<table>
<thead>
<tr>
<th>Title</th>
<th>Stationary isothermic surfaces of the heat flow (Nonlinear Diffusive Systems and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Sakaguchi, Shigeru</td>
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Stationary isothermic surfaces of the heat flow

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1 Introduction

This is based on the author's recent work with R. Magnanini [MS 3]. Let $u = u(x,t)$ be the unique solution of the following problem for the heat equation:

$$\begin{align*}
\partial_t u &= \Delta u \quad \text{in} \quad \Omega \times (0, +\infty), \\
u &= 1 \quad \text{on} \quad \partial \Omega \times (0, +\infty), \\
u &= 0 \quad \text{on} \quad \Omega \times \{0\},
\end{align*}$$

(1.1)

(1.2)

(1.3)

where $\Omega$ is a bounded domain in $\mathbb{R}^N, N \geq 2$.

A conjecture, posed in [Kl] by M.S. Klamkin and referred to by L. Zalcman in [Z] as the Matzoh Ball Soup, was settled affirmatively by G. Alessandrini in [A 1]-[A 2]. In [A 2], under the assumption that every point of $\partial \Omega$ is regular with respect to the Laplacian, it was proved that if all the spatial isothermic surfaces of $u$ are invariant with time then $\Omega$ must be a ball. (Of course, the values of $u$ vary with time on its spatial isothermic surfaces.)

The case where the homogeneous initial data in (1.3) is replaced by a function in the space $L^2(\Omega)$ was also considered in [A 1]-[A 2] and, with the help of J. Serrin's celebrated symmetry theorem for elliptic equations [Ser], was settled in the following terms: if all the spatial isothermic surfaces of the solution $u$ of the heat equation with homogeneous Dirichlet boundary condition and initial data $\varphi \in L^2(\Omega)$ are invariant with time, then either $\varphi$ is an eigenfunction of the Laplacian or $\Omega$ is a ball. The analogous question where condition (1.2) is replaced by the homogeneous Neumann boundary condition was examined and answered positively (see [Sak], Theorem 1) with the aid of the classification theorem for isoparametric hypersurfaces in Euclidean
space due to T. Levi-Civita and B. Segre (see [LC], [Seg]). The method used in [Sak] can be applied to give an alternative proof of Alessandrini's results.

An important observation is that, in order to prove Klamkin's conjecture [Ki], both methods employed in [A1]-[A2] and [Sak] need to assume that infinitely many isothermic surfaces of $u$ are invariant with time. As a natural consequence of this remark, one may wonder if the requirement that a finite number (possibly only one) of level surfaces of $u$ are invariant with time implies that $\Omega$ is a ball.

Our main result in this direction is the following.

**Theorem 1.1** Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the exterior sphere condition and suppose that $D$ is a domain, with boundary $\partial D$, satisfying the interior cone condition, and such that $\overline{D} \subset \Omega$. Assume that the solution $u$ of problem (1.1)-(1.3) satisfies the following condition:

$$u(x, t) = a(t), \quad (x, t) \in \partial D \times (0, +\infty), \quad (1.4)$$

for some function $a : (0, +\infty) \to (0, +\infty)$. Then $\Omega$ must be a ball.

We recall that $\Omega$ satisfies the **exterior sphere condition** if for every $y \in \partial \Omega$ there exists a ball $B_r(z)$ such that $\overline{B_r(z)} \cap \overline{\Omega} = \{y\}$, where $B_r(z)$ denotes an open ball centered at $z \in \mathbb{R}^N$ and with radius $r > 0$. Also, $D$ satisfies the **interior cone condition** if for every $x \in \partial D$ there exists a finite right spherical cone $K_x$ with vertex $x$ such that $K_x \subset \overline{D}$ and $\overline{K_x} \cap \partial D = \{x\}$.

The proof of Theorem 1.1 exploits arguments different from the ones used in [A1]-[A2] and [Sak]. Our technique is essentially based on the following three ingredients. One ingredient is a careful study of the asymptotic behavior of $u(x, t)$ as $t \to 0$ which is based on the results of S. R. S. Varadhan [V] (see also [EI]). The second one is A. D. Aleksandrov's uniqueness theorem [Alek]. A special case of this theorem is the well-known *Soap-Bubble Theorem*. The third one is the following balance law proved in [MS 1]-[MS 2] (see [MS 3] for a shorter proof):

**Theorem 1.2 (balance law)** Let $G$ be a domain in $\mathbb{R}^N$, $N \geq 2$, let $x_0$ be a point in $G$ and set $d_* = \text{dist}(x_0, \partial G)$. Suppose that $v = v(x, t)$ is a solution of the heat equation in $G \times (0, +\infty)$. Then the following hold:
(i) $v(x_0, t) = 0$ for every $t \in (0, +\infty)$ if and only if
$$
\int_{\partial B_{r}(x_0)} v(x, t) dS_x = 0 \text{ for every } (r, t) \in (0, d_*) \times (0, +\infty);
$$

(ii) $\nabla v(x_0, t) = 0$ for every $t \in (0, +\infty)$ if and only if
$$
\int_{\partial B_{r}(x_0)} (x - x_0) v(x, t) dS_x = 0 \text{ for every } (r, t) \in (0, d_*) \times (0, +\infty).
$$

Section 2 is devoted to an outline of the proof of Theorem 1.1. In Section 3, we consider the case where the domain $\Omega$ is unbounded.

2 Outline of the proof of Theorem 1.1

Define the function $W = W(x, s)$ by
$$
W(x, s) = s \int_{0}^{+\infty} u(x, t) e^{-s t} dt, \quad s > 0.
$$
(2.1)

Notice that $W$ is the solution of the following elliptic boundary value problem:
$$
\Delta W - s W = 0 \quad \text{in } \Omega,
$$
(2.2)
$$
W = 1 \quad \text{on } \partial \Omega.
$$
(2.3)

A result in [V] (see also [El]) shows that, as $s \to +\infty$, the function $-\frac{1}{\sqrt{s}} \log W(x, s)$ converges uniformly on $\overline{\Omega}$ to the function $d = d(x)$ defined by
$$
d(x) = \text{dist} (x, \partial \Omega), \quad x \in \Omega.
$$
(2.4)

(Since $\Omega$ enjoys the exterior sphere condition, we can apply the result in [V].) Moreover, if $u$ satisfies (1.4), then for any fixed $s > 0$, $W$ is constant on $\partial D$. Indeed,
$$
W(x, s) = s \int_{0}^{+\infty} \alpha(t) e^{-s t} dt := A(s), \quad x \in \partial D.
$$
(2.5)

Thus, in view of the result in [V], we can define the positive number $R > 0$ by
$$
R = \lim_{s \to +\infty} \left\{ -\frac{1}{\sqrt{s}} \log A(s) \right\}.
$$
(2.6)

In Lemma 2.1 below, we prove analyticity of $\partial D$ and $\partial \Omega$ by using our balance law.
Lemma 2.1 The following assertions hold:

(i) for every \( x \in \partial D \), \( d(x) = R \), where \( d \) is defined by (2.4);

(ii) \( \partial D \) is analytic;

(iii) \( \partial \Omega \) is analytic and \( \partial \Omega = \{ x \in \mathbb{R}^N : \text{dist} (x, D) = R \} \);

(iv) the mapping: \( \partial D \ni x \mapsto y(x) = x - R \nu^*(x) \in \partial \Omega \) is a diffeomorphism, where \( \nu^*(x) \) denotes the interior unit normal vector to \( \partial D \) at \( x \in \partial D \);

(v) for every \( x \in \partial D \), \( \nabla d(y(x)) = \nu^*(x) \) and \( \overline{B_R(x)} \cap \partial \Omega = \{ y(x) \} \);

(vi) let \( \kappa_j(y) \), \( j = 1, \ldots, N - 1 \) denote the \( j \)-th principal curvature at \( y \in \partial \Omega \) of the analytic surface \( \partial \Omega \) with respect to the interior normal direction to \( \partial \Omega \). Then \( \kappa_j(y) < \frac{1}{R} \), \( j = 1, \ldots, N - 1 \), for every \( y \in \partial \Omega \).

Proof. (i) The result in [V] and the definition (2.6) of \( R \) yield this assertion.

(ii) It suffices to show that, for every point \( x \in \partial D \), there exists a time \( t^* > 0 \) such that \( \nabla u(x, t^*) \neq 0 \), since \( u \) is analytic with respect to the space variable.

Assume by contradiction that there exists a point \( x_0 \in \partial D \) such that \( \nabla u(x_0, t) = 0 \) for every \( t > 0 \). Since \( u \) is continuous up to \( \partial \Omega \times (0, +\infty) \), by Theorem 1.2 (ii), we can infer that

\[
\int_{\partial B_R(x_0)} (x - x_0) \cdot u(x, t) \, dS_x = 0 \quad \text{for every } t > 0,
\]

and hence

\[
\int_{\partial B_R(x_0)} (x - x_0) \cdot W(x, s) \, dS_x = 0 \quad \text{for every } s > 0, \tag{2.7}
\]

in view of (2.1).

On the other hand, since \( D \) satisfies the interior cone condition, there exists a finite right spherical cone \( K \) with vertex at \( x_0 \) such that \( K \subset \overline{D} \) and \( \overline{K} \cap \partial D = \{ x_0 \} \). By translating and rotating if needed, we can suppose that \( x_0 = 0 \) and that \( K \) is the set \( \{ x \in B_\rho(0) : x_N < -|x| \cos \theta \} \), where \( \rho \in (0, R) \) and \( \theta \in (0, \frac{\pi}{2}) \).

Since \( K \subset \overline{D} \) and \( \overline{K} \cap \partial D = \{ 0 \} \), proposition (i) implies that

\[
d(x) > R \quad \text{for every } x \in K. \tag{2.8}
\]

The set defined by

\[
V = \{ x \in \partial B_R(0) : x_N \geq R \sin \theta \}, \tag{2.9}
\]
is such that
\[ \partial \Omega \cap \partial B_R(0) \subset V, \]  
(2.10)
because, otherwise, there would be a point in \( K \) contradicting (2.8).

Thus, from (2.10) it follows that we can choose a number \( \delta > 0 \) such that
\[ d(x) \geq 5\delta \quad \text{for every } x \in \partial B_R(0) \cap \{ x_N \leq 0 \}. \]  
(2.11)

Since we know that \(- \frac{1}{\sqrt{s}} \log W(x, s)\) converges uniformly on \( \overline{\Omega} \) to \( d(x) \) as \( s \to +\infty \), we can choose \( s^* > 0 \) such that
\[ \left| - \frac{1}{\sqrt{s}} \log W(x, s) - d(x) \right| < \delta, \]
for every \( x \in \overline{\Omega} \) and every \( s \geq s^* \). This latter inequality, together with (2.9), (2.10), and (2.11), gives, for every \( s \geq s^* \), the following two estimates:
\[ \int_{\partial B_R(0) \cap \{ x_N \leq 0 \}} x_N W(x, s) \, dS_x \geq - \frac{1}{2} R e^{-4\delta\sqrt{s}} \mathcal{H}^{N-1} (\partial B_R(0)), \]  
(2.12)
\[ \int_{V \cap \overline{\Omega}_{2\delta}} x_N W(x, s) \, dS_x \geq R \sin \theta e^{-3\delta\sqrt{s}} \mathcal{H}^{N-1} (V \cap \overline{\Omega}_{2\delta}). \]

Here \( \mathcal{H}^{N-1} (\cdot) \) denotes the \((N-1)\)-dimensional Hausdorff measure and \( \Omega_{2\delta} \) is defined by
\[ \Omega_{2\delta} = \{ x \in \Omega : d(x) < 2\delta \}. \]  
(2.13)

A consequence of (2.12) is that, for every \( s \geq s^* \),
\[ \int_{\partial B_R(0)} x_N W(x, s) \, dS_x \geq \int_{V \cap \overline{\Omega}_{2\delta}} x_N W(x, s) \, dS_x \]
\[ + \int_{\partial B_R(0) \cap \{ x_N \leq 0 \}} x_N W(x, s) \, dS_x \geq R e^{-3\delta\sqrt{s}} \left[ \sin \theta \mathcal{H}^{N-1} (V \cap \overline{\Omega}_{2\delta}) - \frac{1}{2} e^{-\delta\sqrt{s}} \mathcal{H}^{N-1} (\partial B_R(0)) \right]. \]

Therefore, we obtain a contradiction by observing that the first term of this chain of inequalities equals zero, by (2.7), while the last term can be made positive by choosing \( s > 0 \) sufficiently large.

(iii), (iv), and (v) Let
\[ \Gamma = \{ x \in \mathbb{R}^N : \text{dist } (x, D) = R \}. \]
It is clear that $\Gamma \subset \partial \Omega$. Take any point $x \in \partial D$. Then, there exists a unique point $y \in \partial \Omega$ such that $\overline{B_R(x)} \cap \partial \Omega = \{y\}$. Indeed, since $\partial D$ is analytic by (ii), if $\tilde{y} \in \overline{B_R(x)} \cap \partial \Omega$ and $\tilde{y} \neq y$, then
\[
\frac{y - x}{|y - x|} = -\nu^*(x) = \frac{\tilde{y} - x}{|\tilde{y} - x|},
\]
where $\nu^*(x)$ is the interior unit normal vector to $\partial D$ at $x$ — a contradiction. Since $\Omega$ enjoys the exterior sphere property, there exists a ball $B_r(z)$ such that $\overline{B_r(z)} \cap \overline{\Omega} = \{y\}$, and hence $B_r(z) \cap \overline{B_R(x)} = \{y\}$. Therefore,
\[
\text{dist} (z, D) = r + R \quad \text{and} \quad \overline{B_{r+R}(z)} \cap \overline{D} = \{x\}. \tag{2.14}
\]
Let $\kappa_j^*$, $j=1, \ldots, N-1$, denote the principal curvatures of the surface $\partial D$ with respect to the interior normal direction to $\partial D$. Then (2.14) implies that
\[
\kappa_j^*(x) \geq -\frac{1}{r+R}, \quad j = 1, \ldots, N-1.
\]
Since $\kappa_j^* > -\frac{1}{R}$ on $\partial D$, for every $j = 1, \ldots, N-1$, $\Gamma$ is an analytic hypersurface diffeomorphic to $\partial D$ (see [GT], Lemma 14.16), and hence $\Gamma$ equals $\partial \Omega$. Assertions (iii), (iv), and (v) then follow at once.

(vi) Take any point $y \in \partial \Omega$. Propositions (iii) and (iv) imply that there exists a unique $x \in \partial D$ such that $\overline{B_R(y)} \cap \overline{D} = \{x\}$. Since $\partial D$ is analytic, $D$ satisfies the interior sphere condition, that is there exists a ball $B_r(z) \subset D$ such that $\overline{B_r(z)} \cap \partial D = \{x\}$. Therefore,
\[
d(z) = r + R \quad \text{and} \quad \overline{B_{r+R}(z)} \cap \partial \Omega = \{y\}, \tag{2.15}
\]
and consequently
\[
\kappa_j(y) \leq \frac{1}{r+R}, \quad j = 1, \ldots, N-1.
\]
Assertion (vi) is proved. \(\square\)

Let us show that the two functions
\[
W^\pm_\epsilon(x, s) = \exp\{-\sqrt{s(1 \mp \epsilon)} \, d(x)\}, \tag{2.16}
\]
where $d(x)$ is defined by (2.4), provide respectively an upper and a lower barrier for $W$ in $\Omega$ for large values of $s$.

**Lemma 2.2** For every $\epsilon > 0$, there exists a positive number $s_\epsilon$ such that
\[
W^-_\epsilon(x, s) \leq W(x, s) \leq W^+_\epsilon(x, s) \tag{2.17}
\]
for every $x \in \overline{\Omega}$ and every $s \geq s_\epsilon$. 

Proof. Choose a number $\delta > 0$ such that the function $d = d(x)$ defined in (2.4) is of class $C^2$ in the set $\overline{\Omega_\delta}$ where
\begin{equation}
\Omega_\delta = \{ x \in \Omega : d(x) < \delta \}.
\end{equation}
(2.18)

Let $W_\epsilon^\pm(x, s)$ be given by (2.16). A straightforward computation gives
\begin{equation}
\Delta W_\epsilon^\pm - s W_\epsilon^\pm = \mp \varepsilon \sqrt{s} \left\{ \sqrt{s} \pm \frac{\sqrt{1 \mp \varepsilon}}{\varepsilon} \Delta d \right\} W_\epsilon^\pm \text{ in } \Omega_\delta.
\end{equation}
Set $M_\delta = \max_{\overline{\Omega_\delta}} |\Delta d|$. If $s \geq \frac{1+\varepsilon}{\varepsilon^2} M_\delta^2$, then
\begin{equation}
\begin{aligned}
\Delta W_\epsilon^+ - s W_\epsilon^+ &\leq 0 \quad \text{in } \Omega_\delta, \\
\Delta W_\epsilon^- - s W_\epsilon^- &\geq 0
\end{aligned}
\end{equation}
(2.19)

Since the function $-\frac{1}{\sqrt{s}} \log W(x, s)$ converges uniformly on $\overline{\Omega}$ to $d(x)$ as $s \to +\infty$, there exists a number $s^* > 0$ such that
\begin{equation}
-\delta(1 - \sqrt{1 - \varepsilon}) \leq -\frac{1}{\sqrt{s}} \log W(x, s) - d(x) \leq \delta(\sqrt{1 + \varepsilon} - 1), \quad x \in \overline{\Omega},
\end{equation}
for every $s \geq s^*$. Hence, since $d(x) \geq \delta$ for every $x \in \Omega \setminus \Omega_\delta$, we obtain
\begin{equation}
W_\epsilon^-(x, s) \leq W(x, s) \leq W_\epsilon^+(x, s), \quad x \in \Omega \setminus \Omega_\delta,
\end{equation}
(2.20)
for every $s \geq s^*$. Moreover,
\begin{equation}
W_\epsilon^-(x, s) = W(x, s) = W_\epsilon^+(x, s) = 1, \quad x \in \partial \Omega,
\end{equation}
(2.21)
for every $s > 0$, clearly.

Choose $s_\varepsilon = \max(s^*, \frac{1+\varepsilon}{\varepsilon^2} M_\delta^2)$. Then by the comparison principle, from (2.19), (2.20) and (2.21), we have
\begin{equation}
W_\epsilon^-(x, s) \leq W(x, s) \leq W_\epsilon^+(x, s), \quad x \in \Omega_\delta,
\end{equation}
(2.22)
for every $s \geq s_\varepsilon$. Combining (2.22) with (2.20) yields (2.17). □

With the help of Lemma 2.1, we obtain
Lemma 2.3 Let \( x_0 \in \partial D \) and put \( y_0 = y(x_0) \in \partial \Omega \), where \( y(x_0) \) is given in Lemma 2.1 (see (iv) and (v)). Then

\[
\lim_{s \to +\infty} s \frac{N-1}{4} \int_{\partial B_R(x_0)} e^{-\sqrt{s(1+\varepsilon)}} d(x) \, dS_x = \left( \frac{2\pi}{\sqrt{1+\varepsilon}} \right)^{N-2} \left\{ \prod_{j=1}^{N-1} \left[ \frac{1}{R} - \kappa_j(y_0) \right] \right\}^{-\frac{1}{2}},
\]

where \( \kappa_j(y) \), \( j = 1, \ldots, N - 1 \) denotes the \( j \)-th principal curvature at \( y \in \partial \Omega \) of the analytic surface \( \partial \Omega \) with respect to the interior normal direction to \( \partial \Omega \).

**Proof.** In view of proposition (vi) of Lemma 2.1, in order to prove this lemma we can use Laplace's method (see [deB], p. 71 for example) or the stationary phase method (see [Ev], pp. 208 – 217 for example). See [MS 3] for details. \( \square \)

Combining Lemma 2.3 with Lemma 2.2 yields

Lemma 2.4 Let \( x_0 \in \partial D \) and put \( y_0 = y(x_0) \in \partial \Omega \), where \( y(x_0) \) is given in Lemma 2.1 (see (iv) and (v)). Then

\[
\lim_{s \to +\infty} s^{\frac{N-1}{4}} \int_{\partial B_R(x_0)} W(x, s) \, dS_x = (2\pi)^{\frac{N-1}{2}} \left\{ \prod_{j=1}^{N-1} \left[ \frac{1}{R} - \kappa_j(y_0) \right] \right\}^{-\frac{1}{2}}.
\]

The last lemma is

Lemma 2.5 We have

\[
\prod_{j=1}^{N-1} \left[ \frac{1}{R} - \kappa_j(y) \right] = \text{a constant} > 0, \text{ for every } y \in \partial \Omega,
\]

where \( \kappa_j(y) \), \( j = 1, \ldots, N - 1 \) denotes the \( j \)-th principal curvature at \( y \in \partial \Omega \) of the analytic surface \( \partial \Omega \) with respect to the interior normal direction to \( \partial \Omega \). In particular, if \( N = 2 \), \( \Omega \) must be a ball.

**Proof.** Let \( p \) and \( q \) be two distinct points in \( \partial \Omega \). Propositions (iv) and (v) from Lemma 2.1 guarantee that there exist two distinct points \( P, Q \) in \( \partial D \) such that \( p = y(P) \) and \( q = y(Q) \) in (iv).

For \( x \in B_R(0) \), consider the function

\[
v(x, t) = u(x + P, t) - u(x + Q, t).
\]
Then \( v = v(x, t) \) satisfies the heat equation in \( B_R(0) \times (0, +\infty) \) and by (1.4)

\[
v(0, t) = u(P, t) - u(Q, t) = 0,
\]

for every \( t > 0 \). Since \( v \) is continuous up to \( \partial B_R(0) \times (0, +\infty) \), by Theorem 1.2 (i) we obtain

\[
\int_{\partial B_R(0)} v(x, t) \, dS_x = 0
\]

for every \( t > 0 \), and hence

\[
\int_{\partial B_R(P)} u(x, t) \, dS_x = \int_{\partial B_R(Q)} u(x, t) \, dS_x
\]

for every \( t > 0 \). Therefore, in view of (2.1), we have

\[
\int_{\partial B_R(P)} W(x, s) \, dS_x = \int_{\partial B_R(Q)} W(x, s) \, dS_x
\]  \hspace{1cm} (2.27)

for every \( s > 0 \). With the help of Lemma 2.4, by multiplying both sides of (2.27) by \( s^{\frac{N-1}{4}} \), we can take the limits as \( s \to +\infty \). Therefore, since \( p = y(P) \) and \( q = y(Q) \), after some manipulation, we obtain:

\[
\prod_{j=1}^{N-1} \left[ \frac{1}{R} - \kappa_j(p) \right] = \prod_{j=1}^{N-1} \left[ \frac{1}{R} - \kappa_j(q) \right],
\]

that is, (2.25) holds. \( \square \)

We quote A.D. Aleksandrov's uniqueness theorem from [Alek], p. 412, adjusted to our notations. A special case of this theorem is the well-known Soap-Bubble Theorem (see also [R]).

**Theorem 2.6 (Aleksandrov)** Let \( \Phi = \Phi(\kappa_1, \cdots, \kappa_{N-1}) \) be a continuously differentiable function, defined for \( \kappa_1 \geq \cdots \geq \kappa_{N-1} \), and subject to the condition \( \frac{\partial \Phi}{\partial \kappa_i} > 0 \) (\( i = 1, \cdots, N - 1 \)).

Suppose that in \( \mathbb{R}^N \) we have a twice-differentiable closed surface \( S \) without self-intersections and with bounded principal curvatures.

If on the surface \( S \) the function \( \Phi \) of its principal curvatures \( \kappa_1, \cdots, \kappa_{N-1} \) has at all points one and the same value, then \( S \) is a sphere.
Proof of Theorem 1.1. By Lemma 2.5, it suffices to consider the case where $N \geq 3$. We set

$$
\Phi = \Phi(\kappa_1, \cdots, \kappa_{N-1}) = -\prod_{j=1}^{N-1} \left[ \frac{1}{R} - \kappa_j \right]
$$

(2.28)

and observe that

$$
\frac{\partial \Phi}{\partial \kappa_i} > 0 \quad (i = 1, \cdots, N-1), \quad \text{if} \quad \max_{1 \leq j \leq N-1} \kappa_j < \frac{1}{R}.
$$

Since condition (2.25) holds by Lemma 2.5, we infer that the function $\Phi$ is constant on $\partial \Omega$. Therefore, by applying Theorem 2.6 to each connected component of $\partial \Omega$, we conclude that $\partial \Omega$ must be a sphere. □

Remark. The method of proof of Theorem 2.6 is called Aleksandrov's reflection principle or the method of moving planes, which is based on the maximum principle for elliptic partial differential equations of second order. In fact, by using local coordinates, the condition $\Phi(\kappa_1, \ldots, \kappa_{N-1}) = \text{constant}$ on the surface $S$ can be converted into a second order partial differential equation which is of elliptic type, since $\frac{\partial \Phi}{\partial \kappa_i} > 0 \quad (i = 1, \cdots, N-1)$. In the case the function $\Phi$ is given by (2.28), we obtain an equation of Monge-Ampère type.

3 Concluding remarks

By the same method as in the proof of Theorem 1.1, we see that the following theorem also holds.

Theorem 3.1 Let $\Omega$ be an exterior domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the exterior sphere condition and suppose that $D$ is an exterior domain, with boundary $\partial D$, satisfying the interior cone condition, and such that $\overline{D} \subset \Omega$.

Assume that the solution $u$ to problem (1.1)-(1.3) satisfies the condition (1.4) for some function $a : (0, +\infty) \to (0, +\infty)$.

Then $\partial \Omega$ must be a sphere. That is, $\Omega$ must be the exterior of a ball.

Since both $\partial \Omega$ and $\partial D$ are compact, it follows from the barrier arguments with the help of Varadhan's result that inequality (2.17) holds for $x$ in an arbitrary bounded neighborhood of $\partial \Omega$ and for sufficiently large $s$. Therefore, we get the same relation
of the principal curvatures of $\partial \Omega$. Hence each connected component of $\partial \Omega$ is a sphere with the same radius. Moreover, by analyticity, $u(x, t)$ must be radially symmetric in $x$ with respect to each center of each connected component of $\partial \Omega$. Thus $\partial \Omega$ must be a sphere.

Professor Messoud A. Efendiev gave us the following conjecture:

Consider domains $\Omega$ whose boundary $\partial \Omega$ is not compact. In particular, let $\Omega$ be a unbounded domain above a Lipschitz graph $x_N = \varphi(x_1, \ldots, x_{N-1})$ over $\mathbb{R}^{N-1}$. Suppose that there exists an invariant isothermic surface. Then $\partial \Omega$ must be a hyperplane.

Our answer to this conjecture is the following theorem:

**Theorem 3.2** Let $\Omega$ be a unbounded domain above a locally Lipschitz graph $x_N = \varphi(x_1, \ldots, x_{N-1})$ over $\mathbb{R}^{N-1}$ such that

$$\nabla \varphi(x) = o(|x|^{\frac{1}{2}}) \text{ near infinity.} \quad (3.1)$$

Suppose that $\Omega$ satisfies the uniform exterior sphere condition, that is, there exists $r > 0$ such that for every $x \in \partial \Omega$ there exists a ball $B_r(z)$ with $\overline{B_r(z)} \cap \overline{\Omega} = \{x\}$. Assume that there exists a domain $D$ with $\overline{D} \subset \Omega$ such that the solution $u$ to problem (1.1)-(1.3) satisfies the condition (1.4) for some function $a : (0, +\infty) \rightarrow (0, +\infty)$.

Then $\partial \Omega$ must be a hyperplane.

With the help of curvature estimates in a Bernstein's theorem due to L. Caffarelli, L. Nirenberg, and J. Spruck (see Theorem 2" and its proof in [CNS]), we can prove this theorem. The details will be given in a forthcoming paper.

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