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BLOW-UP PROBLEMS FOR SEMILINEAR HEAT EQUATIONS WITH LARGE DIFFUSION

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1. Introduction.

We consider blow-up problems of the solutions of the Cauchy-Neumann problem

$$\text{(P)} \qquad \begin{cases} u_t = D\Delta u + u^p & \text{in} \quad \Omega \times (0,T), \\ \dfrac{\partial}{\partial \nu} u = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = \varphi(x) \geq 0 & \text{in} \quad \Omega, \end{cases}$$

where $D > 0$, $p > 1$, $0 < T < \infty$, Ω is a bounded domain in $\boldsymbol{R}^N$ and ν is the outer unit normal vector to $\partial\Omega$. Throughout this paper we assume that

$$\text{(1.1)} \qquad \varphi \in C(\overline{\Omega}), \qquad \varphi \not\equiv 0, \quad \varphi(x) \geq 0 \quad \text{in} \quad \Omega,$$

for simplicity. (For physical background of this problem, see [BE].) In this paper we study the location of the blow-up set of the solutions u_D for the Cauchy-Neumann problem (P) with large diffusion D. Furthermore we give an estimate of the blow-up time of the solutions u_D.

We denote by T_D the supremum of all σ such that the solution u_D of (P) exists uniquely for all $t < \sigma$. If $T_D < \infty$, we have

$$\lim_{t \uparrow T_D} \max_{x \in \overline{\Omega}} u_D(x,t) = \infty.$$

Then we say that u_D blows up at the time T_D, and call T_D the blow-up time of the solution u_D. We define the blow-up set $B_D(\varphi)$ of the solution u_D by

$$B_D(\varphi) = \{x \in \overline{\Omega} \,|\, \text{there exist } x_k \to x \text{ and } t_k \uparrow T_D \text{ such that } \lim_{k \to \infty} u_D(x_k, t_k) = \infty\}.$$

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F. B. Weissler [W] first proved that some solutions blow up only at a single point for the case $N = 1$. A. Friedman and B. McLeod [FM] proved similar results for more general domains under the Dirichlet boundary condition or the Robin boundary condition. Subsequently, the blow-up sets of the blow-up solutions have been studied by various peoples. Among others, for the case $N = 1$, X. Y. Chen and H. Matano [CM] proved that the blow-up solution blows up at most at finite points in Ω under the Dirichlet boundary condition or the Neumann boundary condition. Furthermore, for the case $N = 1$, F. Merle [16] proved that, for any given finite points $x_1, \ldots, x_k \subset \Omega$, there exists a solution whose blow-up set is exactly $\{x_1, \ldots, x_k\}$. For the case $N \geq 2$ and $\Omega = \boldsymbol{R}^N$, Y. Giga and R. V. Kohn [GK] proved that the blow-up set is bounded if the initial data decays at space infinity. Furthermore, J. J. L. Velázquez [24] proved that the $(n-1)$-dimensional Hausdorff measure of the blow-up set of nontrivial blow-up solution is bounded in compacts sets of $\boldsymbol{R}^N$. (For further results on the blow-up set, see [C], [DL], [L], [Mz], [MY1,2,3], [P] and references given there.) However, for the case $N \geq 2$, it seems to be difficult to study the arrangement of the blow-up set without somewhat strong conditions on the initial data, even for the case that Ω is a cylindrical domain.

Our main interest is to investigate the location of the blow-up set $B_D(\varphi)$ of the solutions of the Cauchy-Neumann problem (P) with large diffusion D. Furthermore, as a by-product, we give an estimate of the blow-up time for sufficiently large D.

We first give an estimate of the blow-up time of the solution u_D for sufficiently large D.

Theorem A. (See [I]). *Consider the Cauchy-Neumann problem* (P) *under the condition* (1.1). *Then $T_D < \infty$. Furthermore there exist constants C and D_0 such that*

$$\left| T_D - (p-1)^{-1}\left(\frac{1}{P_1\varphi}\right)^{p-1} \right| \leq C\frac{\log D}{D}, \qquad P_1\varphi = \frac{1}{|\Omega|}\int_\Omega \varphi dx,$$

for all $D \geq D_0$. Here D_0 depends only on n, Ω, p, and $\|\varphi\|_{L^\infty(\Omega)}$. Here $|\Omega|$ is the Lebesgue measure of Ω.

Next, for the case that Ω is a cylindrical domain, we give a result of the location of the blow-up set $B_D(\varphi)$ the solution u_D for sufficiently large D.

Theorem B. (See [I]). *Let* $\Omega = \Omega' \times (0, L)$*, where* Ω' *is a bounded domain in* $\boldsymbol{R}^{N-1}$ *with smooth boundary* $\partial\Omega'$ *and* $L > 0$*. Consider the Cauchy-Neumann problem* (P) *under the condition* (1.1)*. Assume that*

$$I(\varphi) \equiv \int_\Omega \varphi \cos\left(\frac{\pi}{L} x_N\right) dx \neq 0. \tag{1.2}$$

Then there exists a positive constant D_0 *such that, for any* $D \geq D_0$*, the blow-up set* $B_D(\varphi)$ *of the solution* u_D *of* (P) *satisfies that*

$$B_D(\varphi) \subset \overline{\Omega'} \times \{0\} \qquad \text{if} \quad I(\varphi) > 0$$

and that

$$B_d(\varphi) \subset \overline{\Omega'} \times \{L\} \qquad \text{if} \quad I(\varphi) < 0.$$

Here D_0 *depends only on* n*,* Ω*,* p*,* $I(\varphi)$*, and* $\|\varphi\|_{L^\infty(\Omega)}$*.*

We remark that the condition (1.2) holds for almost all initial data φ *physically.* We may find the similar condition to (1.2) in the Rauch observation, which means that the hot spots of the solutions of the heat equation under the zero Neumann boundary condition move to the boundary, as $t \to \infty$ (see [BB], [K], and [R]).

Next we give a general result of the location of the blow-up set $B_D(\varphi)$ of the solution u_D for sufficiently large D. This is a joint work with Noriko Mizoguchi.

Theorem C. (See [IM1,2]). *Let* Ω *be a bounded domain in* $\boldsymbol{R}^N$ *with* $C^{2,\alpha}$ *boundary* $\partial\Omega$ $(0 < \alpha < 1)$*. Consider the Cauchy-Neumann problem (P) under the condition (1.1) and* $(N-2)p < N+2$*. Assume that* $P_2\varphi \not\equiv 0$ *in* Ω*, where* P_2 *is the projection from* $L^2(\Omega)$ *onto the second Neumann eigenspace. Put*

$$\mathcal{M} = \{x \in \overline{\Omega} : (P_2\varphi)(x) = \max_{y \in \overline{\Omega}} (P_2\varphi)(y)\}.$$

Then, for any $\gamma > 0$*, there exists a positive constant* D_γ *such that*

$$B_D(\varphi) \subset \mathcal{M}_\gamma \equiv \{x \in \overline{\Omega} : dist(x, \mathcal{M}) < \gamma\}$$

for all $D \geq D_\gamma$.

According to the Rauch observation, Kawohl [K] conjected that $M \subset \partial\Omega$ for all convex domains Ω. It is known that this conjecture holds for parallelepipeds, balls, annuli (see [K]), and two dimensional, thin convex polygonal domain with certain symmetry (see [BB]). Furthermore, Burdzy and Werner [BW] gives an example of non-convex domain Ω such that $M \subset \Omega$.

The remainder of paper is organized as follows. In Section 2 we give the outline of the proof of Theorems A and B. In Section 3 we give the outline of the proof of Theorem C.

2. Outline of the proof of Theorems A and B.

Proof of Theorem A. Let G be the Green function of

$$\begin{cases} u_t = \Delta u & \text{in} \quad \Omega \times (0, \infty) \\ \dfrac{\partial}{\partial \nu} u = 0 & \text{on} \quad \partial\Omega \times (0, \infty). \end{cases} \tag{2.1}$$

Let $\{\phi_j\}_{j=1}^\infty$ be a complete orthonormal system of Neumann eigenfunctions for the domain Ω. Let λ_j, $j = 1, 2, \ldots$ be the eigenvalue coresponding to ϕ_j such that $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$. For any $f \in L^2(\Omega)$, we put

$$Q_j f(x) = \sum_{k=1}^{j} (f, \phi_k)_{L^2(\Omega)} \phi_k(x), \qquad j = 1, 2, \ldots.$$

Here we remark that $Q_1 = P_1$. Let D be a sufficiently large and put $t_D = \log D/\lambda_2 D$. Then the solution u_D of (P) satisfies

$$\begin{aligned} u_D(x,t) &= \int_\Omega G(x,y,Dt)\varphi(y)dy + \int_0^t \int_\Omega G(x,y,D(t-s))u(y,s)^p dyds \\ &\equiv J_1(x,t) + J_2(x,t), \end{aligned} \tag{2.2}$$

for all $(x,t) \in \Omega \times (0, T_D)$.

On the other hand, by the comparison principle, we have

$$\|u_D(\cdot,t)\|_{L^\infty(\Omega)} \leq x(t), \tag{2.3}$$

where $x = x(t)$ is the solution of the ordinary differential equation

$$x' = x^p, \qquad x(0) = \|\varphi\|_{L^\infty(\Omega)}. \tag{2.4}$$

By (2.2), (2.3), and $\lim_{D\to\infty} t_D = 0$, we have

$$J_2(x, t_D) = O\left(\frac{\log D}{D}\right) \quad \text{as} \quad D \to \infty. \tag{2.5}$$

Furthermore, since J_1 is a solution of the heat equation, we have

$$\begin{aligned} J_1(x, t_D) &= P_1\varphi + O(e^{-\lambda_2 D t_D}) \\ &= P_1\varphi + O\left(\frac{\log D}{D}\right) \quad \text{as} \quad D \to \infty. \end{aligned} \tag{2.6}$$

By (2.5) and (2.6), we have

$$u_D(x, t_D) = P_1\varphi + O\left(\frac{\log D}{D}\right) \quad \text{as} \quad D \to \infty. \tag{2.7}$$

By (2.7), we compare the solution u_D with the solution $x = x(t)$ of the ordinary differential equaion $x' = x^p$, and may complete the proof of Theorem A. $\square$

Next we give thr outline of the proof of Theorem B. We approximate the solution u_D by the functions $\{Q_j u_D\}_{j=1}^{\infty}$, and obtain the following propositions.

Proposition 2.1. *Let u_D be a solution of* (P) *under the condition* (1.1). *Let $j \in \mathbb{N} \cup \{0\}$ and $0 < \lambda < \lambda_{j+1}$. Then there exist positive constants D_0 and $C = C(N, \Omega)$ such that, if $D \geq D_0$,*

$$\|u_D(\cdot, t) - Q_j u_D(\cdot, t)\|_{C^2(\Omega)} \leq C\left(e^{-D\lambda t} + \frac{1}{D^{1/2}}\right), \qquad \frac{2}{D} \leq t \leq \frac{S}{2}.$$

Here S is the blow-up time of the solution of (2.4).

Proposition 2.2. *Let u_D be a solution of* (P) *under the condition* (1.1). *Then there exist constants C and D_0 such that, if $D \geq D_0$,*

$$\|u_D(\cdot, t) - Q_1 u_D(t)\|_{L^\infty(\Omega)} \leq C\left(e^{-D\lambda t} + \frac{1}{D^{3/2}}\right), \qquad \frac{3}{D} \leq t \leq \frac{S}{2},$$

where $\lambda = \lambda_1/4$.

By Proposition 2.2 and the comparison pinciple, we have the following result.

Proposition 2.3. *Let u_D be a solution of* (P) *under the condition* (1.1). *Then there exist constants C and D_0 such that, if $D \geq D_0$,*

$$\lim_{t \uparrow T_D} \min_{x \in \overline{\Omega}} u_D(x,t) \geq C D^{3/2(p-1)}.$$

By Proposition 2.1, we may prove the monotonicity of the solution u_D in the variable x_N for some time.

Proposition 2.4. *Let u_D be a solution of* (P) *under the condition* (1.1). *Assume $I(\varphi) > 0\,(< 0)$. Then there exist positive constants T and D_0 such that, for all $D \geq D_0$,*

$$\frac{\partial}{\partial x_N} u_D\left(x, \frac{T}{D}\right) < 0\,(> 0), \qquad x \in \Omega. \tag{2.8}$$

Proof. Let $\{\phi_{1,j}\}_{j=1}^{\infty}$ and $\{\phi_{2,j}\}_{j=1}^{\infty}$ be complete orthonormal systems of Neumann eigenfunctions for the domain Ω' and the interval $(0,1)$, respectively. Let $\lambda_{k,j}$ be the eigenvalue corresponding to $\phi_{k,j}$ such that $0 = \lambda_{k,1} < \lambda_{k,2} \leq \lambda_{k,3} \leq \cdots \leq \lambda_{k,j} \leq \cdots$, $k = 1,2$. In this notation we repeat the eigenvalues if needed to take account their multiplicity. Then, by [BB], the family of functions $\{\phi_{1,i}\phi_{2,j}\}_{i,j=1}^{\infty}$ is a complete orthonormal system of Neumann eigenfunctions for D, and the eigenvalue of $\phi_{1,i}\phi_{2,j}$ is $\lambda_{1,i} + \lambda_{2,j}$. Furthermore we have

$$\phi_{1,1} = \frac{1}{|D'|^{1/2}}, \quad \phi_{2,1} = \frac{1}{L^{1/2}}, \quad \phi_{2,j}(x_N) = \sqrt{\frac{2}{L}} \cos\left(\frac{j\pi}{L} x_N\right), \quad j = 1,2,\ldots.$$

Let $j_0 \in \mathbb{N}$ such that $\lambda_{j_0} = \lambda_{2,1} = (\pi/L)^2$. Then $\lambda_j \leq (\pi/L)^2$ for $j = 1,\ldots,j_0 - 1$ and $\lambda_j > (\pi/L)^2$ for $j = j_0 + 1,\ldots$. Furthermore we have

$$\frac{\partial^k}{\partial x_N^k} Q_{j_0} u_D(x,t) = \frac{(u_D(\cdot,t), \phi_{1,0}\phi_{2,1})_{L^2(\Omega)}}{|\Omega'|^{1/2}} \frac{\partial^k}{\partial x_N^k} \phi_{2,1}(x_N), \qquad k = 1,2. \tag{2.9}$$

Put $\lambda = ((\pi/L)^2 + \lambda_{j_0+1})/2$. By Proposition 2.1, there exists a constant C_1 such that the solution u_D satisfies

$$\|u_D(\cdot,\tau) - Q_{j_0} u_D(\cdot,\tau)\|_{C^2(\Omega)}\Big|_{\tau = t/D} \leq C_1\left(e^{-\lambda t} + \frac{1}{D^{1/2}}\right), \quad 2 \leq t \leq \frac{DS}{2}. \tag{2.10}$$

On the other hand, the function $a(t) = (u_D(\cdot,t), \phi_{1,0}\phi_{2,1})_{L^2(\Omega)}$ satisfies

$$\frac{d}{dt} a(t) = -D\left(\frac{\pi}{L}\right)^2 a(t) + \int_D (u_D(x,t))^p \phi_{1,0}\phi_{2,1}\,dx, \qquad 0 < t < T_D.$$

By (3.15), there exists a constant C_2 such that

$$(2.11)\quad \left|a\left(\frac{t}{D}\right)-e^{-(\frac{\pi}{L})^2 t}a(0)\right| = e^{-(\frac{\pi}{L})^2 t}\int_0^{t/D}\int_\Omega e^{D(\frac{\pi}{L})^2 s}(u_D(x,s))^p|\phi_{1,0}\phi_{2,1}|dxds$$
$$\le e^{-(\frac{\pi}{L})^2 t}\int_0^{t/d} e^{D(\frac{\pi}{L})^2 s}\left(\int_\Omega |u_D(x,s)|^{2p}dx\right)^{1/2} ds \le \frac{C_2L^2}{D\pi^2}.$$

for all $0<t<DS/2$. By (2.9)–(2.11) and $a(0)>0$, we have

$$(2.12)\quad \frac{\partial}{\partial x_N}u_D\left(x,\frac{t}{D}\right) \le a\left(\frac{t}{D}\right)\frac{1}{|\Omega'|^{1/2}}\frac{\partial}{\partial x_N}\phi_{2,1}(x)+C_1\left(e^{-\lambda t}+\frac{1}{D^{1/2}}\right)$$
$$\le -\frac{\sqrt{2}\pi}{L^{3/2}|\Omega'|^{1/2}}\left(e^{-\pi^2 t}a(0)-\frac{C_2}{D\pi^2}\right)\sin(\pi x_N)+C_1\left(e^{-\lambda t}+\frac{1}{D^{1/2}}\right)$$

for all $x\in\Omega$ and $2\le t\le DS/2$. By (2.12), $a(0)>0$, and $\lambda>(\pi/L)^2$, there exists a constant T_1 such that, for any $T\ge T_1$, there exists a constant $D_{T,1}$ such that, for all $D\ge D_{T,1}$,

$$(2.13)\quad \frac{\partial}{\partial x_N}u_D\left(x,\frac{T}{D}\right)<0,\qquad x=(x',x_N)\in\Omega\quad\text{with}\quad \min\{x_N,1-x_N\}\ge\frac{1}{8}.$$

Furthermore, by (2.9)–(2.11),

$$\frac{\partial^2}{\partial x_N^2}u_D\left(x,\frac{t}{D}\right)\le -\frac{\pi^2}{L^2}a\left(\frac{t}{D}\right)\phi_{2,1}(x)+C_1\left(e^{-\lambda t}+\frac{1}{D^{1/2}}\right)$$
$$\le -\frac{\sqrt{2}\pi^2}{L^{5/2}|\Omega'|}\left(e^{-\pi^2 t}a(0)-\frac{C_2}{D\pi^2}\right)\cos(\pi x_N)+C_1\left(e^{-\lambda t}+\frac{1}{D^{1/2}}\right)$$

for all $x=(x',x_N)\in\Omega$ with $0<x_N\le 1/4$ and $T\le t\le DS/2$. Similarly in (2.13), there exists a constant T_2 such that, for any $T\ge T_2$, there exists a constant $D_{T,2}$ such that, for all $D\ge D_{T,2}$,

$$(2.14)\quad \frac{\partial^2}{\partial x_N^2}u_D\left(x,\frac{T}{D}\right)<0,\qquad x=(x',x_N)\in\Omega\quad\text{with}\quad 0<x_N\le\frac{1}{4}.$$

Similarly, there exists a constant T_3 such that, for any $T\ge T_3$, there exists a constant $D_{T,3}$ such that, for all $D\ge D_{T,3}$,

$$(2.15)\quad \frac{\partial^2}{\partial x_N^2}u_D\left(x,\frac{T}{D}\right)>0,\qquad x=(x',x_N)\in\Omega\quad\text{with}\quad \frac{3}{4}\le x_N<1,$$

for all $0<\lambda\le\lambda_4$. By (2.13)–(2.15), there exist constants T and D_1 such that

$$\frac{\partial}{\partial x_N}u_D\left(x,\frac{T}{D}\right)<0,\qquad x\in\Omega$$

for all $D \geq D_1$, and the proof of Proposition 2.4 is complete. $\square$

We are ready to complete the proof of Theorem B. We prove Theorem A by applying the arguments of [C] and [FM] together with Propositions 2.2 and 2.4.

Proof of Theorem B. We first assume $I(\varphi) > 0$, and prove Theorem B. By Proposition 2.4, there exist constants T and D_1 such that, $v = \partial u_D/\partial x_N$ satisfies

$$\begin{cases} v_t = D\Delta v + pu_D^{p-1}v & \text{in} \quad \Omega \times (T/D, T_D), \\ v(x,t) = 0 & \text{on} \quad \Gamma_1 \times (T/D, T_D), \\ \frac{\partial}{\partial \nu} v(x,t) = 0 & \text{on} \quad \Gamma_2 \times (T/D, T_D), \\ v(x, T/D) \leq 0 & \text{in} \quad \Omega, \end{cases}$$

for all $D \geq D_1$, where $\Gamma_1 = \Omega' \times \{0, L\}$ and $\Gamma_2 = \partial\Omega' \times (0, L)$. By the maximum principle,

$$\frac{\partial}{\partial x_N} u_D(x,t) = v(x,t) < 0 \qquad \text{in} \quad \Omega \times (0,T) \quad \text{and} \quad \Gamma_2 \times (0,T). \tag{2.16}$$

Assume that $a = (a', a_N) \in B_D(\varphi) \cap (\overline{\Omega'} \times (0,1))$. Let T_* be a constant to be chosen later such that $T/D \leq T_* < T_D$. Put $Q \equiv \Omega' \times (b,c) \times (T_*, T_D)$, where b, $c \in (0, L)$ such that $b < a_N < c$ and $c - b \geq L/2$. Put

$$J(x', x_N, t) = \frac{\partial}{\partial x_N} u_D(x,t) + \epsilon \zeta(x_N)(u_D(x,t))^q, \quad \zeta(s) = \sin\left(\frac{\pi(s-b)}{c-b}\right),$$

where $1 < q < p$ and $\epsilon > 0$ is a positive constant to be chosen later. Then we have

$$J_t - D\Delta J - r(x,t)J = -\epsilon\zeta K(x,t) - \epsilon q(q-1)u_D^{q-2}|\nabla u_D|^2 \leq -\epsilon\zeta K(x,t) \quad \text{in} \quad Q, \tag{2.17}$$

where

(2.18)
$$r(x,t) = -2Dq\epsilon\zeta' u_D^{q-1} + pu_D^{p-1}, \quad K(x,t) = (p-q)u_D^{p+q-1} + D\zeta^{-1}\zeta'' u_D^q - 2Dq\epsilon\zeta' u_D^{2q-1}.$$

On the other hand,

$$\zeta^{-1}\zeta'' = -\left(\frac{\pi}{c-b}\right)^2 \geq -\left(\frac{2\pi}{L}\right)^2.$$

By Proposition 2.3, there exist constants $T_1 \in (T/D, T_D)$ and $D_2 \geq D_1$ such that

$$\frac{p-q}{2}(u_D(x,t))^{p+q-1} \geq D\left(\frac{2\pi}{L}\right)^2 (u_D(x,t))^q, \qquad (x,t) \in \Omega \times (T_1, T_D) \tag{2.19}$$

for all $D \geq D_2$. Furthermore we take a sufficiently small ϵ so that

$$\frac{p-q}{2}(u_D(x,t))^{p+q-1} \geq 2Dq\epsilon|\zeta'|u^{2q-1} \qquad (x,t) \in \Omega \times (T_1, T_D). \tag{2.20}$$

Taking $T_* = T_1$ and $D \geq D_2$, by (2.17)–(2.20), we have

$$\begin{cases} J_t \leq D\Delta J + r(x,t)J & \text{in} \quad Q, \\ J(x,t) < 0 & \text{on} \quad \Omega' \times \{b,c\} \times (T_*, T_D), \\ \frac{\partial}{\partial \nu}J(x,t) = 0 & \text{on} \quad \partial\Omega' \times (b,c) \times (T_*, T_D). \end{cases}$$

By (2.16), taking a sufficiently small ϵ if necessary, we have $J(x, T_*) < 0$, $x \in \Omega' \times (b,c)$. By the maximum principle, we have

$$J(x,t) \leq 0 \qquad \text{for} \quad (x,t) \in \overline{\Omega'} \times (b,c) \times (T_*, T_D). \tag{2.21}$$

By $a = (a', a_N) \in B_D(\varphi)$ and $a_N \in (b,c)$, there exist a sequence $\{(a'_k, a_{kN}, t_k)\}_{k=1}^{\infty}$ and a positive constant δ such that

$$\lim_{k\to\infty}(a'_k, a_{kN}, t_k) = (a', a_N, T_D), \quad \lim_{k\to\infty} u(a'_k, a_{kN}, t_k) = \infty,$$
$$\{(a'_k, a_{kN} + \delta)\}_{k=1}^{\infty} \subset \overline{\Omega'} \times (b,c).$$

By (2.16),

$$\lim_{k\to\infty} u_D(a'_k, a_{kN} + \delta, t_k) = \infty,$$

and by (2.21),

$$\int_{u_D(a'_k, a_{kN}, t_k)}^{u_D(a'_k, a_{kN}+\delta, t_k)} \frac{ds}{s^q} \leq -\epsilon \int_{a_{kN}}^{a_{kN}+\delta} \zeta(s)ds.$$

By $q > 1$, we take the limit as $k \to \infty$ to have

$$0 \leq -\epsilon \int_{a_N}^{a_N+\delta} \zeta(s)ds < 0.$$

This contradiction shows $a \notin B_D(\varphi)$. Therefore we have $(\overline{\Omega'} \times (0,1)) \cap B_D(\varphi) = \emptyset$ for all $D \geq D_2$. Furthermore, if $a \in (\overline{\Omega'} \times \{L\}) \cap B_D(\varphi)$, by (2.16), $(\overline{\Omega'} \times (0,1)) \cap B_D(\varphi) \neq \emptyset$. Therefore we have $(\overline{\Omega'} \times \{L\}) \cap B_D(\varphi) = \emptyset$ for all $D \geq D_2$, and the proof of Theorem B for the case $I(\varphi) > 0$ is complete. By the similar argument as in the proof of Theorem B for the case $I(\varphi) > 0$, we may prove Theorem B for the case $I(\varphi) < 0$. So the proof of Theorem B is complete. □

Remark. Without the condition (1.2), Theorem B does not necessarily hold. In fact, if $\Omega = (0,1)$ and $\varphi(x) = 1 - \cos(2\pi x)$, the solution blows-up only at $\{1/2\}$ for all $D > 0$.

3. Outline of the proof of Theorem C.

In this section we follow the argument of [IM1,2], and give the outline of the proof of Theorem C. Following the argument of [GK], for $b \in \overline{\Omega}$, we put

$$w(y,s) = (T_D - t)^{1/(p-1)} u_D(x,t), \quad y = (T_D - t)^{-1/2}(x-b), \quad s = -\log(T_D - t).$$

Then w satisfies

$$(3.1)\qquad \begin{cases} w_s = D\Delta w - \dfrac{y}{2}\cdot\nabla w - \dfrac{1}{p-1}w + w^p & \text{in } \displaystyle\bigcup_{s_{T_D}<s<\infty} (\Omega_b(s)\times\{s\}), \\ \dfrac{\partial w}{\partial \nu}(y,s) = 0 & \text{on } \displaystyle\bigcup_{s_{T_D}<s<\infty} (\partial\Omega_b(s)\times\{s\}), \\ w(y, s_{T_D}) = T_D^{\frac{1}{p-1}}\phi(T_D^{\frac{1}{2}}y + b) \geq 0 & \text{in } \Omega_b(s_{T_D}), \end{cases}$$

where $s_{T_D} = -\log T_D$ and $\Omega_b(s) = e^{\frac{s}{2}}(\Omega - b) = (T_D - t)^{-\frac{1}{2}}(\Omega - b)$. Define the energy $E_b[w]$ correspondind to (3.1) by

$$E_b[w](s) = \int_{\Omega_b(s)} \left\{ \frac{d}{2}|\nabla w|^2 + f(w) \right\} \rho(y)dy, \qquad s \geq s_{T_D},$$

where

$$f(r) = \frac{1}{2(p-1)}r^2 - \frac{1}{p+1}r^{p+1}, \ r \geq 0, \qquad \rho(y) = \frac{1}{(4\pi D)^{N/2}} \exp\left(-\frac{|y|^2}{4D}\right).$$

Then we have

$$(3.2)\quad E_b[w](s_2) \leq E_b[w](s_1) + \int_{s_1}^{s_2} e^{s/2} \int_{\partial\Omega_b(s)} f(w)\rho(y)\frac{y\cdot\nu}{|y|} d\sigma ds, \quad s_{T_D} \leq s_1 \leq s_2 < \infty.$$

Furthermore we have

Proposition 3.1. *Let Ω be a bounded domain in $\boldsymbol{R}^N$ with $C^{2,\alpha}$ boundary $\partial\Omega$ $(0 < \alpha < 1)$ and $d > 0$. Assume $(N-2)p < N+2$. Then there exists a sequence $\{s_n\}$ with $\lim_{n\to\infty} s_n = \infty$ such that*

$$\lim_{n\to\infty} E_b[w](s_n) = f(\kappa)\chi(b),$$

where $\kappa = (p-1)^{-1/(p-1)}$ and $\kappa = (p-1)^{-1/(p-1)}$, $\chi(b) = 1$ $(b \in \Omega)$, $\chi(b) = 1/2$ $(b \in \partial\Omega)$.

On the other hand, we have

Proposition 3.2. *Let $\lambda_2/2 < \lambda < \lambda_2 < \mu$. Then there exists a positive constants D_1 such that*

$$\text{(i)} \qquad \|P_2 u_D(\cdot,t)\|_{L^\infty(\Omega)} < D^{N+5} P_1 u_D(t) e^{-\lambda D t}$$

$$\text{(ii)} \qquad \|(I-(P_1+P_2))u_D(t)\|_{L^\infty(\Omega)} < D^{N+5} P_1 u_D(t) e^{-\mu D t}$$

for all $t \in [T/4, T-D^{-3}]$ and $D \geq D_1$. Here I is the identity map on $L^2(\Omega)$.

Proposition 3.3. *Let $\lambda_2 < \alpha < 2\lambda_2$. Let $m = dim(P_2 L^2(\Omega))$ and $\{\phi_j\}_{j=1}^m$ be an orthnormal basis of $P_2 L^2(\Omega)$. There are positive constants K and D_2 such that*

$$A_j - KD^{-\frac{\lambda_2}{\alpha}} < \frac{\alpha_j(t)e^{-\lambda_2 D t}}{(P_1 u_D(t))^p} < A_j + KD^{-\frac{\lambda_2}{\alpha}}, \qquad 1 \leq j \leq m,$$

for all $t \in [T/4, T-D^{-3}]$ and $D \geq D_2$, where

$$\int_\Omega u_D(x,t)\phi_j(x)dx, \qquad 1 \leq j \leq m.$$

By using Propositions 3.2 and 3.3, we have the following proposition.

Proposition 3.4. *Let Ω be a bounded domain in $\boldsymbol{R}^N$ with $C^{2,\alpha}$ boundary $\partial\Omega$ $(0 < \alpha < 1)$. Let $b \in B_D(\varphi) \setminus \mathcal{M}_\gamma$. Assume that $P_2\varphi \not\equiv 0$ in Ω. Then there exist positive constants C and D_3 such that*

$$E_b[w](3\log D) \leq f(\kappa) \int_{\Omega_b(3\log D)} \rho(y)dy - Ce^{-\mu D(T_D - D^{-3})}$$

for all $D \geq D_3$. Here μ is the constant given in Proposition 3.2.

Let $b \in B_D(\varphi) \setminus \mathcal{M}_\gamma$. We first consider the case that D is convex. Then we have

(3.3)
$$\begin{aligned}\int_{3\log D}^\infty e^{s/2} \int_{\partial\Omega_b(s)} f(w)\rho(y)\frac{y\cdot\nu}{|y|}d\sigma ds &\leq f(\kappa)\int_{3\log D}^\infty e^{s/2}\int_{\partial\Omega_b(s)} \rho(y)\frac{y\cdot\nu}{|y|}d\sigma ds \\ &= f(\kappa)\int_{3\log D}^\infty \left\{\frac{d}{ds}\int_{\Omega_b(s)} \rho dy\right\} ds \\ &= f(\kappa)\left\{\chi(b) - \int_{\Omega_b(3\log D)} \rho(y)dy\right\}.\end{aligned}$$

By (3.2), (3.3) and Propositions 3.1 and 3.4, we have

$$f(\kappa)\chi(b) \leq f(\kappa)\chi(b) - Ce^{-\mu D(T_D - D^{-3})}$$

for sufficiently large D. This is a contradiction, and we see that $B_D(\varphi) \cap \mathcal{M}_\gamma = \emptyset$ for sufficiently large D. Next we consider the case that D is not convex. Let $\Gamma(x, y, t)$ be the fundamental solution of the Cauchy problem for the heat equation $U_t = \Delta U$ in $\boldsymbol{R}^N \times (0, \infty)$, that is,

$$\Gamma(x, y, t) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

We define an energy of the solutions u_D of (P) as follows:

$$\begin{aligned} &E_D(b, T_D : t) \\ &= (T_D - t)^{\frac{p+1}{p-1}} \int_\Omega \left(\frac{D}{2}|\nabla u_D|^2 - \frac{1}{p+1}u_D^p\right) \Gamma(x, b, D(T-t))dx \\ &\qquad + \frac{1}{2(p-1)}(T_D - t)^{\frac{2}{p-1}} \int_\Omega u_D^2 \Gamma(x, b, D(T_D - t))dx. \end{aligned}$$

Then we have

$$E_b[w](s) = E_D(b, T_D, t), \qquad s = -\log(T-t).$$

Furthermore we modify the energy $E_D(b, T : t)$, and give another energy $F_D^\epsilon(b, T_D : t)$. Let $\epsilon > 0$ and $y \in \overline{\Omega}$. Then we may define a continuous function $h_\epsilon(x, y, t)$ on $\overline{\Omega} \times [0, \infty)$, satisfying

$$\begin{cases} \partial_t h_\epsilon = \Delta_x h & \text{in} \quad \Omega \times (\epsilon, \infty), \\ \dfrac{\partial h}{\partial \nu_x} = -\dfrac{\partial}{\partial \nu_x}\Gamma(x, y : t) & \text{on} \quad \partial\Omega \times (\epsilon, \infty), \\ h_\epsilon(x, y, t) = 0 & \text{in} \quad \Omega \times [0, \epsilon]. \end{cases}$$

Put $G_\epsilon(x, y, t) = \Gamma(x, y, t) + h_\epsilon(x, y, t)$. Then G_ϵ satisfies

$$\begin{cases} \partial_t G_\epsilon(x, y, t) = \Delta_x G_\epsilon(x, y, t) & \text{in} \quad \Omega \times (\epsilon, \infty), \\ \dfrac{\partial}{\partial \nu_x} G_\epsilon(x, y, t) = 0 & \text{on} \quad \partial\Omega \times (\epsilon, \infty), \\ G_\epsilon(x, y, t) = \Gamma(x, y, t) & \text{in} \quad \Omega \times [0, \epsilon] \end{cases}$$

for all $y \in \overline{\Omega}$. By using the function G_ϵ, we modify the energy of the solution u_D introduced by [P], and define an energy $F_d^\epsilon(b, T : t)$ as follows:

$$\begin{aligned} &F_D^\epsilon(b, T_D : t) \\ &= (T_D - t)^{\frac{p+1}{p-1}} \int_\Omega \left(\frac{D}{2}|\nabla u_D|^2 - \frac{1}{p+1}u_D^p\right) G_\epsilon(x, b : D(T_D - t))dx \\ &\qquad + \frac{1}{2(p-1)}(T_D - t)^{\frac{2}{p-1}} \int_\Omega u_D^2 G_\epsilon(x, b : D(T_D - t))dx. \end{aligned}$$

By Proposition 3.1, we see that there exists a positive sequence $\{\epsilon_n\}$ with $\lim_{n\to\infty} \epsilon_n = 0$ such that

$$\lim_{n\to\infty} F_D^{D\epsilon_n}(b, T_D : T_D - \epsilon_n) = f(\kappa)\chi(b).$$

Furthermore, by Propositions 3.2 and 3.3, we have the following estimate, instead of Proposition 3.4,

$$F_D^{D\epsilon_n}[w](b, T_D : T_D - D^{-3}) \leq f(\kappa) - Ce^{-\mu D(T_D - D^{-3})} \tag{3.4}$$

for some constant C. By the same argument as in the one of Poon [P], the energy $F_D^{D\epsilon_n}(b, T_D : t)$ is monotone in $t \in [T_D - D^{-3}, T_D - \epsilon_n]$, and we have

$$f(\kappa)\chi(b) = \lim_{n\to\infty} F_D^{D\epsilon_n}(b, T_D : T_D - \epsilon_n) \leq \lim_{n\to\infty} F_D^{D\epsilon_n}(b, T_D : T_D - D^{-3}),$$

and by (3.4), we obtain

$$f(\kappa)\chi(b) \leq f(\kappa)\chi(b) - Ce^{-\mu D(T_D - D^{-3})}$$

for sufficiently large D. This is a contradiction, and we see that $B_D(\varphi) \cap \mathcal{M}_\gamma = \emptyset$ for sufficiently large D. This completes the proof of Theorem C.

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