On supersingular rank one perturbations of the selfadjoint operators

(Spectral and Scattering Theory and Related Topics)

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On supersingular rank one perturbations of the selfadjoint operators

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1 Introduction.

We shall consider the singular rank one perturbations of the positive self-adjoint operator. First we shall recall the notation of singular rank one perturbation. Let $H$ be a Hilbert space, $A$ a (positive) selfadjoint operator and $H_s := \{u; \|(1 + |A|)^{s/2}u\| < \infty\}$. Assume that $\varphi \in H_{-n} \setminus H_{-n+1}$. We shall put

$$A_\alpha = A + \alpha \langle \cdot, \varphi \rangle \varphi. \quad (1)$$

We call $\langle \cdot, \varphi \rangle \varphi$ "(resp. super)singular rank one perturbation" of $A$ for $n = 1, 2$ (resp. $n \geq 3$). The main purpose is to construct a operator $\mathcal{A}$ corresponding to $A_\alpha$. We shall give a method of the construction of a Hilbert space $\mathcal{H}$ and an operator $\mathcal{A}$ in $\mathcal{H}$ for $n = 3$. (section 2). For $n = 1, 2$ the operator $A_\alpha$ is recognized as a selfadjoint operator by using "restriction and extension theory". (See §2).

Next we shall consider the supersingular rank one perturbation for the selfadjoint operator. There are two approaches for the problem:

1. Using Pontryagin space (Krein space):
   

2.

3. Using Hilbert space:
   
   I. Andronov ([3]), P. Kurasov and K. Watanabe ([16], [17]), P. Kurasov ([15]).

1. In general the norm of the Pontryagin space $\mathcal{P}$ is not positive definite, but they can construct selfadjoint operator $\mathcal{A}$ in $\mathcal{P}$ corresponding $A_\alpha$.
2. We can consider the operator $\mathcal{A}$ corresponding to $A_\alpha$ in the new Hilbert space, but $\mathcal{A}$ is not selfadjoint except for $n = 3$. 
Example 1 In [3] he considered the operator

\[ A = -\Delta \text{ in } L^2(\mathbb{R} \times \mathbb{R}_+) \]

(Neumann condition) and

\[ \varphi = \partial_{x_2} \delta(x_1, x_2). \]

The author can not give completely the articles related to the singular perturbation theory. Many references can be seen in [2].

2 \( H_{-1} \)- and \( H_{-2} \)-perturbation.

In this section we shall review the singular rank one perturbation. \((H_{-1}, H_{-2} \text{-perturbations})\). Let \( \varphi \in H_{-n} \setminus H_{-n+1} \) \((n = 1, 2)\) and \( A^0 \) the restriction of \( A \) to the space

\[
D(A^0) = \{ u \in D(A); \langle u, \varphi \rangle = 0 \}, \\
A^0 u = Au, \ u \in D(A^0).
\]

Then we shall consider the relation between the operator \( A_\alpha \) and the (von Neumann)extension \( A(\theta) \) of \( A_\alpha \). Using "restriction and extension theory" we recognize \( A_\alpha \) as a selfadjoint operator.

Two extension methods:

(I) Direct extension:

\[
D(A_\alpha) \ni U = u + u_1 \frac{A}{A^2 + 1} \varphi, \ u \in H_2, u_1 \in \mathbb{C}, \\
\langle u, \varphi \rangle = -\left( \frac{1}{\alpha} + \frac{A}{A^2 + 1} \langle \varphi, \varphi \rangle \right) u_1, \\
A_\alpha U := Au - u_1 \frac{1}{A^2 + 1} \varphi,
\]

where, if \( \varphi \in H_{-2} \setminus H_{-1} \), then we put

\[
\langle \frac{A}{A^2 + 1} \varphi, \varphi \rangle \equiv c \in \mathbb{R},
\]
(II) von Neumann extension: for $\theta \in [0, \pi)$,

\[
D(A(\theta)) \ni U = \tilde{u} + \tilde{u}_{1} \frac{\sin \theta A - \cos \theta}{A^2 + 1} \varphi, \quad \tilde{u} \in D(A^0), \tilde{u}_{1} \in \mathbb{C}, \quad (4)
\]

\[
A(\theta)U = A^0 \tilde{u} + \tilde{u}_{1} \frac{-\cos \theta A - \sin \theta}{A^2 + 1} \varphi. \quad (5)
\]

**Theorem 1** Let $\varphi \in H_{-1} \setminus H$. Then there exists a bijection between $A_\alpha$ and $A(\theta)$, i.e., if the relation of $\alpha$ and $\theta$, $\theta \in [0, \pi)$, is

\[
\left\langle \frac{1}{A^2 + 1} \varphi, \varphi \right\rangle \cos \theta - (\frac{1}{\alpha} + \left\langle \frac{A}{A^2 + 1} \varphi, \varphi \right\rangle) \sin \theta = 0,
\]

then $A_\alpha = A(\theta)$. For $n = 2$ we put $c \in \mathbb{R}$ in (6) instead of $\left\langle \frac{A}{A^2 + 1} \varphi, \varphi \right\rangle$.

**Proof.** (i) for $A(\theta)$. (von Neumann's method. cf. [21]) Using the deficiency elements $h_{\pm i} = \frac{1}{A \mp i} \varphi$, we put

\[
U = \tilde{u} + \frac{\tilde{u}_{1}}{2} (h_i - e^{2i\theta} h_{-i}), \quad \tilde{u} \in D(A^0), \quad \tilde{u}_{1} \in \mathbb{C},
\]

and define

\[
A(\theta)U = A^0 \tilde{u} + i \frac{\tilde{u}_{1}}{2} (h_i + e^{2i\theta} h_{-i}). \quad (7)
\]

Then $A(\theta)$ is the selfadjoint extension of $A^0$. In particular, $A(0) = A$.

(ii) for $A_\alpha$. (cf. [2].) The domain of $A_\alpha$ is $(A_\alpha U \in H )$

\[
\begin{align*}
U &= u + u_1 \frac{A}{A^2 + 1} \varphi, \quad u \in D(A), \quad u_1 \in \mathbb{C} \\
\left\langle u, \varphi \right\rangle &= -\left(\frac{1}{\alpha} + \left\langle \varphi, \frac{A}{A^2 + 1} \varphi \right\rangle\right) u_1.
\end{align*}
\]

(8)

This means that there is a linear relation in $D(A) \oplus \{ a \frac{A}{A^2 + 1} \varphi \}$.

(iii) Rewriting the element of $D(A(\theta))$ and substituting it to (8), we obtain Theorem.

### 3 $H_{-3}$-perturbation

We assume

\[
A \geq 0.
\]
We shall consider the case of
\[ \varphi \in H_{-3} \setminus H_{-2} \]
and construct the operator corresponding to the operator
\[ A_\alpha = A + \alpha \langle \cdot, \varphi \rangle \varphi \]
in the extended Hilbert space in
\[ \mathcal{H} = H_1 + \mathbb{C}. \]

**Remark 1** If we restrict \( A \) to
\[ D(A^0) = \{ u \in H_3; \langle u, \varphi \rangle = 0 \}, \]
then \( A^0 \) is essentially selfadjoint in \( H \). So any selfadjoint extension of \( A^0 \) is \( A \).

Let \( a_1 \) be a positive constant and put
\[ g_1 = \frac{1}{A + a_1} \varphi. \]

To construct the extended Hilbert space \( \mathcal{H} \) (suitable for \( A_\alpha \)) we put
\[ \mathcal{H}_{pre} = \text{Dom}(A^0) + \mathbb{C} \ni \mathcal{U} = (u, u_1). \]  
(9)

Note that \( \mathcal{H}_{pre} \subset H_3 + \mathbb{C} \). We define the following natural embedding \( \rho \) of the space \( \mathcal{H}_{pre} \) into the space \( H_{-1} \):
\[ \rho : \mathcal{H}_{pre} \rightarrow H_{-1} \]
\[ (u, u_1) \mapsto u + u_1 g_1. \]  
(10)

Then the scalar product in the space \( \mathcal{H}_{pre} \) can be introduced using the following formal calculations where \( b \) is a certain positive constant:
\[
\ll \mathcal{U}, \mathcal{V} \gg = \langle \rho \mathcal{U}, \rho \mathcal{V} \rangle + b \langle \rho \mathcal{U}, A \rho \mathcal{V} \rangle \\
= \langle u + u_1 g_1, v + v_1 g_1 \rangle + b \langle u + u_1 g_1, A(v + v_1 g_1) \rangle \\
= \langle u, v \rangle + b \langle u, Av \rangle + \bar{u}_1 v_1 \langle g_1, g_1 \rangle + b \langle g_1, Ag_1 \rangle + \bar{u}_1 b \langle g_1, v \rangle + v_1 b \langle u, Ag_1 \rangle. 
\]
The last two terms can be simplified taking into account that

$$Ag_1 = -a_1g_1 + \varphi$$

and the fact of $u, v \in H_3 \cap D(A^0)$. Then the scalar product is given by the expression

$$\langle \mathcal{U}, \mathcal{V} \rangle = \langle u, v \rangle + b\langle u, Av \rangle + \bar{u}_1v_1 (\|g_1\|^2 + b\langle g_1, Ag_1 \rangle) + (1 - ba_1)(u_1\langle g_1, v \rangle + v_1\langle u, g_1 \rangle),$$

which can be considered only formally, since the scalar product $\langle g_1, Ag_1 \rangle$ and the norm $\|g_1\|^2$ are not defined (since $\varphi$ is an element from $H_{-3} \setminus H_{-2}$). To define the scalar product we extend $\varphi$ as a bounded linear functional using the equalities

$$\langle g_1, g_1 \rangle = c_1, \quad \langle g_1, Ag_1 \rangle = c_2, \quad \text{(11)}$$

where $c_1$ and $c_2$ are arbitrary positive real constants. In what follows we are going to use the notation

$$d = c_1 + bc_2 \in \mathbb{R}_+.$$

The scalar product determined by the following expression will also be considered:

$$\langle \mathcal{U}, \mathcal{V} \rangle = \langle u, v \rangle + b\langle u, Av \rangle + d\bar{u}_1v_1 + (1 - ba_1) \{\bar{u}_1\langle g_1, v \rangle + v_1\langle u, g_1 \rangle\}. \quad \text{(13)}$$

This formula defines a sesquilinear form on the domain $\text{Dom}(A^0) + \mathbb{C}$. This form defines a scalar product only if it is positive definite.

Let us denote by $\|\mathcal{U}\|^2 = \langle \mathcal{U}, \mathcal{U} \rangle$ the norm associated with the previously introduced scalar product. The space $\mathcal{H}$ with this norm is not complete, and the following lemma describes its completion with respect to this norm.

**Theorem 2** Let $\varphi \in H_{-3} \setminus H_{-2}$, $a_1 > 0$, $g_1 := (A + a_1)^{-1}\varphi$, and

$$\mathcal{H} = H_1 + \mathbb{C}.$$
For \( \mathcal{U} = (u, u_1), \mathcal{V} = (v, v_1) \in \mathcal{H} \) we define
\[
\ll \mathcal{U}, \mathcal{V} \gg = \langle u, v \rangle + b\langle Au, v \rangle + du_1\overline{v}_1 + (1 - ba_1) \{u_1\langle g_1, v \rangle + \overline{v}_1\langle u, g_1 \rangle\} \tag{14}
\]
where \( b > 0, d > 0 \).

If we assume that
\[
d > |1 - ba_1|^2((1 + bA)^{-1}g_1, g_1), \tag{15}
\]
then \( \ll \cdot, \cdot \gg \) is a scalar product on \( \mathcal{H} \) and the norm induced from \( \ll \cdot, \cdot \gg \) is equivalent to the standard norm of \( H_1 \oplus \mathbb{C} \), i.e.,
\[
\ll \mathcal{U}, \mathcal{U} \gg \cong ((1 + b'A)u, u) + d'|u_1|^2.
\]

We omit the proof.

Next we define an operator \( A \) in \( \mathcal{H} \). Let \( a_2 > 0 \),
\[
g_2 = (A + a_2)^{-1}g_1
\]
and define
\[
\text{Dom}(A) \ni \mathcal{U} = (u_r + u_2 g_2, u_1), \tag{16}
\]
\[
A\mathcal{U} = (Au_r - a_1 u_2, u_2 - a_1 u_1) \tag{17}
\]
where \( u_r \in H_3 \) and \( u_2 \in \mathbb{C} \).

**Theorem 3** For \( 0 \leq \theta < \pi \) we define the linear subspace \( D(\theta) \) of \( \text{Dom}(A) \) as follows:
\[
D(\theta) \ni \mathcal{U} = (u_r + u_2 g_2, u_1)
\]
if and only if
\[
b \sin \theta \langle \varphi, u_r \rangle - (a \sin \theta + c \cos \theta)u_1 + b \cos \theta u_2 = 0, \tag{18}
\]
where \( a, c \) are constants determined by \( a_1, a_2, b, d, \varphi \). Let \( A(\theta) \) be the restriction of \( A \) to \( D(\theta) \). Then \( A(\theta) \) is a selfadjoint operator with \( \text{Dom}(A(\theta)) = \)
Remark 2 Using the natural embedding map $\rho$ we have

$$\rho \mathcal{A} \mathcal{U} = Au_r - a_2 u_2 g_2 + (u_2 - a_1 u_1) g_1 \pmod{\varphi}$$

for

$$D(A) \ni \mathcal{U} = (u_r + u_2 g_2, u_1).$$

Because of

$$Ag_1 = \frac{A}{A + a_1} \varphi = \varphi - a_1 g_1,$$
$$Ag_2 = g_1 - a_2 g_2.$$

Proof of Theorem 3.

Step 1: $A$ is symmetric on $D(\theta)$.

We can calculate the boundary form as follows:

$$\ll \mathcal{U}, A \mathcal{V} \gg - \ll A \mathcal{U}, \mathcal{V} \gg = \langle \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} u_r, \varphi \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \\ u_2, v_2 \end{pmatrix} \rangle.$$  \hfill (19)

We assume that the boundary form is written as

$$\alpha \langle u_r, \varphi \rangle + \beta u_2 + \gamma u_1 = 0.$$ \hfill (20)

Combining above two conditions, we have

$$(19) = 0 \ (A \text{ symmetric}) \iff \alpha, \beta, \gamma \in \mathbb{R}, \ \alpha a + \beta b + \gamma c = 0$$

$$\iff \alpha, \beta, \gamma \in \mathbb{R}, \ (a, b, c) \perp (\alpha, \beta, \gamma).$$ \hfill (21)

Hence we can represent $(\alpha, \beta, \gamma)$ by one parameter $\theta$: i.e.,

$$(\alpha, \beta, \gamma) = (b \sin \theta, -a \sin \theta - c \cos \theta, b \cos \theta).$$ \hfill (22)

Therefore $A(\theta)$ is symmetric on $D(\theta)$.

Step 2: $A(\theta)$ is self-adjoint on $D(\theta)$. 
For $\lambda \ll 0$ we prove

$$R(A(\theta) - \lambda) = \mathcal{H},$$

i.e. that for any $\mathcal{V} = (v, v_1) \in \mathcal{H}$ there exists an element $\mathcal{U} = (u_r + u_2g_2, u_1) \in \text{Dom}(A(\theta))$ such that

$$(A(\theta) - \lambda)\mathcal{U} = \mathcal{V}.$$ 

The last equation can be written as

\[
\begin{align*}
(A - \lambda)u_r - (a_2 + \lambda)u_2g_2 &= v; \\
u_2 - (a_1 + \lambda)u_1 &= v_1.
\end{align*}
\]

The first of these equations can be rewritten as

$$u_r - (a_2 + \lambda)u_2 \frac{1}{A - \lambda}g_2 = \frac{1}{A - \lambda}v,$$

which implies

$$\langle u_r, \varphi \rangle - (a_2 + \lambda)(\frac{1}{A - \lambda}g_2, \varphi)u_2 = \langle \frac{1}{A - \lambda}v, \varphi \rangle.$$

$\mathcal{U} = (u_r + u_2g_2, u_1)$ should satisfy the boundary condition (18). Hence $((u_r, \varphi), u_1, u_2) \in \mathbb{C}^3$ solves the system of linear equations

\[
\begin{bmatrix}
1 & 0 & -(a_2 + \lambda)\Phi_\lambda \\
0 & -(a_1 + \lambda) & 1 \\
b \sin \theta & -a \sin \theta - c \cos \theta & b \cos \theta
\end{bmatrix}
\begin{bmatrix}
\langle u_r, \varphi \rangle \\
u_1 \\
u_2
\end{bmatrix}
=
\begin{bmatrix}
\langle \frac{1}{A - \lambda}v, \varphi \rangle \\
v_1 \\
0
\end{bmatrix}
\]

(23)

where we put $\Phi_\lambda = \langle \frac{1}{A - \lambda}g_2, \varphi \rangle$. The determinant of this system is:

$$-(a_1 + \lambda)b \cos \theta - b \sin \theta(a_1 + \lambda)(a_2 + \lambda)\Phi_\lambda + a \sin \theta + c \cos \theta.$$

(24)

(i) $\theta = 0$: Since $b \neq 0$, $\exists \lambda$ such that $-a_1b + c - b\lambda \neq 0$.

(ii) $\theta \neq 0$: We can prove

$$\lim_{\lambda \to -\infty}|\lambda\Phi_\lambda| = \infty$$

because $\varphi \in H_{-3}$. Hence the 2-nd term of (24) is dominant of the determinant. Therefore Theorem has been proved.
Remark 3 From the proof of Theorem we know:

(i) $\mathcal{A}(\theta)$ is semibounded from below.

(ii) For $\theta = 0$: The solution of the linear system (23) is given by

\[
\langle u_r, \varphi \rangle = \frac{(c - b(a_1 + \lambda))(\frac{1}{A - \lambda}v, \varphi) + (a_2 + \lambda)(\frac{1}{A - \lambda}g_2, \varphi)c v_1}{(c - b(a_1 + \lambda)) + (a - b(a_1 + \lambda)(a_2 + \lambda)(\frac{1}{A - \lambda}g_2, \varphi)};
\]

\[
u_1 = \frac{b v_1}{c - b(a_1 + \lambda)};
\]

\[
u_2 = \frac{c v_1}{c - b(a_1 + \lambda)}.
\]

Then the resolvent can be calculated as

\[
\frac{1}{\mathcal{A}(\theta) - \lambda}(v, v_1) = \left( \frac{1}{A - \lambda}v + \left( \frac{1}{A - \lambda}g_1 \right) u_2, u_1 \right). \tag{25}
\]

(iii) $\mathcal{V} = (v, 0) \in H_1 + \mathbb{C}$:

\[
\rho_{\mathcal{A}(\theta) - \lambda}|_{H_1} v = \frac{1}{A - \lambda}v + \frac{b \sin \theta \times}{\cos \theta (c - b(a_1 + \lambda)) + \sin \theta (a - b(a_1 + \lambda)(a_2 + \lambda)(\frac{1}{A - \lambda}g_2, \varphi)} \times
\]

\[
\times \left( \frac{1}{A - \lambda}v, \varphi \right) \left( \frac{1}{A - \lambda}\varphi \right), \tag{26}
\]

(iv) $\theta = 0, v_1 = 0$.

\[
\frac{1}{A - \lambda} \rho|_{H_1} = \rho_{\mathcal{A}(0) - \lambda}|_{H_1}. \tag{27}
\]

Hence

\[
\text{Dom}(\mathcal{A}) \subset \text{Dom}(\mathcal{A}(0))
\]

and the action coincides

\[
\mathcal{A}(0)|_{\text{Dom}(\mathcal{A})} = \mathcal{A}.
\]

Therefore the operator $\mathcal{A}(0)$ should be considered as an unperturbed operator, since this is the unique operator possessing the properties described above. All of the other operators $\mathcal{A}(\theta)$ corresponding to $\theta \neq 0$ are perturbations of $\mathcal{A}(0)$. 

\section{H\textsubscript{−n}-perturbation ($n \geq 4$).}

\[ \varphi \in H\textsubscript{−n} \setminus H\textsubscript{−n+1} \ (n \geq 4). \] For simplicity we confine $n = 4$. For general $n$ see [15]. We put

\[
\begin{align*}
g_1 &= (A + 1)^{-1}\varphi, \\
g_2 &= (A + 1)^{-1}g_1, \\
g_3 &= (A + 1)^{-1}g_2
\end{align*}
\]

and Hilbert space and the scalar product

\[
\mathcal{H} = H_2 \oplus \mathbb{C}^2 \ni \mathcal{U} = (u, u_2, u_1), \mathcal{V} = (v, v_2, v_1),
\]

\[
\ll \mathcal{U}, \mathcal{V} \gg = \langle (A + 1)^2u, u \rangle + u_2\overline{v}_2 + u_1\overline{v}_1
\]

We define the maximal operator $A$ in $\mathcal{H}$ corresponding to $A_\alpha$ as follows:

\[
\text{Dom}(A) = \{ \mathcal{U} = (U_r + u_3g_3, u_2, u_1); \\
U_r \in H_4, u_3, u_2, u_1 \in \mathbb{C} \}
\]

and

\[
A \begin{pmatrix} U_r + u_3g_3 \\
\ u_2 \\
\ u_1 \end{pmatrix} = \begin{pmatrix} AU_r - u_3g_3 \\
\ u_3 - u_2 \\
\ u_2 - u_1 \end{pmatrix}.
\]  \quad (28)

\textbf{Definition 1} Let $T$ be a densely defined closed operator in a Hilbert space.

$T$ is regular $\iff$ D($T$) = D($T^*$) \quad (29)

\textbf{Theorem 4} For $\theta \in [0, \pi)$ let $A(\theta)$ be a restriction of $A$ to

\[
\sin \theta \langle U_r, \varphi \rangle + \cos \theta u_3 - \sin \theta u_2 = 0.
\]  \quad (30)

Then $A(\theta)$ is regular. Conversely any regular restriction of $A$ is given by (30) for some $\theta \in [0, \pi)$. 

Remark 4 (i) The action of the operator $A(\theta)^*$ is given by

$$A(\theta)^* \begin{pmatrix} V_r + v_3 g_3 \\ v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} AV_r - v_3 g_3 \\ v_3 + v_1 - v_2 \\ -v_1 \end{pmatrix}. \quad (31)$$

The real and imaginary parts of the operator $A(\theta)$ are given by

$$A(\theta) = \Re A(\theta) + i\Im A(\theta); \quad (32)$$

$$\begin{pmatrix} U_r + u_3 g_3 \\ u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} AU_r - u_3 g_3 \\ u_3 - u_2 + \frac{1}{2}u_1 \\ \frac{1}{2}u_2 - u_1 \end{pmatrix};$$

$$\Re A(\theta) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}.$$  

The imaginary part of $A(\theta)$ is a bounded operator.

(ii) We can prove that the spectrum of the regular operator $A(\theta)$ is pure real even if the operator is not self-adjoint.

5 Further Results and Problems.

In section 3 we confine the case $a_1, a_2, b, d = 1$. Then the Hilbert space $\mathcal{H}$ and the scalar product are $\mathcal{H} = H_1 \oplus \mathbb{C}$ and $\langle \mathcal{U}, \mathcal{V} \rangle = \langle (1 + A)u, v \rangle + u_1 \overline{v}_1$, respectively. And the condition of the element of $A(\theta)$ is given by, for $\mathcal{U} = (u_r + u_2 g_2, u_1) \in D(A(\theta))$

$$\sin \theta \langle u_r, \varphi \rangle - \sin \theta u_1 + \cos \theta u_2 = 0, \quad (33)$$

and the operator acts as 

$$A(\theta)\mathcal{U} = (Au_r - u_2 g_2, u_2 - u_1).$$
We consider the following selfadjoint operator in $\mathcal{H}$:
\[
AU = (Au, -u_1), \quad u \in H_3, \quad u_1 \in \mathbb{C}.
\]

Then $A$ is selfadjoint and $A \geq -1$. The space $D(A)^*$ of the dual space $D(A)$
with respect to $\langle \cdot, \cdot \rangle$ is $D(A)^* = H_{-1} \oplus \mathbb{C}$. We consider the rank one
perturbation for $A$.

\[
A_\alpha = A + \alpha \langle \cdot, G_1 \rangle G_1,
\]

where $G_1 = (g_1, -1) \in H_{-1} \oplus \mathbb{C}$.

We can see that $A_\alpha$ is selfadjoint if and only if $c \in \mathbb{R}$ is real parameter
and the following relation is satisfied

\[
U \in D(A_\alpha) \iff \begin{cases} 
U = \tilde{U} + aA_{A+1}^{-1}G_1, \tilde{U} \in D(A), \\
\langle \tilde{U}, G_1 \rangle = -(\frac{1}{\alpha} + c)a,
\end{cases}
\]

where

\[
\frac{A}{A^2 + 1}G_1 = (\frac{A}{A^2 + 1}g_1, \frac{1}{2})
\]

We can rewrite the above relation as follows:

\[
\langle u, \varphi \rangle - u_1 = -\left(\frac{1}{\alpha} + c\right)a.
\]

**Theorem 5** There exists one to one correspondence between $A_\alpha$ and $A(\theta)$.

\[
\{A_\alpha\}_{\alpha \in \mathbb{R}} = \{A(\theta)\}_{0 \leq \theta < \pi}.
\]

**Proof.** Since

\[
\frac{A}{A^2 + 1}g_1 = (\frac{A}{A^2 + 1} - \frac{1}{A+1})g_1 + g_2
\]

\[
= (\frac{A}{(A+1)(A+1)}g_1 + g_2,
\]

the element of the domain of $A(\theta)$ can be written as

\[
U = ((u + a\frac{A}{(A+1)(A+1)}g_1), u_1 + \frac{1}{2}a)
\]

\[
= (u_r + ag_2, u_1 + a/2).
\]
Substituting this to (33) we have

$$\sin \theta \langle u + a \frac{A - 1}{A^2 + 1} g_2, \varphi \rangle - \sin \theta (u_1 + a/2) + \cos \theta a = 0. \quad (37)$$

By (36) we obtain

$$\sin \theta a \langle \frac{A - 1}{A^2 + 1} g_2, \varphi \rangle - \sin \theta (1/\alpha + c) a - \sin \theta a/2 + \cos \theta a = 0. \quad (38)$$

Hence

$$\{ A_\alpha \}_{\alpha \in \mathbb{R}} = \{ A(\theta) \}_{0 \leq \theta < \pi}$$

Problem. (1) In this section we began with

$$D(A^0) = \{ u \in H_3; \langle u, \varphi \rangle = 0 \},$$

and

$$D(A^0) = \{ U = (u, u_1) \in D(A); \ll U, G_1 \gg = \langle u, \varphi \rangle - u_1 = 0 \}. \quad (39)$$

We would like to consider the relation of $A^0$ and $A^0$. $A$ is considered as $A : H_s \to H_{s-2}$. Hence by

$$D(A^0) = H_3(+\text{condition}) \oplus \{ 0 \}$$

$$\subset H_3 \oplus \mathbb{C}(+\text{condition}) = D(A^0),$$

we can consider that $\langle u, \varphi \rangle = 0$ and $\langle u, \varphi \rangle - cu_1 = 0$ have some relation. (Because in the case $u_1 = 0$ we can identify.) But $c = 1$? or not?

(2) Are $A(\theta)$ in $\mathcal{H}$ the operators corresponding to the operators constructed by using the Pontryagin space?

References


