

In the following example we get an extension of Theorem 3.

Example 4 *Consider the following problem*

$$x' = P_m(t)x, \quad x(t_0) = x_0 \qquad (P_m)$$

$P_m : \mathbf{R} \to \mathcal{F}_{\mathbf{b}}^{st}$ *such that* $P_m = (-m-q_1, -m+q_2)$ *satisfies*

$$m : \mathbf{R} \times I \to \mathbf{R},\ m(t,\alpha) \geq 0,$$

$$q_i : \mathbf{R} \times I \to \mathbf{R},$$

$$0 \leq q_i(t,\alpha) \leq m(t,\alpha),\ i = 1,2.$$

Theorem 5 *Suppose that for* $\alpha \in I, t_0 \in \mathbf{R}$

$$\lim_{t\to\infty} \int_{t_0}^{t} m(s,\alpha)ds = \infty,$$

$$\lim_{t\to\infty} e^{-\int_{t_0}^{t} m(s,\alpha)ds} \times \int_{t_0}^{t} q(s,\alpha) e^{\int_{t_0}^{s}(2m(r,\alpha)+q(r,\alpha))dr} ds = 0,$$

where $q(t,\alpha) = \max(q_1(t,\alpha), q_2(t,\alpha))$. *Then, if the initial value* $x_0 \in \mathcal{F}_{\mathbf{b}}^{st} \backslash \mathbf{R}$, *for any solution* $x = (x_1, x_2)$ *of* (P_m) *it follows that* $\lim_{t\to\infty} |x_1(t,\alpha) + x_2(t,\alpha)| = 0$ *for* $\alpha \in I$.

In the following example we consider the stability of solutions of fuzzy differential equations.

Example 5 *Let* $P_0(t,\cdot) = (-p_0(t,\cdot), p_0(t,\cdot))$ *satisfy* $p_0(t,\alpha) \geq 0$ *for* $t \in \mathbf{R}, \alpha \in I$. *Consider the following fuzzy initial value problem*

$$x' = P_0(t)x, \quad x(t_0) = x_0. \qquad P_0$$

We treat the following cases (i) -(iv) in order to observe the behaviors of solutions for (P_0).

(i) The relation $x_1(t,\alpha) \geq 0$ for $t \in J, \alpha \in I$ leads to $x_1'(t,\cdot) = -p_0 x_2, x_2'(t,\cdot) = p_0 x_2$, $x_1(t,\alpha) + x_2(t,\alpha) = a_0(\alpha) + b_0(\alpha)$ and the solution $x_2(t,\alpha) = b_0 e^{\int_{t_0}^{t} p_0(s,\alpha)ds}$;

(ii) The relations $x_1(t,\alpha) \leq 0 \leq x_2(t,\alpha)$ and $|x_1(t,\alpha)| \leq x_2(t,\alpha)$ for $t \in J, \alpha \in I$ lead to $x_1'(t,\cdot) = -p_0 x_2, x_2'(t,\cdot) = p_0 x_2$, $x_1(t,\alpha) + x_2(t,\alpha) = a_0(\alpha) + b_0(\alpha)$ and the solution $x_2(t) = b_0 e^{\int_{t_0}^{t} p_0(s,\alpha)ds}$;

(iii) The relations $x_1(t,\alpha) \leq 0 \leq x_2(t,\alpha)$ and $|x_1(t,\alpha)| \geq x_2(t,\alpha)$ for $t \in J, \alpha \in I$ lead to $x_1'(t,\cdot) = p_0 x_1, x_2'(t,\cdot) = -p_0 x_1$, $x_1(t,\alpha) + x_2(t,\alpha) = a_0(\alpha) + b_0(\alpha)$ and the solution $x_1(t) = a_0 e^{\int_{t_0}^{t} p_0(s,\alpha)ds}$;

(iv) When $x_2(t,\alpha) \leq 0$ for $t \in J, \alpha \in I$, we get $x_1'(t,\cdot) = p_0 x_1, x_2'(t,\cdot) = -p_0 x_1, x_1(t,\alpha) + x_2(t,\alpha) = a_0(\alpha) + b_0(\alpha)$ and the solution $x_1(t,\alpha) = a_0 e^{\int_{t_0}^{t} p_0(s,\alpha)ds}$.

Under conditions in Example 5, the zero solution of (P_0) is uniformly stable. The definition of stability is as follows.

Definition 4 *The zero-solution of* (P_0) *is uniformly stable if For each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that each* $t_0 \in \mathbf{R}$ *and each* $x_0 \in \mathcal{F}_{\mathbf{b}}^{st}$ *such that* $d(x_0, 0) \leq \delta$, *each solution* x *of* (P_0) *satisfies* $d(x(t), 0) < \varepsilon$ *for* $t \geq t_0$.

The following conditions are sufficient ones for the stability of the zero solution to (P_0).

Theorem 6 *Assume that there exists an* $M > 0$ *such that* $\limsup_{t\to\infty} \int_{t_0}^{t} p_0(s,\alpha)ds \leq M$ *for* $t \geq t_0 \geq 0, \alpha \in I$ *in Example 5. Then zero solution of* (P_0) *is uniformly stable.*

5 Boundary Value Problems of Fuzzy Differential Equations

Let $J = [a, b] \subset \mathbf{R}$. In this section we consider the following fuzzy differential equastion with fuzzy boundary conditions

$$(F) \qquad x'' = f(t, x, x'),$$

$$(1) \qquad x(a) = A,$$

$$(2) \qquad x(b) = B,$$

where $t \in J$, $x = (x_1, x_2) \in \mathcal{F}_{\mathbf{b}}^{st}, A = (A_1, A_2), B = (B_1, B_2) \in \mathcal{F}_{\mathbf{b}}^{st}$. Then we get ordinary differential equations

$$x_1'' = f_1(t, x_1, x_2, x_1', x_2')$$

$$x_2'' = f_2(t, x_1, x_2, x_1', x_2')$$

$$x_1(a) = A_1, \quad x_2(a) = A_2,$$

$$x_1(b) = B_1, \quad x_2(b) = B_2$$

with conditions that $x_j^{(i)}(t, \cdot), i = 0, 1, 2; j = 1, 2$, satisfy (i) - (iii) of Theorem 1.

By putting $y_1 = x_1', y_2 = x_2'$ we have

$$\begin{pmatrix} x_1' \\ x_2' \\ y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f_1(t, x_1, x_2, y_1, y_2) \\ f_2(t, x_1, x_2, y_1, y_2) \end{pmatrix}.$$

Then, by denoting $z = (x_1, x_2, y_1, y_2)^T \in \mathbf{R}^4$, we get

$$(S) \qquad z' = Bz + F(t, z)$$

$$(C) \qquad \mathcal{L}(z) = (A_1, A_2, B_1, B_2)^T$$

Here

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, F(t, z) = \begin{pmatrix} 0 \\ 0 \\ f_1(t, z) \\ f_2(t, z) \end{pmatrix}$$

and $\mathcal{L}$ is a bounded linear operator from $C(J) \times C(J)$ to $\mathbf{R}^4$ as follows:

$$\mathcal{L}(z) = (x_1(a), x_2(a), y_1(b), y_2(b))^T.$$

In this case we get the fundamental matrix

$$X_B(t) = e^{tB} = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let U_B satisfy

$$\mathcal{L}(X_B(\cdot)z_0) = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & a \\ 1 & 0 & b & 0 \\ 0 & 1 & 0 & b \end{pmatrix} z_0 = U_B z_0$$

for $z_0 \in \mathbf{R}^4$. It follows that

$$U_B^{-1} = \frac{1}{b-a} \begin{pmatrix} b & 0 & -a & 0 \\ 0 & b & 0 & -a \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

We denote a norm in $\mathbf{R}^4$ by $\| z \| = |x_1| + |x_2| + |y_1| + |y_2|$. Then $\| U_B \| = \max(2, a + b)$ and $\| U_B^{-1} \| = \dfrac{b+1}{b-a}$.

In the similar way of discussion in [15] the authors obtain the existence and uniqueness of solutions for boundary value problems of ordinary differential equations

$$(S_n) \qquad x' = D(t)x + F(t, x),$$

$$(C_n) \qquad \mathcal{L}_n(x) = c,$$

where $t \in J, x(t) \in \mathbf{R}^n, c \in \mathbf{R}^n, \mathcal{L}_n : C(J) \to \mathbf{R}^n$ is a bounded linear operator, $D : J \to \mathbf{R}^{n\times n}$ and $F : J \times \mathbf{R}^n \to \mathbf{R}^n$ are continuous. Denote the fundamental matrix of (S_n) by X. Define a constant matrix U with $\mathcal{L}(X(\cdot)x_0) = Ux_0$. Assume that U is nonsingular. Then we have the following existence and uniqueness thereoms.

Theorem 7 (cf. [15]) *Let* $K = e^{\int_a^b \|D(s)\| ds}$ *and* $K_1 = \sup_{a\le s\le t\le b} \| X(t)X^{-1}(s) \|$ *and and let a positive number* δ *satisfy* $\delta < 1/(K \| U^{-1} \|)$. *Assume that* F *satisfies*

$$\liminf_{n\to\infty} \frac{1}{n} \int_a^b \sup_{\|x\|\le n} \| F(s,x) \| ds < \frac{1/K - \delta \| U^{-1} \|}{1 + K_1 \| \mathcal{L} \| \| U^{-1} \|}.$$

If $\| c \| \le \delta$, *then* $((S_n),(C_n))$ *has at least one solution.*

Theorem 8 (cf. [15]) *Let*

$$L(r) = \int_a^b \sup_{\|z_i\|\le r, i=1,2} \| F(s,z_1) - F(s,z_2) \| ds$$

for $r > 0$. *If there exists an* $r_0 > 0$ *such that* $(K_1 \| \mathcal{L} \| + 1)K_1 L(r_0) < 1$ *and* $\| c \| \le r_0$, *then* $((S_n),(C_n))$ *has one and only one solution.*

By applying the above theorems we get the existence ant uniqueness theorems of $((S),(C))$.

Theorem 9 *Let* $\mathbf{R}^2-$ *valud function* $f = (f_1, f_2)^T$ *be continuous on* $J \times \mathbf{R} \times \mathbf{R}$ *and let* $\delta_1 > 0$ *satisfy* $\delta_1 < 1/(e^{(b-a)}\frac{b+1}{b-a})$. *Assume that*

$$\liminf_{n\to\infty} \frac{1}{n} \int_a^b \sup_{\|z\|\le n} (|f_1(s,z)| + |f_2(s,z)|) ds < \frac{1/K - \delta_1 \frac{b+1}{b-a}}{1 + e^{b-a} \| \mathcal{L} \| \frac{b+1}{b-a}}.$$

If $d(z_0, 0) \le \delta_1$, *where* $z_0 = (A_1, A_2, B_1, B_2)^T \in \mathbf{R}^4$, *then* $((S),(C))$ *has at least one solution.*

Theorem 10 *Let* $\mathbf{R}^2-$ *valud function* $f = (f_1, f_2)^T$ *be continuous on* $J \times \mathbf{R} \times \mathbf{R}$ *and let* $L_1(r) = \int_a^b \sup_{d(z_i,0)\le r, i=1,2} (|f_1(s,z_1) - f_1(s,z_2| + |f_2(s,z_1) - f_2(s,z_2|) ds$ *for* $r > 0$. *If there exists an* $r_1 > 0$ *such that* $(e^{b-a} \| \mathcal{L} \| + 1)e^{b-a} L_1(r_1) < 1$ *and* $d(z_0, 0) \le r_1$, *where* $z_0 = (A_1, A_2, B_1, B_2)^T \in \mathbf{R}^4$, *then* $((S),(C))$ *has one and only one solution.*

In the above results we have the following question: Do solutions of $((S),(C))$ are solutions of $((F),(1),(2))$, i.e., solutions of $((S),(C))$ satisfy conditions (i) -(iii) of Theorem 1. In order to gauratee the existence of solutions of $((F),(1),(2))$ with fuzzy numbers we consider the following conditions:

Conditions(FZ) *Let* $d_0 = \min(\delta_1, r_1)$. *Denote* $S_{d_0} = \{x \in \mathcal{F}^{st}_{\mathbf{b}} : d(x,0) \le d_0\}$. *Let* $S_{d_0} = \{x(\alpha) = (x_1(\alpha), x_2(\alpha))^T \in \mathbf{R}^2 : x \in S_{d_0}$ *and* $\alpha \in I\}$. *Let the following estimates* (i)-(iv) *hold for* $0 \le \alpha < \beta < 1$.

(i) $B_2(\alpha) - B_1(\alpha) > A_2(\alpha) - A_1(\alpha)$.

(ii) $B_2(\alpha) - B_1(\alpha) > \int_a^b \sup_{x,y\in S_{d_0}} [f_2(s,x,y) - f_1(s,x,y)] ds$.

(iii) $B_1(\alpha) - B_1(\beta) > A_1(\alpha) - A_1(\beta) + \int_a^b \sup_{x,y\in S_{d_0}} [f_1(s,x(\alpha),y(\alpha),\alpha) - f_1(s,x(\beta),y(\beta),\beta)] ds$.

(iv) $B_2(\alpha) - B_2(\beta) < A_2(\alpha) - A_2(\beta) + \int_a^b \sup_{x,y\in S_{d_0}} [f_2(s,x(\alpha),y(\alpha),\alpha) - f_2(s,x(\beta),y(\beta),\beta)] ds$.

Provided that Condition (FZ) holds, it is expected that sufficient conditions in Theorems 9 and 10 lead to the same conclusion, respectively.

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