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Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity

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1 Introduction

In this paper we study the local smoothing property and Strichartz inequality for n-dimensional Schrödinger equations with potentials which grow super-quadratically at infinity:

$$i\frac{\partial u}{\partial t}=-(1/2)\Delta u+V(x)u, \quad x\in\mathbb{R}^n,\ t\in\mathbb{R}; \quad u(0,x)=u_0(x), \quad x\in\mathbb{R}^n. \tag{1.1}$$

Assumption 1.1. *$V(x)$ is real valued and is of C^∞-class. There exist $m>2$ and $R>0$ such that:*

(1) *For $|x|\geq R$, $D_1\langle x\rangle^m\leq V(x)\leq D_2\langle x\rangle^m$, where $D_1\leq D_2$ are positive constants.*

(2) *For any α, $|\partial_x^\alpha V(x)|\leq C_\alpha\langle x\rangle^{m-|\alpha|}$.*

Under the assumption, the operator $L: u\mapsto -(1/2)\Delta u+V(x)u$ defined on $C_0^\infty(\mathbb{R}^n)$ is essentially selfadjoint in $L^2(\mathbb{R}^n)$ and the solution in $L^2(\mathbb{R}^n)$ of the initial value problem (1.1) is given by $u(t,\cdot)=U(t)u_0$ via the unitary group $U(t)=e^{-itH}$ generated by the unique selfadjoint extension H of L. We shall show that the solution $u(t,\cdot)$, nonetheless, is much smoother than u_0 and $1/m$ times differentiable at almost all time $t\neq 0$. More precisely, we prove the following theorem. We write $\langle A\rangle=(1+|A|^2)^{\frac{1}{2}}$ for a self-adjoint operator A and $D=(D_1,\dots,D_n)$, $D_j=-i\partial/\partial x_j$. $\|\cdot\|_p$ is the norm of Lebesgue space $L^p(\mathbb{R}^n)$ and $\|\cdot\|=\|\cdot\|_2$, $1\leq p\leq\infty$.

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Theorem 1.2. *Let V satisfy* Assumption 1.1 *and* $\Psi \in C_0^\infty(\mathbf{R}^n)$. *Then, for any $T > 0$, there exists a constant $C > 0$ such that*

$$\left(\int_{-T}^{T} \|\Psi(x)\langle D\rangle^{\frac{1}{m}} e^{-itH} u_0\|^2 dt\right)^{\frac{1}{2}} \leq C\|u_0\|, \quad u_0 \in L^2(\mathbf{R}^n). \tag{1.2}$$

Theorem 1.2 is an extension of the one dimensional result by [YZ] to multi-dimensional cases and it is sharp in the sense that the exponent $1/m$ in (1.2) cannot in general be replaced by any larger number. This can be seen by taking the potential $V(x) = \langle x_1\rangle^m + \cdots + \langle x_n\rangle^m$ and the initial state $u_0(x) = e_{i_1}(x_1)\cdots e_{i_n}(x_n)$, where $e_j(x)$ is the j-th eigenfunction of the one dimensional Schrödinger operator $-(1/2)(d^2/dx^2) + \langle x\rangle^m$, and by using the well known result on the asymptotic behavior as $j \to \infty$ of $e_j(x)$ for x in a compact set (see e.g. [YZ]). However, a slightly stronger result $\sup_{x\in\mathbf{R}^1}\left(\int_{-T}^{T} |\Psi(x)\langle D\rangle^{\frac{1}{m}} e^{-itH} u_0(x)|^2 dt\right)^{\frac{1}{2}} \leq C\|u_0\|$ is known in one dimension (see [YZ]).

On the way to the proof of Theorem 1.2 we prove the following Strichartz type inequality with "derivative loss".

Theorem 1.3. *Let V satisfy* Assumption 1.1. *Let $2 \leq p, \theta \leq \infty$ be such that $\dfrac{2}{\theta} = n\left(\dfrac{1}{2} - \dfrac{1}{p}\right)$ and $p \neq \infty$ if $n = 2$. Then, for any $T > 0$ and $\gamma > \dfrac{1}{\theta}\left(\dfrac{1}{2} - \dfrac{1}{m}\right)$ there exists a constant $C > 0$ such that*

$$\left(\int_{-T}^{T} \|e^{-itH} u_0\|_p^\theta dt\right)^{\frac{1}{\theta}} \leq C\|\langle H\rangle^\gamma u_0\|, \quad u_0 \in L^2(\mathbf{R}^n). \tag{1.3}$$

Note that $\|\langle H\rangle^\gamma u_0\| < \infty$ requires u_0 also to decay at inifity: $\langle x\rangle^{m\gamma} u_0 \in L^2(\mathbf{R}^n)$. In one dimension a related result $\|\langle H\rangle^{\theta(m,p)} e^{-itH} u_0(x)\|_{L^p(\mathbf{R}_x, L^2(-T,T))} \leq C\|u_0\|$ is known for a certain $\theta(m,p)$ which is *positive* for any $2 \leq p \leq \infty$ if $m < 4$ and for $\frac{1}{p} > \frac{m-4}{4(m-1)}$ if $m \geq 4$ (see [YZ]). This suggests that Theorem 1.3 is far from best possible. For Schrödinger equations on compact Riemannian manifolds, Strichartz' inequality with sharp derivative loss $\gamma = \frac{1}{2\theta}$ has recently been obtained by [Bu]. See also [Bo1], [Bo2] for related results. Applications of Theorem 1.2 and Theorem 1.3 to the initial value problem for nonlinear Schrödinger equations will be discussed elsewhere.

The estimates of the forms (1.2) and (1.3) have been long known for the free Schrödinger equation in the following stronger forms (see e.g. [Sj], [KY] for (1.4) and [St], [GV], [Y1] for (1.5); the "end-point" case of (1.5), however, has been proved by [KT] only recently) and they have been widely applied, in particular, to nonlinear Schrödinger equations ([K3],

[KPV]) or to the convergence problem ([V]). We write H_0 for $-(1/2)\triangle$ with the domain $D(H_0) = H^2(\mathbb{R}^n)$, where $H^\sigma(\mathbb{R}^n)$ is Sobolov space of order σ.

(1) Local smoothing property: For any $T > 0$ and $\Psi \in C_0^\infty(\mathbb{R}^n)$, there exists $C > 0$ such that

$$\left(\int_0^T \|\Psi(x)\langle D\rangle^{\frac{1}{2}} e^{-itH_0}u_0\|^2 dt\right)^{\frac{1}{2}} \leq C\|u_0\|, \quad u_0 \in L^2(\mathbb{R}^n), \tag{1.4}$$

where T can be set $T = \infty$ if $n \geq 3$.

(2) Strichartz inequality: Let $2 \leq p, \theta \leq \infty$ be such that $\dfrac{2}{\theta} = n\left(\dfrac{1}{2} - \dfrac{1}{p}\right)$ and $p \neq \infty$ if $n = 2$. Then, there exists $C > 0$ such that

$$\left(\int_0^\infty \|e^{-itH_0}u_0\|_p^\theta dt\right)^{\frac{1}{\theta}} \leq C\|u_0\|_2, \quad u_0 \in L^2(\mathbb{R}^n). \tag{1.5}$$

For generalizations of these inequalities to the case with decaying potentials, see e.g. [CS], [BAD] and [Y1].

Before proceeding further, we present here the outlines of the proofs of (1.4) (for $T < \infty$) and (1.5) which explain their "physical contents" because they will guide our proofs of Theorem 1.2 and Theorem 1.3 and "physically explain" why $1/m$ in (1.2) is sharp. We consider along with the equation (1.1) corresponding Newton's equations:

$$\begin{gathered} \dot{q}(t) = p(t), \quad \dot{p}(t) = -\nabla_q V(q), \\ q(0) = y, \quad p(0) = k, \end{gathered} \tag{1.6}$$

and denote their solutions by $(q(t, y, k), p(t, y, k))$. If $V = 0$, $q(t, y, k) = y + tk$ and $p(t, y, k) = k$.

For proving (1.4) for $T < \infty$, we use the formula $e^{itH_0}xe^{-itH_0} = x + tD$ and write

$$\begin{aligned} \int_0^T \|\Psi(x)\langle D\rangle^{1/2}e^{-itH_0}u\|_2^2 dt &= \int_0^T (\langle D\rangle^{1/2}e^{itH_0}\Psi^2(x)e^{-itH_0}\langle D\rangle^{1/2}u, u)dt \\ &= \left(\langle D\rangle^{1/2} \cdot \left\{\int_0^T \Psi^2(x + tD)dt\right\} \cdot \langle D\rangle^{1/2}u, u\right). \end{aligned} \tag{1.7}$$

Here we have $\left|\partial_\xi^\alpha \partial_x^\beta \int_0^T \Psi^2(x + t\xi)dt\right| \leq C_{\alpha\beta}\langle\xi\rangle^{-1}$ for any α, β and $\int_0^T \Psi^2(x + tD)dt$ is a pseudo-differential operator (ΦDO for short) of order -1. Hence, the right hand side of (1.7) is bounded by $C\|u\|^2$ and (1.4) follows. Notice that the identity $e^{itH_0}\Psi^2(x)e^{-itH_0} = \Psi^2(x + tD)$ is nothing but the so called Egorov formula which "quantizes" the map $y \mapsto y + tk$ and the relation $\int_0^T \Psi^2(x + t\xi)dt \sim |\xi|^{-1}$ is a result of the obvious fact that the free particles $y + tk$ with velocity k stay in a compact set for the time $\leq C|k|^{-1}$. Thus, we may consider that the local smoothing inequality (1.4) is nothing but the "quantization" of this obvious fact.

We now turn to the proof of (1.5). For $1 \le p \le \infty$, p' denotes its dual exponent: $1/p + 1/p' = 1$. Because $U_0(t) = e^{-itH_0}$ is unitary and because the integral kernel of $U_0(t)$ is bounded in modulus by a constant times $|t|^{-n/2}$, we have

$$\|U_0(t)u\|_2 = \|u\|_2, \quad \text{and} \quad \|U_0(t)u\|_\infty \le C|t|^{-n/2}\|u\|_1. \tag{1.8}$$

(1.5) then follows by applying the the following result of Keel and Tao [KT]: Let (X, dx) be a measure space and $\{U(t) : t \in \mathbb{R}\}$ a one parameter family of operators acting on complex-value functions on X. Suppose that $\{U(t)\}$ satisfies

$$\|U(t)f\|_2 \le C\|f\|_2, \quad \|U(t)U(s)^* f\|_\infty \le C|t-s|^{-\sigma}\|f\|_1. \tag{1.9}$$

Then, for $2 \le p, \theta \le \infty$ such that $2/\theta = \sigma(1/2 - 1/p)$ and $(p, \theta, \sigma) \ne (\infty, 2, 1)$, there exists a constant $C > 0$ such that $\left(\int_{\mathbb{R}} \|U(t)f\|_p^\theta dt\right) \le C\|f\|_2$ for any $f \in L^2(X)$. Thus, (1.5) is a result of the unitarity and the disspative property (1.8) of e^{-itH_0}.

If $V(x)$ grows at most quadratically at infinity in the sense

$$|\partial_x^\alpha V(x)| \le C_\alpha, \quad 2 \le |\alpha| \le 2(n+2), \tag{1.10}$$

it is shown (cf. [F]) that the fundamental solution (FDS for short) $E(t, x, y)$ for (1.1), viz. the integral kernel of e^{-itH}, can be written for short $0 < |t| < \delta$ in the form

$$E(t, x, y) = \frac{1}{(2\pi i t)^{n/2}} e^{iS(t,x,y)} a(t, x, y), \tag{1.11}$$

where $S(t, x, y)$ is real smooth and $a(t, x, y)$ is smooth and bounded. It follows that $U(t) = \exp(-itH)$ satisfies (1.8) for $|t| < \delta$ and, hence, (1.3) with finite $T > 0$ (note that the time global estimates do not hold in general because eigenfunctions exist for H). Moreover, $e^{itH}\Psi(x)^2 e^{-itH}$ is a ΦDO with principal symbol $\Psi(q(t, x, k))^2$ and, if k is large and $y \in \text{supp } \Psi$, $q(t, y, k) \in \text{supp } \Psi$ for the time $|t| \le C|k|^{-1}$ (see [Y2]). Thus, the local smoothing property (1.2) holds with $m = 2$ as in the case $V = 0$.

When V is superquadratic at infinity, $q(t, y, k)$ as well as $E(t, x, y)$ behave very differently from the case that V grows at most quadrartically at infinity. To see this, we consider $V(x) = \langle x \rangle^m$ in one dimension, $m > 0$. Then, classical particles are subject to periodic motion and, when energy $\sim k^2$ is very large, the periods are given by

$$T(k) \sim 2 \int_{-(k^2/2)^{1/m}}^{(k^2/2)^{1/m}} \frac{dx}{\sqrt{(k^2/2) - |x|^m}} = C_m k^{-1+2/m}, \tag{1.12}$$

Note that, as $k \to \infty$, $T(k) \to \infty$ if $0 < m < 2$ and $T(k) \to 0$ if $m > 2$. Thus, if $m > 2$, for given $t > 0, x$ and y, the equation $x = q(t, y, k)$ for k has infinite number of solutions with arbitrary large $|k|$ and, reflecting this, $E(t,x,y)$ is nowhere C^1 and is not in general bounded at infinity (see [Y4], [MY]). Thus, we cannot expect that (1.4) and (1.5) for the case $m \le 2$ remain to hold for $m > 2$. Actually, the motivation for this work was to understand how this change of properties of $E(t,x,y)$ reflects on the local smoothing property and Strichartz inequality. We expect, nonethless, $1/m$ times differentiability improving (1.2) because of the very relation (1.12) and the "physical" argument given for the free Schrödinger equation: If K is a compact set and the velocity of the particle in K is $\sim k$, it stays in K for $\le C/k$ during one period and its period is $\sim Ck^{-1+2/m}$ for the energy is $\sim k^2$. Hence, it stays in K for $\le CTk^{-2/m}$ during the time $[0,T]$ and we expect differentiabity improving by $1/m$.

The rest of the paper is devoted to the proof of Theorem 1.2 and Theorem 1.3. We display the plan of the paper here outlining the proofs. We observe that we can find the fraction $k^{-2/m}$ mentioned above by looking at the motion of the particle only for one period which is $\sim k^{-1+2/m} \sim \lambda^{-(\frac{1}{2}-\frac{1}{m})}$ if the energy is $\lambda \sim k^2$. Hinted by this, we decompose the solution $u(t) = \sum_{j=0}^{\infty} e^{-itH} u_{0j}$ in such a way that u_{0j} is spectrally localized around $\lambda_j = 2^j$ with respect to H. It actually is easy to see that for proving (1.2) and (1.3), it is sufficient to show respectively

$$\int_0^{\varepsilon h_j} \|\Psi(x)e^{-itH}u_{0j}\|^2 dt \le C\lambda_j^{-1/2}\|u_{0j}\|^2, \tag{1.13}$$

$$\left(\int_0^{\varepsilon h_j} \|e^{-itH}u_{0j}\|_p^\theta dt\right)^{\frac{1}{\theta}} \le C\|u_{0j}\| \tag{1.14}$$

for some $\varepsilon > 0$ and $C > 0$ independent of j, where $h_j \equiv \lambda_j^{-(\frac{1}{2}-\frac{1}{m})}$ is virtually the period of the particle with energy λ_j.

In section 2 we prove some preparatory results such as approximation of $\phi(H)$ by a psuedo-differential operator (ΦDO for short). In section 3, we show that $e^{-itH}\phi_j(H)$, where $\phi_j(H)$ is the spectral localization around $H \sim \lambda_j$ is well approximated, at least for $|t| \le \varepsilon h_j$, by $e^{-itH_j}\phi_j(H)$ generated by $H_j = -(1/2)\Delta + \chi(x/C_1\lambda_j^{1/m})V(x)$. Here $\chi \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function such that $\chi(x) = 1$ when $|x| \le 1$ and $\chi(x) = 0$ if $|x| \ge 2$, and C_1 is large enough so that $|x| \ge C_1\lambda^{1/m}$ implies $V(x) > 5\lambda$ whenever $\lambda > 10^{10}$. The reason behind this is that classical particles of energy λ cannot enter the domain where $V(x) > \lambda$. For proving this and also for obtaining the expression of $e^{itH_j}\Psi^2(x)e^{-itH_j}$ as a ΦDO in section 5, we change the scale of time and convert the equations into the semi-classical form: If $s = t/h$ and $\tilde{H}_j = h_j^2 H_j$, then $e^{-itH_j} = e^{-is\tilde{H}_j/h}$. The point here is that $\tilde{V}_j(x) = h_j^2\chi(x/C_1\lambda_j^{1/m})V(x)$ satisfies the estimate $|\partial_x^\alpha \tilde{V}_j(x)| \le C_\alpha$ for $|\alpha| \ge 2$ with C_α independent of j. It then follows that e^{-itH_j} has the integral kernel $E_j(t,x,y)$ of the form (1.11) for $|t| < \varepsilon h_j$, ε independent of j, and, its phase and amplitude functions are estimated uniformly with respect to j. In particular, $|E_j(t,x,y)| \leq C|t|^{-n/2}$ with j-independent C and this implies (1.14). We give a more precise argument in section 4. In section 5, we use h-ΦDO calculus and express $e^{it\tilde{H}_j/h_j}\Psi^2(x)e^{-it\tilde{H}_j/h_j}$ as a h-ΦDO and prove (1.13) by following the argument for the free Schrödinger equation given above.

Incidentally the fact that the study of $e^{-itH}\phi_j(H)$ for one period of the bicharacteristics $|t| < \varepsilon h_j$ is suffcicient for concluding the sharp local smoothing property is reminiscent of the similar fact for the sharp remainder estimate for the distribution of eigenvalues (see e.g. [Ta]) or the local decay property of the spectral projection operator at high energy ([Y5]) for H. See also [Bu] where similar argument is used for proving Strichatz inequalities for Schrödinger equations on compact manifolds.

2 Preliminaries

We write $S(m,g)$ for Hörmander's symbol class with slowly varying metrics g and g-continuous weight functions $m(x,\xi)$ (cf. [Ho], Chapter 18) and define the ΦDO $p(x,D) = Op(p)$ with symbol $p \in S(m,g)$ (we write $\sigma(P) = p(x,\xi)$ for the symbol of $P = p(x,D)$) by

$$p(x,D)u(x) = Op(p)u(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n\times\mathbf{R}^n} e^{i(x-y)\xi}p(x,\xi)u(y)dyd\xi.$$

We use $S(m,g_0)$ and $S(m,g_1)$ where $g_0 = dx\otimes dx + d\xi\otimes d\xi$ and $g_1 = dx\otimes dx/\langle x\rangle^2 + d\xi\otimes d\xi/\langle\xi\rangle^2$. We recall a positive function m is g_1-continuous if it satisfies $|\partial_x^\alpha\partial_\xi^\beta m(x,\xi)| \leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}m(x,\xi)$ and $p \in S(m,g_1)$ if and only if

$$|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)| \leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}m(x,\xi).$$

We denote by $p\sharp q$ the symbols of $Op(p)Op(q)$. If $p \in S(m_1,g_1)$, $q \in S(m_2,g_1)$, we have

$$p\sharp q - \sum_{|\alpha|<N}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha p(x,\xi)\cdot\partial_x^\alpha q(x,\xi) \in S(\langle x\rangle^{-N}\langle\xi\rangle^{-N}m_1m_2,g_1), \quad N = 1,2,\ldots, \tag{2.1}$$

$$\sigma(p(x,D)^*) - \sum_{|\alpha|<N}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha\partial_x^\alpha p(x,\xi) \in S(\langle x\rangle^{-N}\langle\xi\rangle^{-N}m_1,g_1), \quad N = 1,2,\ldots. \tag{2.2}$$

Similar relations hold for $S(m,g_0)$. The symbol class $S(m,g)$ is Fréchet space with natural seminorms and $p \mapsto p(x,D)$ is continuous from $S(1,g_0)$ or $S(1,g_1)$ to the Banach space of bounded operators in $L^2(\mathbf{R}^n)$.

We begin with the following lemma. We write $a(x,\xi) = (1/2)\xi^2 + V(x)$. We may and do assume in what follows that $V(x) > 1$ without losing the generality.

Lemma 2.1. *Let $\delta > \gamma > 0$ and $\phi, \psi \in C_0^\infty([0,\infty))$ be such that*

$$\operatorname{supp} \psi(t) \subset [0,\gamma), \qquad \phi(t) = 1 \text{ for } t \in [0,\delta].$$

Define $\Phi_\lambda(x,\xi) = \phi(a(x,\xi)/\lambda)$ for $\lambda > 1$. Then for any N, there exists C_N such that

$$\|H^N(1-\Phi_\lambda(x,D))\psi(H/\lambda)H^N\|_{B(L^2)} \le C_N\lambda^{-N}, \tag{2.3}$$

where the constant C_N is independent of $\lambda \ge 1$.

Proof. Write $\tilde{\Phi}_\lambda(x,\xi) = 1 - \Phi_\lambda(x,\xi)$. Take an almost analytic extension $\psi(z)$ of $\psi(t)$ such that $\psi(z)$ is supported by a compact subset of $|z| < \gamma$ and set $\psi_\lambda(z) = \psi(z/\lambda)$. We have

$$\tilde{\Phi}_\lambda(x,D)\psi(H/\lambda) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial \psi_\lambda}{\partial \bar{z}}(z)\tilde{\Phi}_\lambda(x,D)(H-z)^{-1}dz \wedge d\bar{z}. \tag{2.4}$$

We construct a parametrix of $\tilde{\Phi}_\lambda(x,D)(H-z)^{-1}$ for $|z| < \gamma\lambda$. On the support of $\Phi_\lambda(x,\xi)$ we have

$$\lambda^{-1}|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)| \le C_{\alpha\beta}\min(\lambda^{-\min(|\alpha|/2+|\beta|/m,1)}, \langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}) \tag{2.5}$$

with constants $C_{\alpha\beta}$ independent of $\lambda \ge 1$, and $\{\Phi_\lambda(x,\xi), \tilde{\Phi}_\lambda(x,\xi) : \lambda \ge 1\}$ is bounded in $S(1,g)$. We write $b(x,\xi,z) = a(x,\xi) - z$ and define q_0, $q_1, \dots$ inductively by

$$q_0 = \tilde{\Phi}_\lambda/b, \quad q_1 = i\partial_\xi q_0 \cdot \partial_x V/b, \quad q_j = \Big(\sum_{|\alpha|+k=j, |\alpha|\ge 1} \frac{-i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha q_k \cdot \partial_x^\alpha V\Big)/b, \quad j \ge 2. \tag{2.6}$$

It is obvious that q_j are of the forms

$$\sum_{k=1}^{N_j} \frac{a_{jk}(x,\xi)}{(a(x,\xi)-z)^k}$$

and $a_{jk}(x,\xi) = 0$ when $a(x,\xi) \le \delta\lambda$. When $a(x,\xi) > \delta\lambda$ and $|z| < \gamma\lambda$, we have $|b(x,\xi,z)| \ge (\delta-\gamma)\lambda$ and $|\partial_x^\beta\partial_\xi^\alpha b^{-1}| \le C_{\alpha\beta}(a+\lambda)^{-1}\langle x\rangle^{-|\beta|}\langle\xi\rangle^{-|\alpha|}$ with constants $C_{\alpha\beta}$ independent of $|z| \le \gamma\lambda$ and $\lambda \ge 1$. Thus, for $j = 0, 1, \dots$,

$$\{(a+\lambda)q_j : |z| \le \gamma\lambda,\ \lambda \ge 1\} \subset S(\langle x\rangle^{-j}\langle\xi\rangle^{-j}, g) \ \text{ is bounded.} \tag{2.7}$$

Denote $Q_j = Op(q_j)$, $j = 0, 1, \dots$. We have

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial \tilde{\psi}_\lambda}{\partial \bar{z}} Q_j dz \wedge d\bar{z} = 0, \quad j = 0, 1, \dots, \tag{2.8}$$

because integration by parts shows

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial \psi_\lambda}{\partial \bar{z}} \frac{a_{jk}(x,\xi)}{(a(x,\xi)-z)^j} dz \wedge d\bar{z} = \frac{1}{2\pi i (j-1)!}\int_{\mathbb{C}} \frac{\partial^j \psi_\lambda}{\partial \bar{z}\partial z^{j-1}} \frac{a_{jk}(x,\xi)}{a(x,\xi)-z} dz \wedge d\bar{z} \tag{2.9}$$

and, as $\psi^{(j-1)}(z)$ is a almost analytic extension of $\psi^{(j-1)}(x)$, (2.9) is equal to

$$\frac{\lambda^{-(j-1)}}{(j-1)!}\psi^{(j-1)}\left(\frac{a(x,\xi)}{\lambda}\right)a_{jk}(x,\xi)=0.$$

By virtue of the product formula (2.1), we have

$$\begin{array}{lllllll}(q_0+q_1+\cdots)\sharp b & & & & & & \\ = q_0 b - i\partial_\xi q_0\cdot\partial_x V & + & \sum_{|\alpha|=2}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha q_0\cdot\partial_x^\alpha V & + & \sum_{|\alpha|=3}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha q_0\cdot\partial_x^\alpha V & + & \cdots \\ \quad + \; q_1 b & - & i\partial_\xi q_1\cdot\partial_x V & + & \sum_{|\alpha|=2}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha q_1\cdot\partial_x^\alpha\partial_x V & + & \cdots \\ & + & q_2 b & - & i\partial_\xi q_2\cdot V & + & \cdots.\end{array}$$

Hence (2.6) and (2.7) imply that, if we set $R_{\lambda,N}(z,x,D) = \tilde{\Phi}_\lambda(x,D) - (Q_0+Q_1+\cdots+Q_N)(H-z)$, $N=0,1,\ldots$, then $\{R_{\lambda,N}(z,x,\xi) : |z|\le\gamma\lambda, \lambda\ge 1\}$ is bounded in $S(\langle x\rangle^{-(N+1)}\langle\xi\rangle^{-(N+1)}, g)$ and

$$\tilde{\Phi}_\lambda(x,D)(H-z)^{-1} = (Q_0+Q_1+\cdots+Q_N) - R_{\lambda N}(z,x,D)(H-z)^{-1} \tag{2.10}$$

It follows by the continuity property of ΦDOs that

$$\|H^{2N+1}R_{\lambda,(4N+1)m}(z,x,D)H^{2N+1}\|\le C_N,\quad |z|\le\gamma\lambda,\ \lambda\ge 1$$

and by inserting (2.10) into (2.4) and by using (2.8) that

$$\tilde{\Phi}_\lambda(x,D)\psi(H/\lambda) = \frac{-1}{2\pi i}\int_C\frac{\partial\tilde{\psi}_\lambda}{\partial\bar{z}}(z)R_{\lambda,(4N+1)m}(z,x,D)(H-z)^{-1}dz\wedge d\bar{z} \tag{2.11}$$

for any $N=1,2,\ldots$. It then follows that

$$\|H^{2N+1}\tilde{\Phi}_\lambda(x,D)\psi(H/\lambda)H^{2N+1}\|\le C_N\lambda^{-1}\int_{\Omega_\lambda}|\Im z|\|(H-z)^{-1}\||dz\wedge d\bar{z}|\le C_N'\lambda,$$

which implies the lemma because

$$\|H^N\tilde{\Phi}_\lambda(x,D)\psi(H/\lambda)H^N\|\le C_N\lambda^{-N-1}\|H^{2N+1}\tilde{\Phi}_\lambda(x,D)\psi(H/\lambda)H^{2N+1}\|.$$

by virtue of the support property of ψ. ∎

Lemma 2.2. *Let $\phi\in C_0^\infty([0,\infty))$ and $\Psi\in C_0^\infty(\mathbb{R}^n)$. Define, for $\lambda\ge 1$, $\Phi_\lambda(x,\xi) = \phi(a(x,\xi)/\lambda)$ and $K_\lambda(x,\xi) = \Psi(x)^2\phi(a(x,\xi)/\lambda)^2$. Then, there exists a constant $C>0$ such that for any $\lambda\ge 1$*

$$\|\Phi_\lambda(x,D)-\Phi_\lambda(x,D)^*\|_{B(L^2)}\le C\lambda^{-(\frac{1}{2}+\frac{1}{m})}, \tag{2.12}$$

$$\|\Phi_\lambda(x,D)\Psi^2(x)\Phi_\lambda(x,D)^* - K_\lambda(x,D)\|_{B(L^2)}\le C\lambda^{-\frac{1}{2}}. \tag{2.13}$$

Proof. It follows from (2.2) and (2.5) that $\{\sigma(\Phi_\lambda^*)-\Phi_\lambda : \lambda\}$ is bounded in $S(\lambda^{-(1/2+1/m)}, g)$. This implies (2.12). The proof for (2.13) is similar. ∎

We take $\psi_0, \psi \in C_0^\infty(\mathbb{R})$ such that $0 \le \psi_0(x),\ \psi(x) \le 1$, $\quad \operatorname{supp}\psi \subset (2^{-1}, 2)$ and

$$\psi_0(x) + \sum_{j=1}^{\infty} \psi(x/2^j) = 1 \quad \text{for } x \in [0,\infty) \tag{2.14}$$

and set $\psi_j(x) = \psi(x/2^j)$, $j = 1, 2, \ldots$. We let $\phi \in C_0^\infty((1/4, 4))$ be such that $\phi(x) = 1$ for $1/2 < x < 2$ and define, slightly abusing notation, $\Phi_j(x,\xi) = \phi(a(x,\xi)/2^j)$ for $j = 0, 1, \ldots$. Note that $1/2 \le \sum_{j=1}^{\infty} \psi_j(x)^2 \le 1$.

Lemma 2.3. *Let $\Psi \in S(1,g)$. For any $N > 0$ there exists a constant $C_N > 0$ such that*

$$\|\Psi(x,D)u\|^2 \le 72(\|\Psi(x,D)\phi_0(H)u\|^2 + \sum_{j=1}^{\infty} \|\Psi(x,D)\Phi_j(x,D)\psi_j(H)u\|^2) + C_N\|\langle H\rangle^{-N}u\|). \tag{2.15}$$

Proof. Take another $\tilde\psi \in C_0^\infty((1/2,2))$ such that $\psi(x)\tilde\psi(x) = \psi(x)$ and set $\tilde\psi_j(t) = \tilde\psi(t/2^j)$. By virtue of Lemma 2.1, we have for any N,

$$\|H^N(1-\Phi_j(x,D))\tilde\psi_j(H)H^N\|_{B(L^2)} \le C_N 2^{-jN}. \tag{2.16}$$

Write $u_j = \phi_j(H)u$. We have $u = \sum u_j = \sum \tilde\psi_j(H)u_j$ and by virtue of (2.16)

$$\begin{aligned} \|\Psi(x,D)u\|^2 &= \|\sum_{j=0}^{\infty} \Psi(x,D)\tilde\psi_j(H)u_j\|^2 \\ &\le 2\|\sum_{j=0}^{\infty} \Psi(x,D)\Phi_j(x,D)u_j\|^2 + C_N\sum_{j=0}^{\infty} 2^{-jN}\|u_j\|^2 \\ &\le 2\sum_{j,k=0}^{\infty} (\Phi_k(x,D)^*\Psi(x,D)^*\Psi(x,D)\Phi_j(x,D)u_j, u_k) + C_N\|H^{-N}\|^2. \end{aligned} \tag{2.17}$$

Since $\{\Phi_j : j = 1, 2, \ldots\}$ is bounded in $S(1,g)$ and the supports of Φ_j and Φ_k are disjoint from each other if $|j-k| \ge 5$. Hence, we see that $\{\Phi_k(x,D)^*\sharp\Psi(x,D)^*\sharp\Psi(x,D)\sharp\Phi_j(x,D) : |j-k| \ge 5\}$ is bounded in $S(\langle x\rangle^{-N}\langle\xi\rangle^{-N}, g)$ for every $N = 1, 2, \ldots$. It follows that, for any N,

$$\|\langle H\rangle^N \Phi_k(x,D)^*\Psi(x,D)\Psi(x,D)^*\Phi_j(x,D)\langle H\rangle^N\|_{B(L^2)} \le C_N$$

with constant independent of $|j-k| \ge 5$. Thus

$$\begin{aligned} &|\sum_{|j-k|\ge 5} (\Phi_k^*\Psi(x,D)^*\Psi(x,D)\Phi_j(x,D)u_j, u_k)| \\ &\quad \le C_N \sum_{j,k=0}^{\infty} 2^{-N(j+k)}\|u_j\|\|u_k\| \le C_N\|\langle H\rangle^{-N}u\|^2. \end{aligned} \tag{2.18}$$

On the other hand Schwarz inequality implies

$$\begin{aligned}
&|\sum_{|j-k|\le 4}(\Psi(x,D)\Phi_j(x,D)u_j,\Psi(x,D)\Phi_k(x,D)u_k)|\\
&\quad\le 2\sum_{|j-k|\le 4}(\|\Psi(x,D)\Phi_j(x,D)u_j\|^2+\|\Psi(x,D)\Phi_k(x,D)u_k\|^2)\\
&\quad\le 36\sum_{j=0}^{\infty}\|\Psi(x,D)\Phi_j(x,D)u\|^2
\end{aligned}\tag{2.19}$$

The lemma follows by combinig (2.17), (2.18) and (2.19). ∎

3 Approximation of propagator

We let $\chi\in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $\chi(x)=1$ for $|x|\le 1$ and $\chi(x)=0$ for $|x|\ge 2$. We define

$$H_\lambda=-\frac{1}{2}\Delta+V_\lambda(x),\qquad V_\lambda(x)=V(x)\chi(x/C_1\lambda^{\frac{1}{m}}),$$

Lemma 3.1. *Let $\psi\in C_0^\infty((0,\infty))$ be as in* Lemma 2.1. *Then, there exit constants $C_1>0$ and $\varepsilon>0$ such that for any $N,\ell=0,1,\dots$*

$$\sup_{|t|\le\varepsilon h}\|H^\ell(e^{-itH}-e^{-itH_\lambda})\psi(H/\lambda)\|\le C_{N\ell}\lambda^{-N}\tag{3.1}$$

for a positive constant $C_{N\ell}$ independent of $\lambda\ge 1$.

For proving Lemma 3.1, we set $h=\lambda^{-(\frac{1}{2}-\frac{1}{m})}$ and convert the equation (1.1) into the semi-classical form considering h as a semi-classical parameter. Thus, we define,

$$H^h=h^2H=\frac{-h^2}{2}\Delta+h^2V(x),\quad \tilde H^h=h^2H_\lambda=\frac{-h^2}{2}\Delta+h^2V_\lambda(x)\tag{3.2}$$

and write $V^h(x)=h^2V_\lambda(x)$. Then, (3.1) is equivalent to

$$\sup_{|t|\le\varepsilon}\|H^\ell(e^{-itH^h/h}-e^{-it\tilde H^h/h})\psi(H/\lambda)\|\le C_{N\ell}\lambda^{-N}.\tag{3.3}$$

It is important to notice here that

$$|\partial_x^\alpha V^h(x)|\le C_\alpha,\quad |\alpha|\ge 2,\tag{3.4}$$

where C_α is independent of $\lambda>1$. The following theorem is due to Fujiwara ([F]). We write $(q^h(t,y,k),p^h(t,y,k))$ for the solutions of Newton's equations

$$\begin{gathered}\dot q(t)=p(t),\quad \dot p(t)=-\nabla_qV^h(q),\\ q(0)=y,\quad p(0)=k,\end{gathered}\tag{3.5}$$

corresponding to the Hamiltonian $\tilde H^h$.

Theorem 3.2. *There exists $\varepsilon > 0$ independent of $h > 1$ such that the following statements are satisfied.*

(1) *For every $x, y \in \mathbb{R}^n$ and $0 < |t| < \varepsilon$, there exists a unique $k = k^h(t,x,y)$ such that $x = q^h(t,y,k)$; $s \mapsto q^h(s) = q^h(s,y,k^h(t,x,y))$ is a unique solution of (3.5) such that $q^h(t) = x$ and $q^h(0) = y$.*

(2) *Define $S^h(t,x,y)$ for $0 < |t| < \varepsilon$ and $x, y \in \mathbb{R}^n$ by*

$$S^h(t,x,y) = \int_0^t \{(1/2)\dot{q}^h(s)^2 - V^h(q^h(s))\}ds. \tag{3.6}$$

Then $S^h(t,x,y)$ is real C^∞ and satisfies

$$\left|\partial_x^\alpha \partial_y^\beta \left(S^h(t,x,y) - \frac{(x-y)^2}{2t}\right)\right| \le C_{\alpha\beta}|t|, \quad |\alpha+\beta| \ge 2. \tag{3.7}$$

(3) *For $0 < |t| < \varepsilon$, the integral kernel $E^h(t,x,y)$ of $e^{-it\tilde{H}^h/h}$ can be written in the form*

$$E^h(t,x,y) = \frac{1}{(2\pi i t h)^{n/2}} e^{iS^h(t,x,y)/h} a^h(t,x,y) \tag{3.8}$$

and $a^h(t,x,y)$ satisfies

$$|\partial_x^\alpha \partial_y^\beta (a^h(t,x,y) - 1)| \le C_{\alpha\beta}|th|, \quad |\alpha+\beta| \ge 0. \tag{3.9}$$

(4) *For $\ell = 0, 1, \ldots$, there exists a constant C_ℓ such that*

$$\sum_{|\alpha|+|\beta|\le \ell} \|x^\alpha \partial_x^\beta e^{-it\tilde{H}^h/h} u\| \le C_\ell \sum_{|\alpha|+|\beta|\le \ell} \|x^\alpha \partial_x^\beta u\|. \tag{3.10}$$

(5) *The constants $C_{\alpha\beta}$ and C_ℓ of (3.7), (3.9) and (3.10) do not depend on $h > 1$.*

Recall that $S^h(t,x,y)$ is a generating function of the flow determined by (3.5):

$$\frac{\partial S^h}{\partial x}(t, q^h(t,y,k), y) = p(t,y,k), \quad \frac{\partial S^h}{\partial y}(t, q^h(t,y,k), y) = -k. \tag{3.11}$$

We need the following lemma.

Lemma 3.3. *Let $\nu = th$ and $\tilde{S}^h(t,x,y) = tS^h(t,x,y)$, where S^h is defined by (3.6). Then, there exist $C_1 > 0$ and $\varepsilon > 0$ such that the following estimates are satisfied for (t,x,z,y,ξ) such that*

$$\Phi_\lambda(z, \xi/\nu) \ne 0, \quad |x| \ge C_1 \lambda^{\frac{1}{m}}, \quad y \in \mathbb{R}^n, \quad |t| \le \varepsilon : \tag{3.12}$$

(1) $\quad \left|\dfrac{\partial \tilde{S}^h}{\partial z}(t,x,z) + \xi\right| \ge \dfrac{1}{10}(|x| + C_1 \lambda^{\frac{1}{m}}).$

(2) $$\left|\frac{\partial \tilde{S}^h}{\partial x}(t,x,z)\right| \le 2\left|\frac{\partial \tilde{S}^h}{\partial z}(t,x,z)+\xi\right|.$$

(3) $$\left|\frac{\partial \tilde{S}^h}{\partial z}(t,x,z)+\xi\right| + |z-y| \ge 100^{-1}(|x|+|y|+|z|+C_1\lambda^{\frac{1}{m}}).$$

Proof. Write $k=\xi/t$ for $t\neq 0$. When $\Phi_\lambda(z,\xi/\nu)\neq 0$, we have $|\xi|\le 6|\nu|\sqrt{\lambda}=6|t|\lambda^{\frac{1}{m}}$, $|k|=|\xi/t|\le 6\lambda^{\frac{1}{m}}$ and $|z|\le C_0\lambda^{\frac{1}{m}}$ for some constant C_0. Since $|\partial_x V^h(x)|\le C\lambda^{\frac{1}{m}}$, where $C=D_2(4C_1)^{m-1}$ depends only on C_1, we have

$$|q^h(t,z,k)|=\left|z+tk-\int_0^t(t-s)\partial_x\tilde{V}_h(q^h(s,z,k))ds\right|\le C_0\lambda^{\frac{1}{m}}+6\varepsilon\lambda^{\frac{1}{m}}+3\varepsilon^2C\lambda^{\frac{1}{m}}.$$

We choose $C_1\ge(2D_2/D_1)^m$ such that $10^3C_0<C_1$ and then $0<\varepsilon<1$ such that $10^3(6+3C)\varepsilon<C_1$. We have

$$|q^h(t,z,k)|\le 100^{-1}C_1\lambda^{\frac{1}{m}}. \tag{3.13}$$

Let $\tilde{x}=q^h(t,z,k)$, $k=\xi/t$, so that $(\partial\tilde{S}^h/\partial z)(t,\tilde{x},z)=-\xi$ (see (3.11)). Then, taking $\varepsilon>0$ smaller if necessary, we have from (3.7) and (3.13) that

$$\begin{aligned}\left|\frac{\partial \tilde{S}^h}{\partial z}(t,x,z)+\xi\right| &= \left|\frac{\partial \tilde{S}^h}{\partial z}(t,x,z)-\frac{\partial \tilde{S}^h}{\partial z}(t,\tilde{x},z)\right|\\ &=\left|\int_0^1\frac{\partial^2\tilde{S}^h}{\partial x\partial z}(t,\theta x+(1-\theta)\tilde{x},z)d\theta\cdot(x-\tilde{x})\right|\ge\frac{1}{2}|x-\tilde{x}|\ge 8^{-1}(|x|+C_1\lambda^{\frac{1}{m}})\end{aligned}$$

if $|x|\ge C_1\lambda^{\frac{1}{m}}$ and (1) follows. By virtue of (3.11) and the conservation law of energy, we have

$$\frac{1}{2}\left(\frac{\partial S^h}{\partial x}\right)(t,x,z)^2+\tilde{V}_h(x)=\frac{1}{2}\left(\frac{\partial S^h}{\partial z}\right)(t,x,z)^2+\tilde{V}_h(z).$$

If $|x|\ge C_1\lambda^{\frac{1}{m}}$ and $|z|\le C_0\lambda^{\frac{1}{m}}$, we have $\tilde{V}_h(z)\le\tilde{V}_h(x)$. Hence,

$$\left|\frac{\partial \tilde{S}^h}{\partial x}(t,x,z)\right|\le\left|\frac{\partial \tilde{S}^h}{\partial z}(t,x,z)\right|\le\left|\frac{\partial \tilde{S}^h}{\partial z}(t,x,z)+\xi\right|+|\xi|$$

Since $|\xi|\le 6|t|\lambda^{\frac{1}{m}}\le 100^{-1}(|x|+C_1\lambda^{\frac{1}{m}})$ if $\varepsilon<10^{-3}$, statement (2) follows from (1). By the choice of C_1, we have $|z|\le C_0\lambda^{\frac{1}{m}}\le 10^{-3}C_1\lambda^{\frac{1}{m}}$ and $10^{-1}|x|-|z|\ge 10^{-2}(|x|+|z|)$. It follows from (1) that the left hand side of (3) is bounded from below by

$$\begin{aligned}10^{-1}(|x|+C_1\lambda^{\frac{1}{m}})+|z-y| &\ge 10^{-1}(|x|+C_1\lambda^{\frac{1}{m}})+|y|-|z|\\ &\ge 100^{-1}(|x|+|y|+|z|+C_1\lambda^{\frac{1}{m}}).\end{aligned}$$

Proof of Lemma 3.1. By virtue of Lemma 2.1 and (3.10), it suffices to show

$$\sup_{|t|\le\varepsilon} \|H^\ell(e^{-itH^h/h} - e^{-it\tilde{H}^h/h})\Phi_\lambda(x,D)\| \le C_{N\ell}\lambda^{-N} \tag{3.14}$$

Duhamel formula yields

$$H^\ell(e^{-itH^h/h} - e^{-it\tilde{H}^h/h})\Phi_\lambda(x,D)u = -ih\int_0^t H^\ell e^{-i(t-s)H}(V - V_\lambda)e^{-is\tilde{H}^h/h}\Phi_\lambda(x,D)uds$$

and the operator $H^\ell(V-V_\lambda)$ can be written in the form $\sum_{|\alpha|\le 2\ell} c_\alpha(x)\partial_x^\alpha$ where $c_\alpha(x)$ are supported by $\{x : |x| \ge C_1\lambda^{1/m}\}$ and are bounded by $C\langle x\rangle^{m(\ell+1)}$. Hence, it suffices for proving the lemma to show that, for any M and $|\alpha|\le\ell$,

$$\int_0^t \|\chi_{|x|\ge C_1\lambda^{1/m}}\langle x\rangle^M\partial_x^\alpha e^{-it\tilde{H}^h/h}\Phi_\lambda(x,D)u\|dt \le C_{M\ell}\lambda^{-N} \tag{3.15}$$

Introduce a new parameter $\nu = th$ and write $tS^h = \tilde{S}^h$. Then,

$$\begin{aligned} &e^{-it\tilde{H}^h/h}\Phi_\lambda(x,D)u(x) \\ &= \frac{1}{(2\pi i\nu)^{n/2}(2\pi\nu)^n}\int e^{i(\tilde{S}^h(t,x,z)+(z-y)\xi)/\nu}a^h(t,x,z)\Phi_\lambda(z,\xi/\nu)u(y)dyd\xi dz. \end{aligned} \tag{3.16}$$

We differentiate the right hand side of (3.16) by ∂_x^α and multiply by $\langle x\rangle^M$. This will produce several terms of the form

$$\frac{\langle x\rangle^M}{(2\pi i\nu)^{n/2}(2\pi\nu)^n}\int e^{iJ(t,x,z,y,\xi)/\nu}\prod_{j=1}^{\ell}\left(\frac{i}{\nu}\frac{\partial^{\alpha_j}\tilde{S}^h}{\partial x^{\alpha_j}}\right)\frac{\partial^\beta a^h}{\partial x^\beta}(t,x,z)\Phi_\lambda(z,\xi/\nu)u(y)dyd\xi dz, \tag{3.17}$$

where $\alpha_1+\cdots+\alpha_\ell+\beta=\alpha$ and $\alpha_j\ne 0$, and

$$J(t,x,z,y,\xi) = \tilde{S}^h(t,x,z) + (z-y)\xi.$$

When $|x|\ge C_1\lambda^{\frac{1}{m}}$, $\Phi(z,\xi/\nu)\ne 0$ and $|t|<\varepsilon$, we have by virtue of Lemma 3.3

$$\left|\frac{\partial J}{\partial z}\right| \ge \frac{1}{10}(|x| + C_1\lambda^{\frac{1}{m}}), \qquad \left|\frac{\partial J}{\partial z}\right| + \left|\frac{\partial J}{\partial \xi}\right| \ge 10^{-3}(|x|+|y|+|z|+C_1\lambda^{\frac{1}{m}}). \tag{3.18}$$

Define

$$L_0 = -i\left(\frac{\partial J}{\partial z}\right)^{-2}\frac{\partial J}{\partial z}\frac{\partial}{\partial z}, \quad L_1 = -i\left\{\left(\frac{\partial J}{\partial z}\right)^2 + \left(\frac{\partial J}{\partial \xi}\right)^2\right\}^{-1}\left\{\frac{\partial J}{\partial z}\frac{\partial}{\partial z} + \frac{\partial J}{\partial \xi}\frac{\partial}{\partial \xi}\right\}.$$

First order differential operators L_0 and L_1 satisfy

$$\nu L_0 e^{iJ/\nu} = \nu L_1 e^{iJ/\nu} = e^{iJ/\nu}.$$

We apply to (3.17) ℓ times integration by parts by using L_0 and then N times integration parts by using L_1. The factor $\nu^{-\ell}$ in the integrand of (3.17) is cancelled by ν^{ℓ} produced by L_0^{ℓ} and we obtain

$$\begin{aligned}(3.17) &= \frac{i^n\nu^N\langle x\rangle^M}{(2\pi i\nu)^{3n/2}}\int\{L_0^{\ell}L_1^N e^{iJ/\nu}\}b^h(t,x,z,\xi)u(y)dyd\xi dz \\ &= \frac{i^n\nu^N\langle x\rangle^M}{(2\pi i\nu)^{3n/2}}\int e^{iJ/\nu}(L_1^*)^N(L_0^*)^{\ell}\{b^h(t,x,z,\xi)\}u(y)dyd\xi dz \qquad (3.19)\\ &= \frac{1}{(2\pi\nu i)^{n/2}}\int e^{i\tilde{S}^h(t,x,z)/\nu}F(x,z,\nu)dz.\end{aligned}$$

Here L_0^* and L_1^* are the transporse of L_0 and L_1, respectively:

$$L_0 = i\frac{\partial}{\partial z}\cdot\left(\frac{\partial J}{\partial z}\right)^{-2}\frac{\partial J}{\partial z},\quad L_1 = i\left\{\frac{\partial}{\partial z}\cdot\frac{\partial J}{\partial z}+\frac{\partial}{\partial\xi}\cdot\frac{\partial J}{\partial\xi}\right\}\left\{\left(\frac{\partial J}{\partial z}\right)^2+\left(\frac{\partial J}{\partial\xi}\right)^2\right\}^{-1}$$

and $b^h(t,x,z,\xi)$ and $F(t,x,z,\nu)$ are defined by

$$b^h(t,x,z,\xi) = \prod_{j=1}^{\ell}\left(i\frac{\partial^{\alpha_j}\tilde{S}^h}{\partial x^{\alpha_j}}\right)\frac{\partial^{\beta}a^h}{\partial x^{\beta}}\Phi_\lambda(z,\xi/\nu),$$

$$F(t,x,z,\nu) = \frac{\langle x\rangle^M\nu^N}{(2\pi\nu)^n}\int e^{i(z-y)\xi/\nu}\{(L_1^*)^N(L_0^*)^{\ell}b^h(t,x,z,\xi)\}u(y)dyd\xi \qquad (3.20)$$

Recall that Φ_λ is bounded in $S(1,g)$, hence

$$\nu^{|\beta|}|(\partial_z^{\alpha}\partial_\xi^{\beta})\Phi_\lambda(z,\xi/\nu)| \le C_{\alpha\beta}\langle z\rangle^{-|\alpha|}\langle\xi/\nu\rangle^{-|\beta|/2}; \qquad (3.21)$$

(3.7) implies that the second or higher derivatives of J with respect to (x,z,y,ξ) are bounded uniformly with respect to $0<|t|<\varepsilon$. It then follows by the help of (1) and (2) of Lemma 3.3 that

$$|\partial_x^{\alpha}\partial_z^{\beta}(L_0^*)^{\ell}b^h(t,x,z,\xi)| \le C_{\alpha\beta}$$

and then, by virtue of (3.18),

$$c^h(t,x,z,\xi) = (L_1^*)^N\{(L_0^*)^{\ell}b^h(t,x,z,\xi)$$

satisfies

$$\nu^N|\partial_x^{\alpha}\partial_z^{\beta}c^h(t,x,z,\xi)| \le C_{\alpha\beta N}(|x|+|y|+|z|+C_1\lambda^{\frac{1}{m}})^{-N} \qquad (3.22)$$

with constants $C_{\alpha\beta}$ independent of (t,x,z,ξ) and $\lambda\ge 1$. Since $c^h(t,x,z,\xi)$ is supported by $|\xi|\le C\lambda^{1/2}\nu$, we obtain, by replacing N by $4N$, $N\ge n$, that,

$$\begin{aligned}|\partial_x^{\alpha}\partial_z^{\beta}F(t,x,z,\nu)| &\le \frac{C_{\alpha\beta N}\langle x\rangle^M}{(2\pi\nu)^n}\int_{|\xi|\le C\lambda^{1/2}\nu}\langle\xi/\nu\rangle^{|\beta|}(|x|+|y|+|z|+C_1\lambda^{\frac{1}{m}})^{-4N}|u(y)|dyd\xi \\ &\le C_N\langle x\rangle^{M-N}\langle z\rangle^{-N}\lambda^{-\frac{N}{m}}\frac{1}{(2\pi\nu)^n}\int_{|\xi|\le C\lambda^{1/2}\nu}\langle\xi/\nu\rangle^{|\beta|}d\xi\int_{\mathbf{R}^n}\langle y\rangle^{-N}|u(y)|dy \\ &\le C_N\langle x\rangle^{M-N}\langle z\rangle^{-N}\lambda^{-\left(\frac{N}{m}-\frac{n+|\beta|}{2}\right)}\|u\|_2.\end{aligned}$$

Thus, if we set $G(t,x,z,\nu) = F(t,x,z,\nu)\langle z\rangle^n$, we have for any $N > \max(M,n)$ that

$$|\partial_x^\alpha \partial_z^\beta G(t,x,z,\nu)| \le C_{\alpha\beta}\lambda^{(n-\frac{N}{m})}\|u\|_2, \quad |\alpha|,|\beta| \le n \tag{3.24}$$

Hence, applying the L^2 continuity property of oscillatory integral operators to

$$\frac{1}{(2\pi\nu)^{n/2}}\int e^{i\tilde{S}^h(t,x,z)/\nu}F(t,x,z,\nu)dz = \frac{1}{(2\pi\nu)^{n/2}}\int e^{i\tilde{S}^h(t,x,z)/\nu}G(t,x,z,\nu)f(z)dz,$$

$f(z) = \langle z\rangle^{-n}$, we see from (3.24) that

$$\|(3.17)\| \le C_N\lambda^{(n-\frac{N}{m})}\|u\|_2\|f\|_2 \le C_N'\lambda^{(n-\frac{N}{m})}\|u\|$$

This ends the proof of Lemma 3.1. ∎

4 Proof of Strichartz inequality

We prove Theorem 1.3 in this section. We use the notation of the previous sections. Thus $\{\psi_j\}$ is the partition of unity of (2.14), $u_{0j} = \psi_j(H)u_0$ so that $u_0 = \sum_{j=0}^\infty u_{0j}$ and $\Phi_j(x,\xi) = \phi(a(x,\xi)/2^j)$. When $\lambda_j = 2^j$, we set the semi-classical parameter h_j by

$$h_j = \lambda_j^{-(\frac{1}{2}-\frac{1}{m})} = 2^{-j(\frac{1}{2}-\frac{1}{m})}$$

and denote $H_j = H^{h_j}$ and $\tilde{H}_j = \tilde{H}^{h_j}$, where H^h and $\tilde{H}^h$ are the operators defined by (3.2).

Lemma 4.1. *Let $p \in [2,\infty)$, $\theta \in (2,\infty]$ be such that $0 \le \frac{2}{\theta} = n\left(\frac{1}{2}-\frac{1}{p}\right) < 1$. Then, there exists a constant $\varepsilon > 0$ and $C > 0$ independent of $j = 0,1,\dots$ such that*

$$\left(\int_{|t|\le\varepsilon h_j} \|e^{-itH}u_{0j}\|_p^\theta dt\right)^{1/\theta} \le C\|u_{0j}\|_2. \tag{4.1}$$

Proof. By the elliptic estimate and the Sobolev embedding theorem, we have $\|u\|_p \le C_p\|H^n u\|_2$ for any $1 \le p \le \infty$ and (4.1) holds for $j = 0$. We let $j \ge 1$. We have by Minskowski inequality

$$\begin{aligned}&\left(\int_{|t|\le\varepsilon h_j} \|e^{-itH}u_{0j}\|_p^\theta dt\right)^{1/\theta}\\ &\le \left(\int_{|t|\le\varepsilon h_j} \|e^{-itH_j}u_{0j}\|_p^\theta dt\right)^{1/\theta} + \left(\int_{|t|\le\varepsilon h_j} \|(e^{-itH}-e^{-itH_j})u_{0j}\|_p^\theta dt\right)^{1/\theta}\end{aligned} \tag{4.2}$$

By virtue of Lemma 3.1, We have

$$\begin{aligned}&\sup_{|t|\le\varepsilon h_j}\|(e^{-itH}-e^{-itH_j})u_{0j}\|_p\\&\le C\sup_{|t|\le\varepsilon h_j}\|H^n(e^{-itH}-e^{-itH_j})\tilde{\psi}_j(H)u_{0j}\|_2\le C_N2^{-jN}\|u_{0j}\|_2.\end{aligned}\tag{4.3}$$

Recall that $e^{-itH_j}=e^{-i(t/h_j)\tilde{H}_j/h_j}$ and $e^{-it\tilde{H}_j/h_j}$ has the integral kernel given by (3.8) with h_j in replace of h. Thus, e^{-itH_j} also has smooth integral kernel $\tilde{E}_j(t,x,y)$ which satisfies

$$|\tilde{E}_j(t,x,y)|\le C|t|^{-n/2},\quad |t|\le\varepsilon h_j$$

with j-independent constant C. Thus, e^{-itH_j} satisfies (1.8) with constant independent of j and the theorem of Keel-Tao mentioned in the introduction implies

$$\left(\int_{|t|\le\varepsilon h_j}\|e^{-itH_j}u_{0j}\|_p^\theta dt\right)^{1/\theta}\le C\|u_{0j}\|_2.\tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we obtain for (4.1). ∎

Proof of Theorem 1.3. Given $T>0$, find $L_j\equiv[T/\varepsilon h_j]+1\le C_\varepsilon 2^{j(\frac12-\frac1m)}$ number of points

$$0=t_0<t_1<\ldots<t_{L_j}=T$$

such that $|t_k-t_{k-1}|<\varepsilon h_j$. Then, Lemma 4.1 implies

$$\begin{aligned}\int_0^T\|e^{-itH}u_{0j}\|_p^\theta dt&=\sum_{k=1}^{L_j}\int_{t_{k-1}}^{t_k}\|e^{-itH}u_{0j}\|_p^\theta dt\\&=\sum_{k=1}^{L_j}\int_0^{t_k-t_{k-1}}\|e^{-itH}e^{i(t_k-t_{k-1})H}u_{0j}\|_p^\theta dt\\&\le\sum_{k=1}^{L_j}C\|u_{0j}\|_2^\theta\le C_\varepsilon 2^{j(\frac12-\frac1m)}\|u_{0j}\|_2^\theta\le C_\varepsilon\|\langle H\rangle^{\frac1\theta(\frac12-\frac1m)}u_{0j}\|^\theta.\end{aligned}$$

Minkowski's inequlity and Schwatz' inequality then imply

$$\left(\int_0^T\|e^{-itH}u_0\|_p^\theta dt\right)^{1/\theta}\le C\sum_{j=0}^\infty\|\langle H\rangle^{\frac1\theta(\frac12-\frac1m)}u_{0j}\|\le C\|\langle H\rangle^\gamma u_0\|$$

for any $\gamma>\frac1\theta\left(\frac12-\frac1m\right)$. This concludes the proof of Theorem 1.3.

5 Proof of local smoothing property

In this section we prove Theorem 1.2. We use the notation of the previous section. In particular, $\lambda_j = 2^j$, $h_j = 2^{-j(\frac{1}{2}-\frac{1}{m})}$ is the corresponding semi-classical parameter and $U_j(t) = e^{-i(t/h_j)\tilde{H}_j/h}$. We fix a function $\Psi \in C_0^\infty(\mathbb{R}^n)$.

Lemma 5.1. *Suppose that there exists a constant C independent of $j = 0, 1,$ ldots and $u_0 \in L^2(\mathbb{R}^n)$ such that*

$$\int_0^{\varepsilon h_j} \|\Psi(x)\Phi_j(x,D)^* e^{-itH_j} u_{0j}\|^2 dt \le C\lambda_j^{-1/2} \|u_{0j}\|^2. \tag{5.1}$$

Then Theorem 1.2 *follows.*

Proof. We have from (5.1) and Lemma 3.1

$$\begin{aligned}
&\int_0^{\varepsilon h_j} \|\Psi(x)\Phi_j(x,D)^* e^{-itH} u_{0j}\|^2 dt \\
&\le \int_0^{\varepsilon h_j} \|\Psi(x)\Phi_j(x,D)^* e^{-itH_j} u_{0j}\|^2 dt + \int_0^{\varepsilon h_j} \|\Psi(x)\Phi_j(x,D)^* (e^{-itH} - e^{-itH_j})\tilde{\Psi}_j(H) u_{0j}\|^2 dt \\
&\le C\lambda_j^{-1/2} \|u_{0j}\|^2 + C_N \lambda_j^{-N}.
\end{aligned}$$

As in the proof of Theorem 1.3, we take $L_j \le C_\varepsilon \lambda_j^{(\frac{1}{2}-\frac{1}{m})}$ number of points $0 = t_0 < t_1 < \ldots < t_{L_j} = T$ such that $|t_k - t_{k-1}| < \varepsilon h_j$. It then follows that

$$\begin{aligned}
\int_0^T \|\Psi(x)\Phi_j(x,D)^* e^{-itH} u_{0j}\|^2 dt &= \sum_{k=1}^{L_j} \int_{t_{k-1}}^{t_k} \|\Psi(x)\Phi_j(x,D)^* e^{-itH} u_{0j}\|^2 dt \\
&= \sum_{k=1}^{L_j} \int_0^{t_k - t_{k-1}} \|\Psi(x)\Phi_j(x,D)^* e^{-itH} e^{i(t_k - t_{k-1})H} u_{0j}\|^2 dt \\
&\le \sum_{k=1}^{L_j} C\lambda_j^{-1/2} \|u_{0j}\|_2 \le C_\varepsilon \lambda_j^{-1/m} \|u_{0j}\|^2.
\end{aligned} \tag{5.2}$$

Summing up (5.2) with respect to $j = 0, 1, \ldots$ anf applying (2.15), we conclude that

$$\begin{aligned}
&\int^T \|\Psi(x) e^{-itH} u_0\|^2 dt \le C \sum_{j=0}^{\infty} \int^T \|\Psi(x)\Phi_j(x,D)^* e^{-itH} u_{0j}\|^2 dt + C_{N,T} \|\langle H\rangle^{-N} u_0\|^2 \\
&\sum_{j=0}^{\infty} C_\varepsilon \lambda_j^{-1/m} \|u_{0j}\|^2 + C_{N,T} \|\langle H\rangle^{-N} u_0\|^2 \le C\|\langle H\rangle^{-1/2m} u_0\|^2,
\end{aligned}$$

which implies Theorem 1.2. ∎

We prove (5.1). Define $K_j(x,\xi) = \Psi(x)^2\Phi_j(x,\xi)^2$. We have by virtue of (2.13) that

$$\|K_j(x,D) - \Phi_j(x,D)\Psi(x)^2\Phi_j(x,D)^*\|_{B(L^2)} \le C\lambda_j^{-1/2}.$$

Introducing the semiclassical parameter h_j and the operator $\tilde{H}_j$ again, we rewrite (5.1)

$$\begin{aligned}
&\int_0^{\epsilon h_j} \|\Psi(x)\Phi_j(x,D)^* e^{-itH_j}u_{0j}\|^2 dt \\
&= h_j \int_0^{\epsilon} \|\Psi(x)\Phi_j(x,D)^* e^{-it\tilde{H}_j/h_j}u_{0j}\|^2 dt \\
&\le h_j \int_0^{\epsilon} (e^{it\tilde{H}_j/h_j}K_j(x,D)e^{-it\tilde{H}_j/h_j}u_{0j}, u_{0j})dt + Ch_j\lambda_j^{-1/2}.
\end{aligned} \tag{5.3}$$

We write $K_j(x,D)$ in the form of h-ΦDO by changing $\xi \to \xi/h_j$:

$$\begin{aligned}
K_j(x,D)u(x) &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}\Psi^2(x)\phi^2\left(\frac{\xi^2/2+V(x))}{\lambda_j}\right)u(y)dyd\xi \\
&= \frac{1}{(2\pi h_j)^n}\int e^{i(x-y)\xi/h_j}\Psi^2(x)\phi^2\left(\frac{\xi^2/2+V^{h_j}(x))}{\lambda_j^{\frac{2}{m}}}\right)u(y)dyd\xi \\
&= \tilde{K}_j(x,h_jD)u(x),
\end{aligned}$$

where $\tilde{K}_j(x,\xi) = \Psi^2(x)\phi^2((\xi^2/2+V^{h_j}(x))/\lambda_j^{\frac{2}{m}})$. Notice that we have replaced $h_j^2V(x)$ by $V^{h_j}(x)$ as they agree on the support of Ψ. It is obvious that $\{\tilde{K}_j(x,\xi) : j = 1,2,\ldots\}$ is a bounded set of $S(1,g_0)$, where $g_0 = dx^2 + d\xi^2$. We compute

$$K_j(t,x,h_jD) = e^{it\tilde{H}_j/h_j}K_j(x,D)e^{-it\tilde{H}_j/h_j}$$

following the standard procedure in h-ΦDO (see e.g. [Ro]). We have

$$\begin{aligned}
0 &= \frac{d}{dt}\{e^{-it\tilde{H}_j/h_j}K_j(t,x,h_jD)e^{it\tilde{H}_j/h_j}\} \\
&= e^{-it\tilde{H}_j/h_j}\left(\frac{\partial K_j}{\partial t}(t,x,h_jD) - \frac{i}{h_j}[\tilde{H}_j, K_j(t,x,h_jD)]\right)e^{it\tilde{H}_j/h_j}
\end{aligned}$$

We ansatz that $K_j(t,x,h_jD)$ is an h-ΦDO and that it has an expansion

$$K_j(t,x,h_jD) \sim \sum_{n=0}^{\infty} h_j^n K_{jn}(t,x,h_jD).$$

Denote $\tilde{H}_j(x,\xi) = \xi^2/2 + V^{h_j}(x)$. Then, the symbol of the h-ΦDO in the brackets on the right is given by

$$\frac{\partial K_j}{\partial t}(t,x,\xi) - \frac{\partial\tilde{H}_j}{\partial\xi}\frac{\partial K_j}{\partial x} + \frac{\partial\tilde{H}_j}{\partial x}\frac{\partial K_j}{\partial\xi} + \sum_{|\alpha|\ge 2} h_j^{|\alpha|-1}\frac{(-i)^{|\alpha|+1}}{\alpha!}\left(\frac{\partial^\alpha\tilde{H}_j}{\partial\xi^\alpha}\frac{\partial^\alpha K_j}{\partial x^\alpha} - \frac{\partial^\alpha\tilde{H}_j}{\partial x^\alpha}\frac{\partial^\alpha K_j}{\partial\xi^\alpha}\right)$$

We determine K_{jn} by inserting $K_j(t,x,\xi) = \sum_{n=0}^{\infty} h_j^n K_{jn}(t,x,\xi)$ into the right hand side, collecting the terms with the same order in h and set them $= 0$. The result is

$$\frac{\partial K_{j0}}{\partial t}(t,x,\xi) - \frac{\partial \tilde{H}_j}{\partial \xi}\frac{\partial K_{j0}}{\partial x} + \frac{\partial \tilde{H}_j}{\partial x}\frac{\partial K_{j0}}{\partial \xi} = 0 \tag{5.4}$$

and for $n = 1, 2, \dots$

$$\begin{aligned} &\frac{\partial K_{jn}}{\partial t}(t,x,\xi) - \frac{\partial \tilde{H}_j}{\partial \xi}\frac{\partial K_{jn}}{\partial x} + \frac{\partial \tilde{H}_j}{\partial x}\frac{\partial K_{jn}}{\partial \xi} \\ &\quad + \sum_{k+|\alpha|=n+1, |\alpha|\geq 2} \frac{(-i)^{|\alpha|+1}}{\alpha!}\left(\frac{\partial^\alpha \tilde{H}_j}{\partial \xi^\alpha}\frac{\partial^\alpha K_{jk}}{\partial x^\alpha} - \frac{\partial^\alpha \tilde{H}_j}{\partial x^\alpha}\frac{\partial^\alpha K_{jk}}{\partial \xi^\alpha}\right) = 0 \end{aligned} \tag{5.5}$$

Solve (5.4) and (5.5) inductively with the initial condition

$$K_{j0}(0,x,\xi) = \tilde{K}_j(x,\xi), \quad K_{jn}(0,x,\xi) = 0, \quad n = 1, 2, \dots.$$

We denote the solutions of the initial value problem (3.5) with $h = h_j$ by $(q^j(t,y,k), p^j(t,y,k))$. Since the map $(x,\xi) \to (q^j(t,x,\xi), p^j(t,x,\xi))$ is a global differomorphism and the derivatives of $(q^j(t,x,\xi), p^j(t,x,\xi))$ with respect to (x,ξ) are bounded uniformly with respect to $|t| < \varepsilon$ and $j = 1, 2, \dots$, we find that

$$K_{j0}(t,x,\xi) = \tilde{K}_j(q^j(t,x,\xi), p^j(t,x,\xi)) \tag{5.6}$$

solves the equation (5.4) and $\{K_{j0} : j = 0, 1, \dots\}$ is bounded in $S(1, g_0)$. Evidently $K_{j0}(t,x,\xi) = 0$ unless $(q^j(t,x,\xi), p^j(t,x,\xi)) \in \operatorname{supp} \tilde{K}_j$.

The equation (5.5) for $n = 1$ can be written in the form

$$\begin{aligned} &\frac{d}{dt}K_{j1}(t, q^j(-t,x,\xi), p^j(-t,x,\xi)) = R_{j1}(t, q^j(-t,x,\xi), p^j(-t,x,\xi)) \\ &\quad \equiv \sum_{|\alpha|=2} \frac{i}{\alpha!}\left(\frac{\partial^\alpha \tilde{H}_j}{\partial \xi^\alpha}\frac{\partial^\alpha K_{j0}}{\partial x^\alpha} - \frac{\partial^\alpha \tilde{H}_j}{\partial x^\alpha}\frac{\partial^\alpha K_0}{\partial \xi^\alpha}\right)(t, q^j(-t,x,\xi), p^j(-t,x,\xi)) \end{aligned}$$

and may be solved in the form

$$K_{j1}(t, q^j(-t,x,\xi), p^j(-t,x,\xi)) = \int_0^t R_{j1}(s, q^j(-s,x,\xi), p^j(-s,x,\xi))ds$$

or

$$K_{j1}(t,x,\xi) = \int_0^t R_{j1}(s, q^j(t-s,x,\xi), p^j(t-s,x,\xi))ds.$$

Again $\{K_{j1}(t,x,\xi) : j = 1, 2, \dots, \ |t| < \varepsilon\}$ is bounded in $S(1, g_0)$ and $K_{j1}(t,x,\xi) = 0$ unless $(q^j(t,x,\xi), p^j(t,x,\xi)) \in \operatorname{supp} \tilde{K}_j$. The latter can be seen from (5.6) and $K_{j0}(s, q^j(t-s,x,\xi), p^j(t-s,x,\xi)) = K_{j0}(t,x,\xi)$ which follows from the group property of the flow

$(y,k) \mapsto (q^j(t,y,k), p^j(t,y,k))$. We succesively solve the equation (5.5) for $n = 2,3,\dots$ in a similar fashion and find that solutions $K_{j0}, K_{j1}, \dots$ satisfy

$$\{K_{jn}(t,x,\xi) : j = 1,2,\dots,\ |t| < \varepsilon\} \text{ is bounded in } S(1,g_0),\quad n = 0,1,\dots, \tag{5.7}$$

$$K_{jn}(t,x,\xi) = 0 \text{ if } (q^j(t,x,\xi), p^j(t,x,\xi)) \notin \operatorname{supp} \tilde{K}_j. \tag{5.8}$$

We define

$$K_j^N(t,x,\xi) = \sum_{n=0}^{N} h_j^n K_{jn}(t,x,\xi).$$

Lemma 5.2. *Let $K_j^N(t,x,\xi)$ be defined as above. Then, there exists $\varepsilon > 0$ such that the following estimates are satisfied:*

(1) *For any $N = 1,2,\dots$, there exists a constant C_N such that for $j = 1,2,\dots$,*

$$\sup_{|t|\le\varepsilon} \|e^{it\tilde{H}_j/h_j} K_j(x,D) e^{-it\tilde{H}_j/h_j} - K_j^N(t,x,h_jD)\|_{B(L^2)} \le C_N h_j^{N+1}. \tag{5.9}$$

(2) *For any $N = 1,2,\dots$ and α,β, there exists a constant $C_{\alpha\beta N}$ such that for $j = 1,2,\dots$*

$$\left\| \int_0^\varepsilon K_j^N(t,x,h_jD)dt \right\| \le C_{\alpha\beta N} \lambda_j^{-\frac{1}{m}}. \tag{5.10}$$

Proof. By construction and the symbol calculus for h-ΦDO ([Ro]), it is standard to see that

$$\frac{\partial K_j^N}{\partial t}(t,x,h_jD) - \frac{i}{h_j}[\tilde{H}_j, K_j^N(t,x,h_jD)] \in OpS(h_j^{N+1}, g_0)$$

uniformly with respect to j and $|t| < \varepsilon$. Hence,

$$\|e^{-it\tilde{H}_j/h_j} K_j^N(t,x,h_jD) e^{it\tilde{H}_j/h_j} - K_j(x,h_jD)\| \le C_N h_j^{N+1}$$

with j independent constant C_N. The statement (1) follows. For proving (5.10), it suffices to show

$$\left| \int_0^\varepsilon \partial_\xi^\alpha \partial_x^\beta K_j^N(t,x,\xi)dt \right| \le C_{\alpha\beta N} \lambda_j^{-\frac{1}{m}}. \tag{5.11}$$

By virtue of (5.7) and (5.8), we know that $|\partial_\xi^\alpha \partial_x^\beta K_j^N(t,x,\xi)| \le C_N$ with C_N independent of j, $|t| < \varepsilon$ and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and that $K_j^N(t,x,\xi) = 0$ unless $\Psi(q^j(t,x,\xi)) \ne 0$. Thus, for proving (5.11), it clearly suffices to show by replacing $\varepsilon > 0$ by a smaller constant if necessary, that there exists a constant $C > 0$ independent of j such that

$$\tilde{K}_j(q^j(0,x,\xi), p^j(0,x,\xi)) \ne 0, \quad \text{then} \quad \tilde{K}_j(q^j(t,x,\xi), p^j(t,x,\xi)) = 0 \quad \text{for} \quad C\lambda_j^{-\frac{1}{m}} < |t| < \varepsilon.$$

This, however, is almost evident. First, we remark that $|\partial_x V^{h_j}(x)| \le C\langle x\rangle$ with j independent constant $C > 0$. It follows that $1 + |\dot{q}^j(t)| + |\dot{p}^j(t)| \le C(1 + |q^j(t)| + |p^j(t)|)$ and

$$\sup_{|t|\le\varepsilon}(1 + |q^j(t)| + |p^j(t)|) \le (1 + |q^j(0)| + |p^j(0)|)e^{C\varepsilon} \le C\lambda_j^{\frac{1}{m}}.$$

The last inequality holds because $\tilde{K}_j((q^j(0), p^j(0)) \ne 0$ implies $p^j(0)^2/2 + V^{h_j}(q^j(0)) \sim \lambda_j^{\frac{2}{m}}$ and $q^j(0) \in \text{supp } \Psi$. Thus, $|p^j(0)| \ge C\lambda_j^{\frac{1}{m}}$ and

$$\sup_{|t|\le\varepsilon} |p(t) - p(0)| \le \int_0^\varepsilon |\partial_q \tilde{V}_{h_j}(q(s))|ds \le C\varepsilon\lambda_j^{\frac{1}{m}} \le 10^{-3}|p(0)|$$

if $\varepsilon > 0$ is sufficiently small. Thus, $p(t)$ changes its direction and the magnitude only by a small fraction and we clearly have $q^j(t) \notin \text{supp } \Psi$ if $|t| \ge 100\text{diam}(\text{supp } \Psi)/|p(0)|$ when $|t| < \varepsilon$. ∎

Completion of the proof of Theorem 1.2. By virtue of (5.9) and (5.10), we have

$$\begin{aligned} &\left|\textstyle\int_0^\varepsilon (e^{it\tilde{H}_j/h_j} K_j(x, D)e^{-it\tilde{H}_j/h_j} u_{0j}, u_{0j})dt\right| \\ &\quad \le C_N h_j^N + \left|\left(\textstyle\int_0^\varepsilon K_j^N(t, x, h_j D)dt \cdot u_{0j}, u_{0j}\right) dt\right| \le C\lambda_j^{-1/m}. \end{aligned}$$

We apply this to the right of (5.3) and obtain

$$\int_0^{\varepsilon h_j} \|\Psi(x)\Phi_j(x, D)^* e^{-itH_j} u_{0j}\|^2 dt \le C h_j \lambda_j^{-1/2} = C\lambda_j^{-1/m} \tag{5.12}$$

which implies Lemma 5.1, hence, Theorem 1.2. ∎

References

[BAD] M.Ben-Artzi and A.Devinatz, *Local smoothing and convergence properties of Schrödinger type equations,* J. Funct. Anal. **101** (1991), 231-254.

[Bo1] J. Bourgain, *Fourier transform restriction phenomona for certain lattice subsets and application to nonlinear evolution equations I, Schrödinger equations,* Geom. and Funct. Anal. **3** (1993), 107–156.

[Bo2] J. Bourgain, *Exponential sums and nonlinear Schrödinger equations,* Geom. and Funct. Anal. **3** (1993), 157–178.

[Bu] N. Burg, P. Gérad and N. Tzvetkov, *Strichartz inequlities and the nonlinear Schrödinger equation on compact manifolds,* preprint (2001), Université Paris Sud.

[CS] P.Constantin and J.C.Saut, *Local smoothing properties of Schrödinger equations,* Indiana Univ. Math. J. **38** (1989), 791–810.

[F] D. Fujiwara, *Remarks on convergence of the Feynman path integrals*, Duke Math. J. **47** (1980), 41-96.

[GV] J. Ginibre and G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Comm. Math. Phys. **144** (1992), 163-188.

[Ho] L. Hörmander, *The Analysis of Linear Partial Differential Operators III*, Springer Verlag, Berlin-Heidelberg-New York-Tokyo (1985).

[K3] T. Kato, *Nonlinear Schrödinger equations*, Lect. Notes for Physics **345** "Schrödinger Operators" (1988).

[KT] M. Keel and T. Tao, *Endpoint Strichartz inequlities*, Amer. J. Math. **120** (1998) 955-980.

[KY] T. Kato and K. Yajima, *Some examples of smooth operators and the associated smoothing effect*, Rev. Math. Phys. **1** (1989) 481-496.

[KPV] C. E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J.. **40** (1991), 33–69.

[MY] A. Martinez and K. Yajima, *On the Fundamental Solution of Semiclassical Schrödinger Equations at Resonant Times*, to appear in Commun. Math. Phys.

[Ro] D. Robert, *Autour de l'approximation semi-classique*, Progress in Mathematics **68**, Basel, Birkheuser (1987).

[Sj] P.Sjölin, *Regularity of solutions to the Schrödinger equations*, Duke Math. J. **55** (1987), 699–715.

[St] R. S. Strichartz, *Restrictions of Fourier transforms to a quadratic surface and decay of solutions of wave equations*, Duke Math. J. **44** (1977), 704–714.

[Ta] H. Tamura, Asymptotic formulas with sharp remainder estimates for eigenvalues of elliptic operators second order, Duke Math. J. **49** (1982), 87–111.

[V] L. Vega, *Schrödinger equations: Pointwise convergence to the initial data*, Proc. A. M. S. **102** (1988), 874-878.

[Y1] K. Yajima, *Existence of evolution for time depe. lent Schrödinger equations*, Commun. Math. Phys. **110** (1987), 415–426.

[Y2] K.Yajima, *On smoothing property of Schrödinger propagators*, Lecture Notes in Mathematics, **1450**, 20–35.

[Y3] K. Yajima, *Schrödinger evolution equation with magnetic fields*, J. d'Analyse Math. **56** (1991), 29–76.

[Y4] K.Yajima, *Smoothness and non-smoothness of the fundamental solution of time dependent Schrödinger equations*, Commun. Math. Phys. **181** (1996), 605–629.

[Y5] K. Yajima, *Rate of decay at high energy of local spectral projections assiciated with Schrödinger operators*, J. Math. Soc. Japan **41** (1989), 117–142.

[YZ] K.Yajima and G. P. Zhang, *Smoothing property for Schrödinger equations with potential superquadratic at infinity*, Commun. Math. Phys. **221** (2001), 573–590.