数理解析研究所講究録 1258 巻 2002 年 49-71

EXISTENCE AND LONGTIME BEHAVIOR OF SOLUTIONS OF A NONLINEAR REACTION-DIFFUSION SYSTEM ARISING IN THE MODELING OF BIOFILMS

M. A. EFENDIEV, H. J. EBERL, AND S. V. ZELIK

ABSTRACT. A nonlinear density-dependent system of diffusion-reaction equations describing the spatial spreading of biomass during the development of microbial films is analysed. It comprises two non-standard diffusion effects, degeneracy as in the porous medium equation and fast diffusion. The existence of a unique bounded solution and a global attractor is proved in dependence of the boundary conditions. This is achieved by studying an auxiliary approximating sequence of systems of non-degenerate evolution equations and the construction of a Lipschitz continuous semigroup by passing to the limit in the approximation parameter. Numerical examples are given that illustrate the main result of this paper.

INTRODUCTION.

Biofilms play a very important role in many scientific and technological areas. Consequently, they are studied in many disciplines and biofilm research is a truly interdisciplinary research topic. Biofilms are the most succesful life form on earth growing virtually everywhere, where nutrients are available to feed bacteria. In fact, most bacteria live in biofilm colonies and only a small minority appears as suspended planktonic organisms. Biofouling, biocorrosion, and bacterial infections are harmful impacts of biofilms. On the other hand, benefical properties of biofilms are used in enviromental engineering for wastewater treatment, groundwater protection, and soil remediation, where the sorption properties of microbial films play a key role in self-purification. The microorganisms in a biofilm are embedded in a polymeric matrix. This slime layer provides protection to the bacteria and vivid microbial communities can develop.

The first generation of mathematical models for biofilms was based on the assumption that biofilms develop in flat homogeneous layers and not much attention was brought to the actual biofilms structure. These models serve well for the purpose of engineering applications on the macro-scale, *i.e.* on the reactor level. However, they cannot be used to explain the sometimes highly irregular shape of microbial communities and the behavior of biofilms on the meso-scale, *i.e.* the biofilm itself. Since this first generation of biofilm models based on the seminal work [WG86] explicitly takes advantage of the one-dimensionality of the model setup (the biofilm can only grow perpendicularly to the substratum), a generalization of this approach to the spatially heterogeneous case is not possible. Therefore new model concepts became necessary. The big challenge in biofilm modelling is to describe the spatial spreading mechanism for biomass. The *a priori* postulations for most spatial biofilms, derived from experimental evidence, are:

(i) Spatial spreading of biomass does not take place if the biomass density is low but only if it is approaches a known maximum value.
(ii) The biomass density does not exceed this upper bound.
(iii) The model should be compatible with established biofilm reaction kinetics.
A characteristic property of biofilms is that new biomass is produced as long as there are nutrients available to feed on. Therefore, as a consequence of *(iii)*, the upper bound for biomass mass cannot stem from the kinetic reactions, but must be governed by the mechanism describing spatial spreading of biomass.

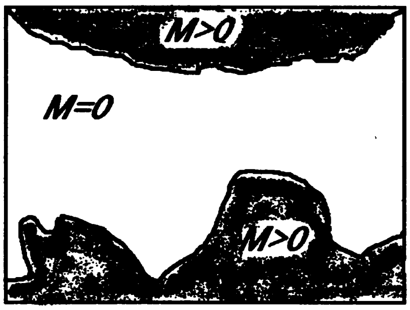

Fig. 1: Schematic sketch of the biofilm model setup. The actual biofilm is the biomass occupied region $\Omega_2(t) = \{x \in \Omega \mid M(t,x) > 0\}$ *the region* $\Omega_1(t) = \{x \in \Omega \mid M(t,x) = 0\}$ *without biomass represents the liquid region.*

A highly nonlinear reaction-diffusion model for the independent model variables biomass density and nutrient concentration was formulated in [EPL01]. The actual shape of the biofilm is the region $\Omega_2(t)$ where the biomass density is positive, while the region $\Omega_1(t)$ without biomass is the liquid region (see Figure 1.). In this mono-substrate/mono-species prototype model, biomass acts as a predator and nutrients are their prey. While the evolution equation for nutrients is a standard semi-linear reaction-diffusion equation, the spatial spreading mechanism for biomass shows two non-standard effects: the density-dependent biomass diffusion coefficient vanishes in the absence of biomass present and it has a singularity where the biomass density takes it maximum possible value. Thus, the model equation for bimoass formation comprises degeneracy as known from the porous medium equation, as well as fast diffusion. Both equations are coupled through reaction terms described by a Monod function.

In first *ad hoc* numerical simulations of the formation of three-dimensional biofilm structures, it was illustrated that this model obeys the above *a priori* postulations, and that its qualitative behavior with respect to the formation of rough or smooth biofilm surfaces agrees with experimental expectations.

The biofilm growth model suggested in [EPL01] reads:

$$
(0.1) \qquad \begin{cases} \partial_t S = d_1 \Delta_x S - K_1 \frac{SM}{K_4+S}, \quad x \in \Omega, \\ \partial_t M = d_2 \nabla_x \left(\frac{M^b}{(1-M)^a} \nabla_x M \right) - K_2 M + K_3 \frac{SM}{K_4+S} \\ S\big|_{\partial\Omega} = 1, \quad M\big|_{\partial\Omega} = 0, \\ S\big|_{t=0} = S_0, \quad M\big|_{t=0} = M_0, \end{cases}
$$

where $\Omega \subset\subset \mathbb{R}^n$ is a bounded domain with the piecewise smooth boundary $\Gamma = \partial\Omega$, Δ_x is the Laplacian with respect to spatial variables $x = (x^1, \cdots, x^n)$, ∇_x is the gradient/divergence operator, $S(t,x)$ is the unknown substrate concentration, $M(t,x)$ is the unknown biomass density and $d_1 > 0$, $d_2 > 0$, $K_1 \geq 0$, $K_2 \geq 0$, $K_3 \geq 0$, $K_4 > 0$, $a \geq 1$ and $b \geq 1$ are given constants with the following meaning:

d_1	substrate diffusion coefficient;
d_2	biomass diffusion coefficient;
K_1	maximum specific consumption rate;
K_2	biomass decay rate;
K_3	maximum specific growth rate;
K_4	Monod half saturation constant (relative to S_0);
a, b	biomass spreading parameters.

In this formulation, both dependent variables are dimensionless: S is scaled with respect to the bulk concentration and M is relative to the maximum biomass density. Therefore, it is required *a priori* that the initial data (S_0, M_0) satisfy

$$(0.2) \qquad S_0, M_0 \in L^\infty(\Omega), \quad 0 \leq S_0(x) \leq 1, \quad 0 \leq M_0(x) \leq 1, \quad x \in \Omega,$$

and seek for the solution $(S(t), M(t))$ of (0.1) satisfying

$$(0.3) \qquad S(t), L(t) \in L^\infty(\Omega), \quad 0 \leq S(t,x) \leq 1, \quad 0 \leq M(t,x) \leq 1.$$

In the present paper, we give a mathematical study of problem (0.1). In Section 1, an auxiliary non-degenerate second order parabolic system is introduced that approximates the original degenerate problem (0.1) and several useful uniform (with respect to the approximating parameter) estimates are derived for solutions of this problem. Then, in Section 2, we prove the existence of a solution of the original problem (0.1) satisfying (0.3) and we verify its uniqueness. Moreover, it is proven that (0.1) generates a Lipschitz continuous semigroup in the appropriate phase space and that this semigroup possesses a compact global attractor $\mathcal{A}$.

Further types of boundary conditions for the biomass concentration M are briefly considered in Section 3. In particular, we consider mixed homogeneous Dirichlet-Neumann boundary conditions, *i.e.* the boundary Γ is splitted into two parts Γ_D and Γ_N with Dirichlet boundary conditions for M imposed on Γ_D and Neumann boundary conditions on Γ_N. In the case where $\Gamma_D \neq \varnothing$, the situation occurs to be very closely related to the case of Dirichlet boundary conditions everywhere, and we obtain existence of a global solution of system (0.1), its uniqueness, and the existence of a global attractor $\mathcal{A}$ in the same way as for pure Dirichlet boundary conditions.

In contrast to this, the case of Neumann boundary conditions is more complicated: The diffusion mechanism appears to be not strong enough to preserve the *a priori* upper bound $M(t,x) \leq 1$ for the biomass concentration. In fact, in Section 3 we give a set of parameters d_1, d_2, K_1, K_2, K_3 and K_4 for which for every non-trivial initial data M_0 the corresponding solutions reach the singularity at $M \equiv 1$ in a finite time. Therefore, in this case there exist no global solutions of (0.1), except the trivial one $M \equiv 0$. This is an important result that corresponds to the case of a closed biofilm reactor if nutrients are not limited; eventually the entire reactor space will be filled due to the steady production of new biomass in the system.

Finally, some numerical illustrations of solutions of problem (0.1) are given in Section 4.

§1 AUXILIARY APPROXIMATION EQUATIONS AND UNIFORM A PRIORI ESTIMATES.

In this section, we approximate the degenerate problem (0.1) by a sequence of nondegenerate parabolic systems with smooth classical solutions and we derive several useful uniform (with respect to the approximating parameters) *a priori* estimates for the solutions of these auxiliary problems. In the next section, we will construct a Lipschitz continuous semigroup associated with the degenerate problem (0.1) by passing to the limit in the auxiliary approximation equations.

More precisely, for every $R > 1$, we consider the following regular second order parabolic system:

$$\begin{cases} \partial_t S = d_1 \Delta_x S - K_1 \frac{SM}{K_4+S}, \\ \partial_t M = d_2 \nabla_x (f_R(M) \nabla_x M) - K_2 M + K_3 \frac{SM}{K_4+S}, \\ S\big|_{\partial\Omega} = 1, \ \ S\big|_{t=0} = C_0, \ \ M\big|_{\partial\Omega} = 0, \ \ M\big|_{t=0} = M_0, \end{cases} \tag{1.1}$$

where the function $f_R(z)$ is defined by

$$f_R(z) := \begin{cases} (z+1/R)^b/(1-z)^a, & \text{if } z \le 1-1/R, \\ R^a, \ \ \text{if } z > 1-1/R. \end{cases} \tag{1.2}$$

In the sequel of this section we derive several uniform (with respect to R) estimates for solutions of (1.1) that are necessary for passing to the limit $R \to \infty$.

Proposition 1.1. *Let assumptions* (0.2) *hold. Then, for every* $t \ge 0$*, the solution* $(S(t), M(t))$ *of* (1.1) *satisfies*

$$0 \le S(t) \le 1, \ \ 0 \le M(t) \le 1 + CR^{-a}, \tag{1.3}$$

where R is large enough and C is independent of R. Moreover, the following estimate is valid:

$$\int_t^{t+1} \|\partial_t S(s)\|^2_{H^{-1}(\Omega)}\, ds + \frac{t}{t+1}\|S(t)\|^2_{H^1(\Omega)} \le C, \tag{1.4}$$

and if, in addition, $S_0 \in H^1(\Omega)$ *and* $S_0\big|_{\partial\Omega} = 1$*, then*

$$\|\partial_t S(t)\|^2_{H^{-1}(\Omega)} + \|S(t)\|^2_{H^1(\Omega)} \le C \tag{1.5}$$

where the constant C is independent of R.

Proof. First, we observe that second order parabolic system (1.1) is regular and, consequently, the existence of a solution can be proved in a classical way if the estimates for L^∞-norm are known *a priori*, see e.g. [LSU67]. Therefore, it only remains to obtain such estimates using a comparison principle.

It is obvious that $S(t)$ and $M(t)$ are nonnegative, and the upper bound $S(t) \le 1$ is an immediate corollary of the first equation of (1.1) by comparison with $S(t) \equiv 1$. In order to derive the upper bounds of (1.3) for the function $M(t)$ we introduce a barrier function

$$M_\nu(x) := 1 + \nu v(x), \tag{1.6}$$

where $v(x) \geq 0$ solves

$$\Delta_x v = -1, \quad v\big|_{\partial\Omega} = 0. \tag{1.7}$$

Then, obviously

$$1 \leq M_\nu(x) \leq 1 + C\nu, \tag{1.8}$$

where C is independent of ν. Furthermore, on one hand, we have

$$M_0(x) \leq M_\nu(x) \quad \text{and} \quad M(t)\big|_{\partial\Omega} \leq M_\nu\big|_{\partial\Omega}, \tag{1.9}$$

and on the other hand we obtain

$$\nabla_x(f_R(M_\nu)\nabla_x M_\nu) - K_2 M_\nu + K_3\frac{M_\nu S(t)}{K_4 + S(t)} \leq d_2 R^a \nu - K_2 + 2K_3 \leq 0 \tag{1.10}$$

if $R > 1$ is large enough and $\nu \sim R^{-a}$. Thus, applying the comparison principle to the second equation (1.1) and using (1.9) and (1.10), we prove that

$$M(t) \leq M_\nu(t) \leq 1 + CR^{-a},$$

which finishes the proof of (1.3). In order to verify (1.4) and (1.5), we rewrite the first equation of (1.1) as follows:

$$\partial_t S - \Delta_x S = h(t) := K_1\frac{M(t)S(t)}{K_4 + S(t)}, \quad S\big|_{\partial\Omega} = 1. \tag{1.11}$$

We note that (1.3) yields

$$\|h(t)\|_{L^\infty(\Omega)} \leq 2K_1, \quad t \in \mathbb{R}_+. \tag{1.12}$$

Applying now the standard L^2-regularity result to equation (1.11) and taking into account (1.12), we derive estimates (1.4) and (1.5). Hence, Proposition 1.1 is proved.

The next proposition provides the Lipschitz continuity in $L^1(\Omega)$-norm which is standard for non-degenerate second order parabolic equations (see e.g. [BrC79], [BeG95] and [FlS96]).

Proposition 1.2. *Let $(S_1(t), M_1(t))$ and $(S_2(t), M_2(t))$ be two solutions of* (1.1) *with initial data satisfying* (0.2). *Then, the following estimate is valid:*

$$\begin{aligned} &\|S_1(t) - S_2(t)\|_{L^1(\Omega)} + \|M_1(t) - M_2(t)\|_{L^1(\Omega)} \leq \\ &\leq e^{(K_1+K_2+K_3)t}\left(\|S_1(0) - S_2(0)\|_{L^1(\Omega)} + \|M_1(0) - M_2(0)\|_{L^1(\Omega)}\right). \end{aligned} \tag{1.13}$$

Proof. Substracting equations (1.1) for $S_2(t)$ and $M_2(t)$ from the corresponding equations (1.1) for $S_1(t)$ and $M_1(t)$), multiplying them by $\mathrm{sgn}(S_1(t) - S_2(t))$ and $\mathrm{sgn}(M_1(t) - M_2(t))$ respectively, summing the obtained equations and using the Kato inequality (see [CHA87]), we derive in a standard way (see the proof of Theorem 2.2 below) that

$$\begin{aligned} &\partial_t(\|S_1(t) - S_2(t)\|_{L^1(\Omega)} + \|M_1(t) - M_2(t)\|_{L^1(\Omega)}) \leq \\ &\leq (K_1 + K_2 + K_3)(\|S_1(t) - S_2(t)\|_{L^1(\Omega)} + \|M_1(t) - M_2(t)\|_{L^1(\Omega)}). \end{aligned} \tag{1.14}$$

Applying Gronwall's inequality to relation (1.14), we obtain (1.13) which proves Proposition 1.2.

The following proposition gives uniform (with respect to R) estimates for the function

$$F_R(M) := \int_0^M f_R(v)\,dv. \tag{1.15}$$

Proposition 1.3. *Let the above assumptions hold. Then, the following estimate is valid:*

$$\|F_R(M(T))\|^2_{H^1(\Omega)} + \int_T^{T+1} \left(|\partial_t M(t)|^2, f_R(M(t))\right) dt \le \le C\left(\|F_R(M(0))\|^2_{H^1(\Omega)} + 1\right), \tag{1.16}$$

where $F_R(M)$ is defined by (1.15) *and the constant C is independent of R and initial data (S_0, M_0) satisfying* (0.2).

Proof. Let us multiply the second equation of (1.1) by $F_R(M(t))$ and integrate over $x \in \Omega$. Then, taking into account inequalities (1.3) and the Friedrichs inequality, we have

$$\partial_t \left(\Phi_R(M(t)), 1\right) + 3\alpha\|F_R(M(t))\|^2_{H^1(\Omega)} \le C \tag{1.17}$$

where

$$\Phi_R(M) := \int_0^M F_R(v)\, dv, \tag{1.18}$$

and where positive constants C and α are independent of R.

Let us now multiply the second equation of (1.1) by $\partial_t F_R(M(t))$ and integrate over $x \in \Omega$. Then, after some obvious transformations, we have

$$\left(|\partial_t M(t)|^2, f_R(M(t))\right) + \partial_t \left[\frac{1}{2}\|F_R(M(t))\|^2_{H^1(\Omega)} + \right. \\ \left. + K_2\left(\int_0^{M(t)} v f_R(v)\, dv, 1\right) - K_3\left(\frac{S(t)}{K_4 + S(t)}, \int_0^{M(t)} v\, f_R(v)\, dv\right)\right] = \\ = K_4\left(\frac{\partial_t S(t)}{(K_4+S)^2}, \int_0^{M(t)} v\, f_R(v)\, dv\right) \le \alpha\|F_R(M(t))\|^2_{H^1(\Omega)} + C\|\partial_t S(t)\|^2_{H^{-1}(\Omega)}, \tag{1.19}$$

Summing now inequalities (1.17) and (1.19) and denoting

$$G(t) := \frac{1}{2}\|F_R(M(t))\|^2_{H^1(\Omega)} + K_2\left(\int_0^{M(t)} v f_R(v)\, dv, 1\right) - \\ - K_3\left(\frac{S(t)}{K_4 + S(t)}, \int_0^{M(t)} v\, f_R(v)\, dv\right) + \left(\Phi_R(M(t)), 1\right), \tag{1.20}$$

we derive that

$$\partial_t G(t) + \alpha G(t) + \alpha\|F_R(M(t))\|^2_{H^1(\Omega)} + \\ + \left(|\partial_t M(t)|^2, f_R(M(t))\right) \le H(t) := \alpha K_2\left(\int_0^{M(t)} v f_R(v)\, dv, 1\right) - \\ - \alpha K_3\left(\frac{S(t)}{K_4 + S(t)}, \int_0^{M(t)} v\, f_R(v)\, dv\right) + C\left(1 + \|\partial_t S(t)\|^2_{H^{-1}(\Omega)}\right). \tag{1.21}$$

Then, on one hand, using (1.3) and Schwartz' inequalty, we have

$$\frac{1}{4}\|F_R(M(t))\|^2_{H^1(\Omega)} - C_1 \le G(t) \le \|F_R(M(t))\|^2_{H^1(\Omega)} + C_1, \tag{1.22}$$

where C_1 is independent of R. On the other hand, one obtains analogously

$$H(t) \le \alpha\|F_R(M(t))\|^2_{H^1(\Omega)} + C_2(1 + \|\partial_t S(t)\|^2_{H^{-1}(\Omega)}), \tag{1.23}$$

where C_2 is also independent of R. Applying now Gronwall's inequality to relation (1.21) and using (1.22), (1.23), and (1.4), we finally derive estimate (1.16). This finishes the proof of Proposition 1.3.

Corollary 1.1. *Under the assumptions of Proposition 1.3 the following estimate holds:*

$$\|\partial_t M(t)\|^2_{H^{-1}(\Omega)} \le C\left(1 + \|F_R(M(0))\|^2_{H^1(\Omega)}\right), \tag{1.24}$$

where the constant C is independent of R.

This estimate (1.24) is an immediate corollary of (1.16) and the second equation of (1.1).

In conclusion we derive the uniform (with respect to R) smoothing property for solutions of (1.1).

Proposition 1.4. *Let the above assumptions hold. Then, there exist positive numbers $\kappa = \kappa(a)$ and $C = C(a)$, which are independent of R and of the initial data (S_0, M_0) satisfying* (0.2), *such that*

$$\|F_R(M(t))\|^2_{H^1(\Omega)} \le C\frac{t^\kappa + 1}{t^\kappa}, \quad t > 0. \tag{1.25}$$

Proof. Because of (1.16), it is sufficient to prove (1.25) for $t \le 1$ only. Multiplying now the second equation of (1.1) by $[F_R(M(t))]^\delta$, where $\delta > 0$ will be fixed below, denoting

$$\Phi_{\delta,R}(M) := \int_0^M [F_R(v)]^\delta \, dv \tag{1.26}$$

and taking into account (1.3), we obtain after standard calculations

$$\partial_t \Phi_{\delta,R}(M(t)) + \alpha\|F_R(M(t))^{\delta+1}\|_{L^1(\Omega)} + \alpha\|F_R(M(t))^{(\delta+1)/2}\|^2_{H^1(\Omega)} \le C, \tag{1.27}$$

where positive constants C and α are independent of R. Let us first set $\delta = \delta_0 := a^{-1}$. Then, as it is not difficult to verify,

$$\Phi_{\delta_0,R}(M_0) \le C_1 \tag{1.28}$$

is valid for every M_0 satisfying (0.2) and the constant C_1 is independent of R. Integrating now (1.27) by t and taking into account (1.28), we derive

$$\int_0^1 \|F_R(M(t))^{(\delta_0+1)/2}\|^2_{H^1(\Omega)} + \|F_R(M(t))^{\delta_0+1}\|_{L^1(\Omega)} \, dt \le C_2, \tag{1.29}$$

where the constant C_2 is independent of R.

We now assume that $\delta > \delta_0$. Then, multiplying (1.27) by t^N, integrating by t and using the obvious inequality

$$\Phi_{\delta_0,R}(M(t)) \le C[F_R(M(t))]^\delta,$$

we derive the reccurrent relation

$$\begin{aligned}(1.30)\quad \int_0^1 t^N \left(\|F_R(M(t))^{(\delta+1)/2}\|^2_{H^1(\Omega)} + \|F_R(M(t))\|^{\delta+1}_{L^{\delta+1}(\Omega)}\right) dt \le \\ \le C_3 \left(1 + \int_0^1 t^{N-1}\|F_R(M(t))\|^\delta_{L^\delta(\Omega)}\, dt\right).\end{aligned}$$

Starting with (1.29) and iterating estimate (1.30) if necessary, we have

$$(1.31)\qquad \int_0^1 t^{N_a}\|F_R(M(t))\|^2_{H^1(\Omega)}\, dt \le C_4,$$

where $N_a \in \mathbb{N}$ and $C_4 > 0$ are independent of R.

Now the proof of Proposition 1.4 can be finished. We note that, for every $t \in (0,1)$, (1.31) implies the existence of $T_0 \in [t/2, t]$ such that

$$(1.32)\qquad \|F_R(M(T_0))\|^2_{H^1(\Omega)} \le 2C_4 t^{-N_a-1}$$

Estimate (1.25) is now an immediate corollary of (1.32) and (1.16) (in which we replace the initial time $t = 0$ by $t = T_0$). Moreover, we can set $\kappa_a := (N_a + 1)$. Hence, Proposition 1.4 is proved.

The next proposition gives uniform (with respect to R) estimates for the $H^s(\Omega)$-norms of solutions $S(t)$, $M(t)$ of (1.1) for a sufficiently small positive s.

Proposition 1.5. *Let the above assumptions hold. Then, the following estimate is valid for solutions $(S(t), M(t))$ of problem (1.1):*

$$(1.33)\qquad \|M(t)\|_{H^s(\Omega)} \le C\left(\|F_R(M(t))\|_{H^1(\Omega)} + 1\right),$$

where $s < \frac{1}{b+1}$ and the constant C is independent on R.

Proof. It follows from the definition of $F_R(M)$ that

$$\begin{aligned}(1.34)\quad \|M(t)^{1+b}\|^2_{H^1(\Omega)} \le C\int_{x\in\Omega} M(t,x)^{2b}|\nabla_x M(t,x)|^2\, dx \le \\ \le C\int_{x\in\Omega} [f_R(M(t,x))]^2|\nabla_x M(t,x)|^2\, dx\, dt \le C\|F_R(M(t)\|^2_{H^1(\Omega)},\end{aligned}$$

where the constant C is independent of R. Since the L^∞-norm of $M(t)$ is uniformly bounded (due to (1.3)), estimate (1.33) is an immediate corollary of (1.34) and of the standard description of fractional order Sobolev spaces $H^s(\Omega)$ (see e.g. [Tri78] for details). Hence, Proposition 1.5 is proved.

Corollary 1.2. *Let assumption* (0.2) *hold. Then, the following estimate is valid for the solution* $(S(t), M(t))$ *of problem* (1.1)*:*

$$\|S(t)\|^2_{H^1(\Omega)} + \|M(t)\|^2_{H^s(\Omega)} \le C\frac{t^\kappa + 1}{t^\kappa}, \tag{1.35}$$

where $\kappa > 0$ *is the same as in Proposition 1.4,* $s < \frac{1}{b+1}$ *and the constant* C *is independent of* R.

Indeed, estimate (1.35) is an immediate corollary of estimates (1.4), (1.25) and (1.33).

Let us consider, in conclusion of this section, the case where the L^∞-norm of the initial data M_0 is separated from 1:

$$M_0(x) \le 1 - \delta, \quad \delta > 0 \tag{1.36}$$

The next proposition shows that (1.36) preserves under the temporal evolution governed by equations (1.1).

Proposition 1.6. *Let assumptions* (0.2) *and* (1.36) *hold. Then, there exists* $\mu = \mu(\delta) > 0$ *such that, for a sufficiently large* $R > 0$*, the following estimate is valid for the solution* $(S(t), M(t))$ *of* (1.1)*:*

$$0 \le M(t,x) \le 1 - \mu \tag{1.37}$$

Proof. We are going to apply the comparison principle to the second equation of (1.1). To this end, we rewrite this equation as follows:

$$\begin{cases} \partial_t M - d_2\Delta_x\left(F_R(M)\right) = h(t) := K_3\frac{S(t)M(t)}{K_4+S(t)} - K_2M(t), \\ M\big|_{\partial\Omega} = 0, \quad M\big|_{t=0} = M_0 \end{cases} \tag{1.38}$$

and note that with (1.3)

$$\|h(t)\|_{L^\infty(\Omega)} \le P, \tag{1.39}$$

for an appropriate positive constant P independent of R. We also observe that assumption (1.36) implies that

$$\|F_R(M_0)\|_{L^\infty(\Omega)} \le P_1, \tag{1.40}$$

where the constant P_1 depends on $\delta > 0$, but is independent of R.

Let us now introduce the function $V(x) \in L^\infty(\Omega) \cap H^1_0(\Omega)$ as a solution of the following elliptic boundary value problem:

$$-d_2\Delta_x V = P, \quad V\big|_{\partial\Omega} = P_1, \tag{1.41}$$

and, finally, we define the barrier function $M_\delta(t,x) = M_{\delta,R}(x)$ as follows:

$$M_\delta(t,x) := F_R^{-1}(V(x)). \tag{1.42}$$

Then, on the one hand, (1.40) and (1.41) imply that

$$(1.43)\qquad M(0,x)\le M_\delta(x),\quad M(t,x)\big|_{\partial\Omega}\le M_\delta(t,x)\big|_{\partial\Omega}$$

(since the function $F_R(z)$ is monotonic and $V(x)\ge P_1$) and, on the other hand, (1.39), (1.41) and (1.42) imply that

$$(1.44)\qquad \partial_t M_\delta(t,x)-d_2\Delta_x\left(F_R(M_\delta(t,x))\right)\le h(t,x)$$

Thus, according to the comparison principle

$$(1.45)\qquad M(t,x)\le M_\nu(t,x),\quad \text{for all}\quad (t,x)\in\mathbb{R}_+\times\Omega.$$

It remains to note that $\lim_{R\to\infty}F_R(1)=+\infty$ (here we have used the assumption that the exponent $a\ge 1$) and, consequently, the fact that $\|V\|_{L^\infty(\Omega)}<\infty$, together with (1.42), imply that there exists a positive constant $\mu=\mu(\delta)$ independent of R such that

$$(1.46)\qquad M_\delta(x)\le 1-\mu,$$

if R is large enough. Estimates (1.45) and (1.46) finish the proof of Proposition 1.6.

§2 Degenerate parabolic system: existence of solutions, their uniqueness and longtime behavior.

In this section, we establish existence and uniqueness of solutions of the degenerate parabolic system (0.1) and prove that the associated semigroup possesses a global attractor in the appropriate phase space. We start with an existence theorem for more smooth initial data.

Theorem 2.1. *Let the initial data (S_0,M_0) satisfy the conditions*

$$(2.1)\qquad \begin{cases} 1.\ S_0\in L^\infty(\Omega)\cap H^1(\Omega),\quad 0\le S_0(x)\le 1,\quad S_0\big|_{\partial\Omega}=1,\\ 2.\ M_0\in L^\infty(\Omega),\quad F(M_0)\in H^1_0(\Omega),\\ 3.\ M_0\ge 0,\quad \|M_0\|_{L^\infty(\Omega)}<1,\end{cases}$$

with

$$(2.2)\qquad F(u)=F_\infty(u):=\int_0^u\frac{v^b}{(1-v)^a}\,dv,\quad 0\le u<1.$$

Then, there exists a solution $(S(t),M(t))$ of problem (0.1) *(in the sense of distributions) belonging to the following class:*

$$(2.3)\qquad \begin{cases} 1.\ \ S,M\in L^\infty(\mathbb{R}_+\times\Omega)\cap C([0,\infty),L^2(\Omega)),\\ 2.\ \ S,F(M)\in L^\infty(\mathbb{R}_+,H^1(\Omega))\cap C([0,\infty),L^2(\Omega)),\\ 3.\ \ 0\le S(t,x),M(t,x)\le 1,\quad \|M\|_{L^\infty(\mathbb{R}_+\times\Omega)}<1.\end{cases}$$

Moreover, the following estimates hold:

$$(2.4)\quad \|S(t)\|^2_{H^1(\Omega)}+\|F(M(t))\|^2_{H^1(\Omega)}\le C\left(\|S(0)\|^2_{H^1(\Omega)}+\|F(M(0))\|^2_{H^1(\Omega)}+1\right)$$

$$(2.5)\quad \|S(t)\|^2_{H^1(\Omega)} + \|\partial_t S(t)\|^2_{H^{-1}(\Omega)} + \|F_R(M(t))\|^2_{H^1(\Omega)} + \\ + \|M(t)\|^2_{H^s(\Omega)} + \|\partial_t M(t)\|^2_{H^{-1}(\Omega)} \le C\frac{t^\kappa + 1}{t^\kappa}, \quad t > 0,$$

where the constants C and $\kappa \geq 1$ are independent of (C_0, M_0).

Proof. For every $R > 1$, we consider the solution $(S_R(t), M_R(t))$ of the auxiliary problem (1.1). Then, due to Propositions 1.1 and 1.6, we have

$$(2.6)\qquad 0 \le S_R(t,x), M_R(t,x) \le 1 \text{ and } \|M_R(t,x)\|_{L^\infty(\Omega)} < 1-\mu,$$

for some positive μ depending on $\|M_0\|_{L^\infty(\Omega)}$. Moreover, due to Propositions 1.1 , 1.3 and 1.5 and Corollary 1.1, the following estimate is valid:

$$(2.7)\quad \|S_R(t)\|^2_{H^1(\Omega)} + \|\partial_t S_R(t)\|^2_{H^{-1}(\Omega)} + \|F_R(M_R(t))\|^2_{H^1(\Omega)} + \\ + \|M_R(t)\|^2_{H^s(\Omega)} + \|\partial_t M_R(t)\|^2_{H^{-1}(\Omega)} \le C(1 + \|F(M_0)\|^2_{H^1(\Omega)} + \|S_0\|^2_{H^1(\Omega)}),$$

where $s < \frac{1}{b+1}$ and the constant C is independent of R. As usual (see e.g. [Dub65], [Lio69]), the uniform estimate (2.7) implies that there exist a sequence $R_n \to \infty$ and a pair of functions $(S(t), M(t))$ such that

$$(2.8)\qquad S_R \to S \text{ and } M_R \to M \text{ strongly in } C_{loc}(\mathbb{R}_+, L^2(\Omega)).$$

We claim that $(S(t), M(t))$ is a desired solution of (0.1). Indeed, in order to pass to the limit $R_n \to \infty$ (in the sense of distributions) in equations (1.1), it is sufficient to verify that,

$$(2.9)\qquad F_R(M_R(t)) \to F(M(t)) \text{ in } D'(\mathbb{R}_+ \times \Omega)$$

(passing to the limit $R_n \to \infty$ in the remaining terms is completely standard; the rigorous proof is omitted here). Since H^1-norms of $F_R(M_R(t))$ are uniformly bounded (due to estimate (2.7)) it is sufficient to verify that

$$(2.10)\qquad F_R(M_R(t,x)) \to F(M(t,x)) \text{ for almost all } (t,x) \in R_+ \times \Omega$$

(see [Lio69]). Let us prove (2.10). To this end, we recall that, because of (2.8), we may assume without loss of generality that

$$(2.11)\qquad M_R(t,x) \to M(t,x) \text{ for almost all } (t,x) \in \mathbb{R}_+ \times \Omega.$$

Then, splitting the difference $F_R(M_R) - F(M)$ as follows:

$$(2.12)\quad |F_R(M_R(t,x)) - F(M(t,x))| \le |F_R(M_R(t,x)) - F_R(M(t,x))| + \\ + |F_R(M(t,x)) - F(M(t,x))|$$

and taking into account that $0 \le M_R(t,x) \le 1-\mu$ and that the family of functions $F_R(z)$ is uniformly continuous on the interval $[0, 1-\mu]$, we derive that each term in the right-hand side of (2.12) tends to zero almost everywhere. Thus, we have proved (2.9) and, therefore, $(S(t), M(t))$ is indeed a solution of problem (0.1). The fact that this solution belongs to the class (2.3) and estimate (2.4) are immediate corollaries of (2.6) and (2.7). Estimate (2.5) is also an obvious corollary of (2.7) and the smoothing estimate (1.25). Hence, Theorem 2.1 is proved.

The next theorem establishes the uniform Lipschitz continuity of solutions of class (2.3) with respect to initial data.

Theorem 2.2. *Let* $(S_1(t), M_1(t))$ *and* $(S_2(t), M_2(t))$ *be two solutions of* (0.1) *belonging to the class* (2.3). *Then, the following estimate is valid:*

$$\begin{aligned}(2.13)\quad &\|S_1(t) - S_2(t)\|_{L^1(\Omega)} + \|M_1(t) - M_2(t)\|_{L^1(\Omega)} \le \\ &\le e^{(K_1+K_2+K_3)t} \left(\|S_1(0) - S_2(0)\|_{L^1(\Omega)} + \|M_1(0) - M_2(0)\|_{L^1(\Omega)}\right).\end{aligned}$$

In particular, the solution of (0.1) *is unique in the class* (2.3).

Proof. Let $U(t) := M_1(t) - M_2(t)$ and $V(t) := S_1(t) - S_2(t)$. Then, these functions satisfy the following equations

$$(2.14)\quad \begin{cases} \partial_t V - d_1\Delta_x V = h_1(t) := -K_1\frac{S_1(t)M_1(t)}{K_4+S_1(t)} + K_1\frac{S_2(t)M_2(t)}{K_4+S_2(t)}, \\ \partial_t U - d_2\Delta_x\left(l(t)U\right) = h_2(t) := K_3\frac{S_1(t)M_1(t)}{K_4+S_1(t)} - K_3\frac{S_2(t)M_2(t)}{K_4+S_2(t)} - K_2U(t), \end{cases}$$

where

$$(2.15)\qquad l(t,x) := \int_0^1 F'(sM_1(t,x) + (1-s)M_2(t,x))\,ds.$$

The derivation of estimate (2.13) is based on the following lemma.

Lemma 2.1. *Let the above assumptions hold. Then, the following estimates are valid for every* $T \ge 0$*:*

$$(2.16)\qquad \begin{cases} \|U(T)\|_{L^1(\Omega)} - \|U(0)\|_{L^1(\Omega)} \le \int_0^T |h_2(t,x)|\,dx\,dt, \\ \|V(T)\|_{L^1(\Omega)} - \|V(0)\|_{L^1(\Omega)} \le \int_0^T |h_1(t,x)|\,dx\,dt. \end{cases}$$

Although the assertion of Lemma 2.16 is more or less standard in the theory of second order degenerate parabolic equations (see *e.g.* [BrC79] or [BeG95]), we give the proof of the more complicated first estimate of (2.16) for the convenience of the reader. To this end, we need to consider the following auxiliary linear parabolic problem:

$$(2.17)\qquad \partial_t\phi = (d_2 l(T-t) + \varepsilon)\Delta_x\phi,\ \ \phi\big|_{\partial\Omega} = 0,\ \ \phi\big|_{t=0} = \phi_0,$$

where the function $l(t,x)$ is defined as above by (2.15), $\varepsilon > 0$ is a small regularising parameter, and $T > 0$ is another fixed parameter.

Lemma 2.2. *For every* $\phi_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$, *problem* (2.17) *possesses a unique solution* $\phi(t)$ *belonging to the class*

$$(2.18)\quad \phi \in L^\infty([0,T]\times\Omega) \cap L^\infty([0,T], H_0^1(\Omega)),\ \ \Delta_x\phi \in L^2([0,T]\times\Omega),\ \ T \in \mathbb{R}_+$$

and the following estimates are valid:

$$(2.19)\qquad \begin{cases} 1.\ \|\phi(t)\|_{L^\infty(\Omega)} \le \|\phi_0\|_{L^\infty(\Omega)}, \\ 2.\ \|\phi(t)\|^2_{H^1(\Omega)} + 2\varepsilon\int_0^t \|\Delta_x\phi(s)\|^2_{L^2(\Omega)}\,ds \le \|\phi_0\|^2_{H^1(\Omega)}. \end{cases}$$

Proof of Lemma 2.2. Since solutions (S_1, M_1) and (S_2, M_2) belong to the class (2.3), the function $l(t) = l(t,x)$ satisfies

$$(2.20)\qquad l \in L^\infty(\mathbb{R}_+\times\Omega)\ \text{ and }\ l(t,x) \ge 0.$$

Thus, (2.17) is a nondegenerate (since $\varepsilon > 0$) linear second order parabolic equation. Applying now the maximum principle to equation (2.17), we derive the first estimate of (2.19). Multiplying equation (2.17) by $\Delta_x\phi(t)$, integrating over (t,x), and taking (2.20) into account, we derive the second estimate of (2.19). The existence of a solution and its uniqueness can be verified in a standard way (based on a priori estimates (2.19), see [LSU67]). Hence, Lemma 2.2 is proved.

Proof of Lemma 2.1. Let $\phi_0 \in C_0^\infty(\Omega)$ be an arbitrary function, let $T>0$ and $\varepsilon>0$ be fixed parameters, and let $\phi(t)$ be a solution of problem (2.17). Multiplying now the second equation of (2.14) by $\phi_T(t) := \phi(T-t)$, integrating over $(t,x)\in[0,T]\times\Omega$ and integrating by parts, we have

$$\begin{aligned}(2.21)\quad (U(T),\phi_0) - (U(0),\phi(T)) + \\ +\int_0^T (\partial_t\phi(t) - (d_1 l(T-t)+\varepsilon)\Delta_x\phi(t), U(T-t))\,dt = \\ = \int_0^T (U(T-t),\varepsilon\Delta_x\phi(t))\,dt + \int_0^T (h_2(t),\phi(T-t))\,dt.\end{aligned}$$

Using now the first estimate of (2.19) and taking into account that $\phi(t)$ solves (2.17), we derive

$$\begin{aligned}(2.22)\quad (U(T),\phi_0) \le \|\phi_0\|_{L^\infty(\Omega)}\left(\|U(0)\|_{L^1(\Omega)} + \int_0^T \|h_2(t)\|_{L^1(\Omega)}\,dt\right) + \\ +\varepsilon^{1/2}\left(\int_0^T \|U(t)\|^2_{L^2(\Omega)} + \varepsilon\|\Delta_x\phi(t)\|^2_{L^2(\Omega)}\,dt\right).\end{aligned}$$

Passing now to the limit $\varepsilon\to 0^+$ in (2.22) and using the second estimate of (2.19), we obtain

$$(2.23)\qquad (U(T),\phi_0) \le \|\phi_0\|_{L^\infty(\Omega)}\left(\|U(0)\|_{L^1(\Omega)} + \int_0^T \|h_2(t)\|_{L^1(\Omega)}\,dt\right),$$

which is valid for every $\phi_0\in C_0^\infty(\Omega)$. Approximating now an arbitrary function $\phi_0\in L^\infty(\Omega)$ by a sequence $\phi_0^k\in C_0^\infty(\Omega)$ such that

$$\|\phi_0^k\|_{L^\infty(\Omega)} \le \|\phi_0\|_{L^\infty(\Omega)} \quad\text{and}\quad \lim_{k\to\infty}\|\phi_0-\phi_0^k\|_{L^1(\Omega)} = 0,$$

we derive that (2.23) is valid for every $\phi_0\in L^\infty(\Omega)$. Setting $\phi_0(x) := \operatorname{sgn} U(T,x)$ in (2.23), we obtain the first estimate of (2.16). The second estimate of (2.16) can be proved analogously. Hence, Lemma 2.2 is proved.

Now it is not difficult to complete the proof of Theorem 2.2. Indeed, since $0\le S_i(t), M_i(t)\le 1$, $i=1,2$, one obviously obtains

$$(2.24)\quad \|h_1(t)\|_{L^1(\Omega)} + \|h_2(t)\|_{L^1(\Omega)} \le (K_1+K_2+K_3)\left(\|U(t)\|_{L^1(\Omega)} + \|V(t)\|_{L^1(\Omega)}\right).$$

Summing now the first and the second estimate of (2.16) and using (2.24), we derive

$$\begin{aligned}(2.25)\quad \|U(T)\|_{L^1(\Omega)} + \|V(T)\|_{L^1(\Omega)} \le \|U(0)\|_{L^1(\Omega)} + \|V(0)\|_{L^1(\Omega)} + \\ +(K_1+K_2+K_3)\int_0^T \left(\|U(t)\|_{L^1(\Omega)} + \|V(t)\|_{L^1(\Omega)}\right)dt\end{aligned}$$

The proof of Theorem 2.2 is finished by applying Gronwall's inequality to (2.25).

Let us now set

$$\mathbb{V}_{smooth} := \{(S_0, M_0) \in L^\infty(\Omega) \times L^\infty(\Omega) \ : \ (S_0, M_0) \ \text{satisfies (2.1)}\}.$$

According to Theorems 2.1 and 2.2, equation (0.1) generates a uniformly Lipschitz continuous (in $L^1(\Omega) \times L^1(\Omega)$) semigroup on $\mathbb{V}_{smooth}$:

$$\mathcal{S}_t : \mathbb{V}_{smooth} \to \mathbb{V}_{smooth}, \ \ \mathcal{S}_t(S_0, M_0) := (S(t), M(t)), \tag{2.26}$$

where $(S(t), M(t))$ solves (0.1). This unique solution (in the class (2.3)) can be obtained via

$$S(t) = L^2(\Omega) - \lim_{R\to\infty} S_R(t), \ \ M(t) = L^2(\Omega) - \lim_{R\to\infty} M_R(t), \tag{2.27}$$

where $(S_R(t), M_R(t))$ is the corresponding solution of auxiliary problem (1.1).
With the additional definition

$$\mathbb{V} := \{(S_0, M_0) \in L^1(\Omega) \times L^1(\Omega) \ : (S_0, M_0) \ \text{satisfies (0.2)}\}. \tag{2.28}$$

one obviously obtains

$$\mathbb{V} = \Big[\mathbb{V}_{smooth}\Big]_{L^1(\Omega)\times L^1(\Omega)}, \tag{2.29}$$

where $[\cdot]_V$ denotes the closure in the space V. Therefore, due to estimate (2.13), semigroup (2.26) can be extended in a unique way to the semigroup $\mathcal{S}_t$ acting in the space $\mathbb{V}$ preserving the uniform Lipschitz continuity (2.13). This extension is given by the following expression:

$$\begin{aligned} \mathcal{S}_t(S_0, M_0) := L^1(\Omega) - \lim_{k\to\infty} \mathcal{S}_t(S_0^k, M_0^k), \ \ (S_0^k, M_0^k) \in \mathbb{V}_{smooth} \ \text{ and} \\ (S_0, M_0) = L^1(\Omega) \times L^1(\Omega) - \lim_{k\to\infty} (S_0^k, M_0^k). \end{aligned} \tag{2.30}$$

The next theorem shows that this extension also gives a solution of problem (0.1).

Theorem 2.3. *Let $(S_0, M_0) \in \mathbb{V}$ and let $(S(t), M(t)) := \mathcal{S}_t(S_0, M_0)$. Then,*

$$S, M \in L^\infty(\mathbb{R}_+ \times \Omega) \cap C([0,\infty), L^1(\Omega)). \tag{2.31}$$

Moroeover,

$$\text{mes}\{x \in \Omega \ : M(t,x) = 1\} = 0, \quad \text{for every } t > 0, \tag{2.32}$$

where mes$\{V\}$ *denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$,*

$$\begin{aligned} \|S(t)\|^2_{H^1(\Omega)} + \|\partial_t S(t)\|^2_{H^{-1}(\Omega)} + \|F(M(t))\|^2_{H^1(\Omega)}+ \\ + \|M(t)\|^2_{H^s(\Omega)} + \|\partial_t M(t)\|^2_{H^{-1}(\Omega)} \le C\frac{t^\kappa + 1}{t^\kappa}, \ \ t > 0, \end{aligned} \tag{2.33}$$

where $0 < s < \frac{1}{b+1}$, $\kappa > 0$, *and* $C > 0$ *are fixed constants which are independent of* $(S_0, M_0) \in \mathbb{V}$, *and the functions* $(S(t), M(t))$ *solve* (0.1) *in the sense of distributions.*

Proof. Let $(S_0^k, M_0^k) \in \mathbb{V}_{smooth}$ be an approximating sequence for (S_0, M_0) and let $(S^k(t), M^k(t))$ be the corresponding solutions of (0.1). Then, due to Theorem 2.1, $S^k, M^k \in C([0,\infty), L^1(\Omega))$. Consequently the $L^1(\Omega)$-limit functions $(S(t), M(t))$ also belong to this space. Hence, (2.31) is proved. In order to prove (2.32), we fix an arbitrary $t > 0$ and note that, due to (2.13)

$$\begin{aligned} (2.34) \quad & \|M^k(t) - M(t)\|_{L^1(\Omega)} + \|S^k(t) - S(t)\|_{L^1(\Omega)} \le \\ & \le e^{(K_1+K_2+K_3)t} \left(\|M_0^k - M_0\|_{L^1(\Omega)} + \|S_0^k - S_0\|_{L^1(\Omega)}\right) \to 0 \end{aligned}$$

as $k \to 0$. Therefore, without loss of generality, we may assume that

$$(2.35) \qquad M^k(t,x) \to M(t,x), \quad \text{for almost all } x \in \Omega.$$

Therefore, using (2.5) and the fact that $F(z)$ is monotonic, we derive that for every $\delta > 0$

$$(2.36) \qquad \text{mes}\{x : M^k(t,x) \ge 1-\delta\} \le C_t |F(1-\delta)|^{-2},$$

where C_t depends on $t > 0$, but is independent of k. We fix also an arbitrary continuous function $Q_\delta(z)$ such that $Q_\delta(z) = 0$, for $z \le 1 - 2\delta$, $Q_\delta(z) = 1$, for $z \ge 1 - \delta$, and $Q_\delta(z) \in [0,1]$, for $z \in [1-2\delta, 1-\delta]$. Then, according to Lebesgue's dominated convergence theorem, one has

$$\begin{aligned} (2.37) \quad & \text{mes}\{x : M(t,x) \ge 1-\delta\} \le \int_\Omega Q_\delta(M(t,x))\,dx = \\ & = \lim_{k\to\infty} \int_\Omega Q_\delta(M^k(t,x))\,dx \le \limsup_{k\to\infty} \text{mes}\{x : M^k(t,x) \ge 1-2\delta\} \le \\ & \le C_t |F(1-2\delta)|^{-2}. \end{aligned}$$

Passing to the limit $\delta \to 0^+$ in (2.37) and noting that $F(z) \to +\infty$ as $z \to 1^-$ and $M(t,x) \le 1$, we obtain (2.32). As a next step we verify that

$$(2.38) \qquad F(M^k(t,x)) \rightharpoonup F(M(t,x))) \text{ weakly in } H^1(\Omega).$$

Due to Theorem 2.1 the sequence $F(M^k(t,x))$ is uniformly (with respect to k) bounded in $H^1(\Omega)$. Therefore, it remains to prove that

$$(2.39) \qquad F(M^k(t,x)) \to F(M(t,x)), \text{ for almost all } x \in \Omega.$$

This, however, is an immediate corollary of (2.32) and (2.35) since $F \in C([0,1), R)$. Thus, convergence (2.38) is also verified.

(2.34) and (2.38) permit to pass to the limit $k \to \infty$ in equations (0.1) in a standard way and to derive that the limit functions $(S(t), M(t))$ satisfy (0.1) in the sense of distributions. Passing to the limit $k \to \infty$ in estimate (2.5) for solutions $(S^k(t), M^k(t))$, we finally derive estimate (2.33) which finishes the proof of Theorem 2.3.

Corollary 2.1. *Let $(S_0, M_0) \in \mathbb{V}$ and let $(S_R(t), M_R(t))$ be the corresponding solution of the auxiliary problem* (1.1). *Then, the solution $(S(t), M(t)) := \mathcal{S}_t(S_0, M_0)$ of problem* (0.1) *can be found by*

$$S(t) = L^1(\Omega) - \lim_{R\to\infty} S_R(t), \quad M(t) = L^1(\Omega) - \lim_{R\to\infty} M_R(t). \tag{2.40}$$

In other words, for every $(S_0, M_0) \in \mathbb{V}$ solutions of auxiliary problems (1.1) *converge to the corresponding solution of* (0.1) *which is constructed in Theorem 2.3.*

Proof. Let $(S_0^k, M_0^k) \in \mathbb{V}_{smooth}$ be an approximating sequence for (S_0, M_0) and let $(S^k(t), M^k(t))$ and $(S_R^k(t), M_R^k(t))$ be the corresponding solutions of (1.1) and (0.1), respectively. Then, we split the difference between $(S(t), M(t))$ and $(S_R(t), M_R(t))$ as follows:

$$\begin{aligned} \|S(t) - S_R(t)\|_{L^1(\Omega)} + \|M(t) - M_R(t)\|_{L^1(\Omega)} \le \\ \le [\|S(t) - S^k(t)\|_{L^1(\Omega)} + \|M(t) - M^k(t)\|_{L^1(\Omega)}] + \\ + [\|S_R(t) - S_R^k(t)\|_{L^1(\Omega)} + \|M_R(t) - M_R^k(t)\|_{L^1(\Omega)}] + \\ + [\|S_R^k(t) - S^k(t)\|_{L^1(\Omega)} + \|M_R^k(t) - M^k(t)\|_{L^1(\Omega)}] =: I_1 + I_2 + I_3. \end{aligned} \tag{2.41}$$

Because of (1.13) and (2.13), for every $\varepsilon > 0$ we may fix $k = k(\varepsilon)$ such that

$$I_1 + I_2 \le \varepsilon,$$

for every $R > 1$. Due to (2.27), for a fixed $k = k(\varepsilon)$, we may find $R = R(\varepsilon)$ such that $I_3 \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, Corollary 2.1 is proved.

We conclude this section by constructing the global attractor for the semigroup $\mathcal{S}_t$ associated with equation (0.1) in the space $\mathbb{V}$ endowed by the $L^1(\Omega)$-topology. For the convenience of the reader, we recall that a set $\mathcal{A}$ is called a global attractor of the semigroup $\mathcal{S}_t$ if the following conditions are satisfied:

1. The set $\mathcal{A} \subset \mathbb{V}$ is compact in $L^1(\Omega)$.
2. The set $\mathcal{A}$ is strictly invariant: $\mathcal{S}_t\mathcal{A} = \mathcal{A}$.
3. The set $\mathcal{A}$ is an attracting set for $\mathcal{S}_t$, *i.e.* for every neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in the $L^1(\Omega)$-topology there exists $T = T(\mathcal{O})$ such that

$$\mathcal{S}_t\mathbb{V} \subset \mathcal{O}(\mathcal{A}), \quad \text{for all } t \ge T \tag{2.42}$$

(here we adopted the classical definition of an attractor to the case where the phase space is initially bounded, see [BaV92] and [Tem88] for details).

Theorem 2.4. *The semigroup $\mathcal{S}_t$ associated with equation* (0.1) *possesses a global attractor $\mathcal{A}$ in $\mathbb{V}$ endowed by the $L^1(\Omega)$-topology.*

Proof. According to the attractor's existence theorem for abstract semigroups, see [BaV92], we have to verify the following two properties:

1. The operators $\mathcal{S}_t$ are continuous with respect to initial data, for every fixed $t \ge 0$.
2. The semigroup $\mathcal{S}_t$ possesses a compact (in $L^1(\Omega)$-topology) attracting set.

In our case, the first condition is obviously satisfied due to (2.13). Moreover, due to estimate (2.33), the set

$$B_L := \{(S_0, M_0) \in \mathbb{V} : \|S_0\|^2_{H^1(\Omega)} + \|M_0\|^2_{H^s(\Omega)} \le L^2\}, \tag{2.43}$$

where $0 < s < \frac{1}{b+1}$ is fixed, is a compact attracting (and even absorbing) set for $\mathcal{S}_t$ if L is large enough. Thus, the second condition is verified as well. Hence the semigroup $\mathcal{S}_t$ associated with equation (0.1) possesses a global attractor $\mathcal{A}$ in $L^1(\Omega)$-topology. Theorem 2.4 is proved.

§3 Other boundary conditions.

In this section, we briefly consider the further sets of boundary conditions for biomass density M at $\Gamma = \partial\Omega$. To be more precise, we assume that there is a splitting of the boundary Γ into two piecewise smooth submanifolds $\Gamma_D \subset \Gamma$ and $\Gamma_N \subset \Gamma$ such that

$$\Gamma = \Gamma_N \cup \Gamma_D \cup \partial\Gamma, \qquad \partial\Gamma = \partial\Gamma_D = \partial\Gamma_N, \quad \Gamma_N \cap \Gamma_D = \varnothing. \tag{3.1}$$

Then, we impose the Dirichlet boundary conditions on the part Γ_D of the boundary $\partial\Omega$ and Neumann boundary conditions on Γ_N:

$$M\big|_{\Gamma_D} = 0, \ \partial_n M\big|_{\Gamma_N} = 0. \tag{3.2}$$

There are two basically different cases to consider:

1. $\Gamma_D \neq \varnothing$ which corresponds to Dirichlet boundary conditions if $\Gamma_N = \varnothing$ or mixed Dirichlet-Neumann boundary conditions if $\Gamma_N \neq \varnothing$.
2. $\Gamma_D = \varnothing$ corrresponding to pure Neumann boundary conditions.

In the first case the first eigenvalue of the Laplacian $-\Delta_x$ in Ω with boundary conditions (3.2) is strictly positive and, consequently, repeating word by word the corresponding proofs given above for the case with pure Dirichlet conditions, we extend all results of Sections 1 and 2 to the case of mixed Dirichlet-Neumann boundary conditions.

In particular, the following result is obtained.

Theorem 3.1. *Let $\Gamma_D \neq \varnothing$. Then problem* (0.1) *with Dirichlet boundary conditions $S\big|_{\partial\Omega} = 1$ for the nutrient concentration and mixed boundary conditions* (3.2) *for the biomass concentration M generates a semigroup $S_t : \mathbb{V} \to \mathbb{V}$ in the phase space $\mathbb{V}$ defined by* (2.28) *that is uniformly Lipschitz continuous in the $L^1(\Omega)$-topology and possesses a global attractor $\mathcal{A}$ in the $L^1(\Omega)$-topology.*

In contrast to this, in the second case of pure Neumann boundary conditions ($\Gamma_D = \varnothing$), the first eigenvalue of the Laplacian is equal to zero and, consequently, there are no reasons to expect that for all initial data $(S_0, M_0) \in \mathbb{V}$

$$M(t,x) < 1, \quad \text{for almost all } (t,x), \tag{3.3}$$

In the sequel, we show that (3.3) is indeed violated for an appropriate choice of initial data and parameters K_1, K_2, K_3, d_1 and d_2. Thus, in contrast to the case 1, for Neumann boundary conditions, the biomass concentration may reach the singular point at $M(T) \equiv 1$ in finite time (and we have the so-called quenching phenomenon). In order to show this, we integrate the second equation of (0.1) with respect to $x \in \Omega$. Then, we have

$$\partial_t \langle M(t) \rangle = \frac{1}{|\Omega|}\int_\Omega \left(K_3\frac{S(t,x)}{K_4 + S(t,x)} - K_2\right) M(t,x)\,dx \geq \theta(S(t))\,\langle M(t)\rangle\,, \tag{3.4}$$

where

$$\theta(S(t)) := \inf_{x\in\Omega}\left(K_3\frac{S(t,x)}{K_4 + S(t,x)} - K_2\right), \quad \langle M(t)\rangle := \frac{1}{|\Omega|}\int_{x\in\Omega} M(t)\,dx.$$

$$\langle M(t)\rangle \geq \langle M(0)\rangle\, e^{\int_0^t \theta(S(s))\,ds}. \tag{3.5}$$

From the other side, we derive from the first equation of (0.1) and from the maximum principle that

$$S(t) \geq \widehat{S}(t), \tag{3.6}$$

where $\widehat{S}(t)$ is a solution of the following equation

$$\partial_t \widehat{S} = d_1 \Delta_x \widehat{S} - \frac{K_1}{K_4+1}\widehat{S}, \quad \widehat{S}\big|_{\partial\Omega} = 1, \quad \widehat{S}(0) = S_0. \tag{3.7}$$

We assume now that

$$\frac{K_3}{1+K_4} - K_2 > 0. \tag{3.8}$$

Otherwise, (3.5) implies that the value $\langle M(t)\rangle$ tends exponentially to zero and, consequently, the biomass amount decays exponentially with respect to time. It is not difficult to see that in this case all solutions of (0.1) with (S_0, M_0) exist globally in time and the quenching phenomenon does not occur.

The next proposition shows that this is not the case if (3.8) is satisfied.

Proposition 3.1. *Let* (3.8) *be satisfied. Then, there exist the initial data* (S_0, M_0) *belonging to* $\mathbb{V}_0$ *such that*

$$0 \leq S_0 < 1, \quad \textit{and} \quad 0 \leq M_0 < 1$$

and the corresponding solution $(S(t), M(t))$ *quenches in finite time, i.e. there exists* $T = T(S_0, M_0)$ *such that*

$$\langle M(t)\rangle < 1, \quad \textit{for } t < T, \textit{ and } \lim_{t\to T^-} \langle M(t)\rangle = 1. \tag{3.9}$$

Proof. Applying the maximum principle to (3.7) and using (3.6) yields

$$S(t) \geq \inf_{x\in\Omega}\{S_0(x)\} e^{-\frac{K_1 t}{K_4+1}}. \tag{3.10}$$

We also note that $\theta(1) > 0$ because of (3.8). Therefore, according to (3.10) there exist $T > 0$ and $0 < S_0 < 1$ (which is sufficiently close to 1 in the L^∞-norm) such that

$$\theta(S(t)) > \delta_0 > 0, \quad \text{for all } t \leq T, \tag{3.11}$$

for a sufficiently small positve δ_0. Estimate (3.5) now yields

$$\langle M(t)\rangle \geq e^{\delta_0 t}\langle M_0\rangle, \quad 0 \leq t \leq T. \tag{3.12}$$

Estimate (3.12) shows that, for all initial data M_0 with $\langle M_0\rangle$ sufficiently close to 1, quenching occurs in finite time. Hence Proposition 3.1 is proved.

Moreover, the next Proposition shows that there exist positive coefficients K_1, K_2, K_3, K_4, d_1, and d_2 such that any solution of (0.1) with Neumann boundary conditions for M quenches in finite time.

Poposition 3.2. *Let $\bar{S}(x)$ be a solution of the following elliptic boundary problem:*

$$d_1\Delta_x\bar{S}(x) = \frac{K_1}{K_4+1}\bar{S}(x), \quad \bar{S}|_{\partial\Omega} = 1. \tag{3.13}$$

Let also

$$\inf_{x\in\Omega}\left(K_3\frac{\bar{S}(x)}{K_4+\bar{S}(x)} - K_2\right) > 0. \tag{3.14}$$

Then for every initial data $(S_0, M_0) \in \mathbb{V}$ with $\langle M_0\rangle \neq 0$, the solution $(S(t), M(t))$ of problem (0.1) with Neumann boundary conditions for M quenches in finite time, i.e. (3.9) is satisfied for some $T = T(S_0, M_0) < \infty$.

Proof. According to the maximum principle

$$\|\widehat{S}(t) - \bar{S}\|_{L^\infty(\Omega)} \leq \|S_0 - \bar{S}\|_{L^\infty(\Omega)}e^{-\delta_0 t},$$

holds for some positive constant δ_0. Therefore, if (3.14) is satisfied, then

$$\theta(S(t)) \geq \theta(\widehat{S}(t)) > \delta_1 > 0, \quad \text{for all } t \geq T_0, \tag{3.15}$$

where $T_0 = T_0(S_0)$ is an appropriate time. With estimates (3.15) and (3.5) the proof of Proposition 3.2 is finished.

Remark 3.1. It can be proven (analogously to Theorem 2.1) that condition (3.9) determines the existence interval for the solution $(S(t), M(t))$, namely, that problem (0.1) with Neumann boundary conditions for M is locally solvable for every $(S_0, M_0) \in \mathbb{V}$ with $\langle M_0\rangle < 1$.

Remark 3.2. We finally note that the solution $\bar{S}(x)$ of problem (3.13) tends to 1 in the $L^\infty(\Omega)$-norm if $\frac{K_1}{d_1} \to 0$. Consequently, if (3.8) holds then condition (3.14) is also valid for a sufficiently small $\frac{K_1}{d_1}$.

§4 Some illustrations of model behavior

Some numerical simulations are shown in order to illustrate the results obtained above. For simplicity of the visual presentation we restrict ourselves to the one-dimensional case. That is, we consider the interval $\Omega = [0, L]$. The initial biomass seed M_0 varies for the different cases and so do the boundary conditions. For the sake of comparability, the same model parameters were chosen in all three examples. In all three examples, the biomass density does not exceed the upper bound as proven in section 2 and 3. For some fully three-dimensional simulations we refer to [EPL01].

Since basic model equation (0.1) stems from spatio-temporal biofilm modeling, the examples presented here are chosen accordingly. The following notation is introduced for this section:

$$\begin{cases} \Omega_1(t) := \{x \in \Omega \mid M(t,x) = 0\} \\ \Omega_2(t) := \{x \in \Omega \mid M(t,x) > 0\} \end{cases} \tag{4.1}$$

That is, $\Omega_2(t)$ describes the actual biofilm structure and $\Omega_1(t)$ describes the liquid region of Ω. $\Omega_2(0)$ describes the initial seed of biomass. We show three different *(a) Development of a regular homogeneous biofilm structure:* As a first examp we consider $\Omega_2(0)$ symmetric around the center of the interval. On both sides tv closed intervals are specified:

(4.2a) $\Omega_2(0) = [0.3L, 0.35L] \cup [0.425L, 0.475L], \cup[0.525L, 0.575L] \cup [0.65L, 0.7L]$

The initial conditions are

(4.2b) $S(0,x) = S_0(x) = 1, \quad \text{and} \quad M(0,x) = M_0(x) = \begin{cases} 0 & \text{for} \quad x \in \Omega_1(0) \\ 0.87 & \text{for} \quad x \in \Omega_2(0) \end{cases}$

Symmetric boundary conditions are specified for the dissolved substrate

(4.2c) $$S(t,0) = S(t,L) = 1$$

and the boundary conditions for biomass read

(4.2d) $$\frac{\partial M}{\partial x}\,|_{x=0}= 0, \quad M(t,L) = 0$$

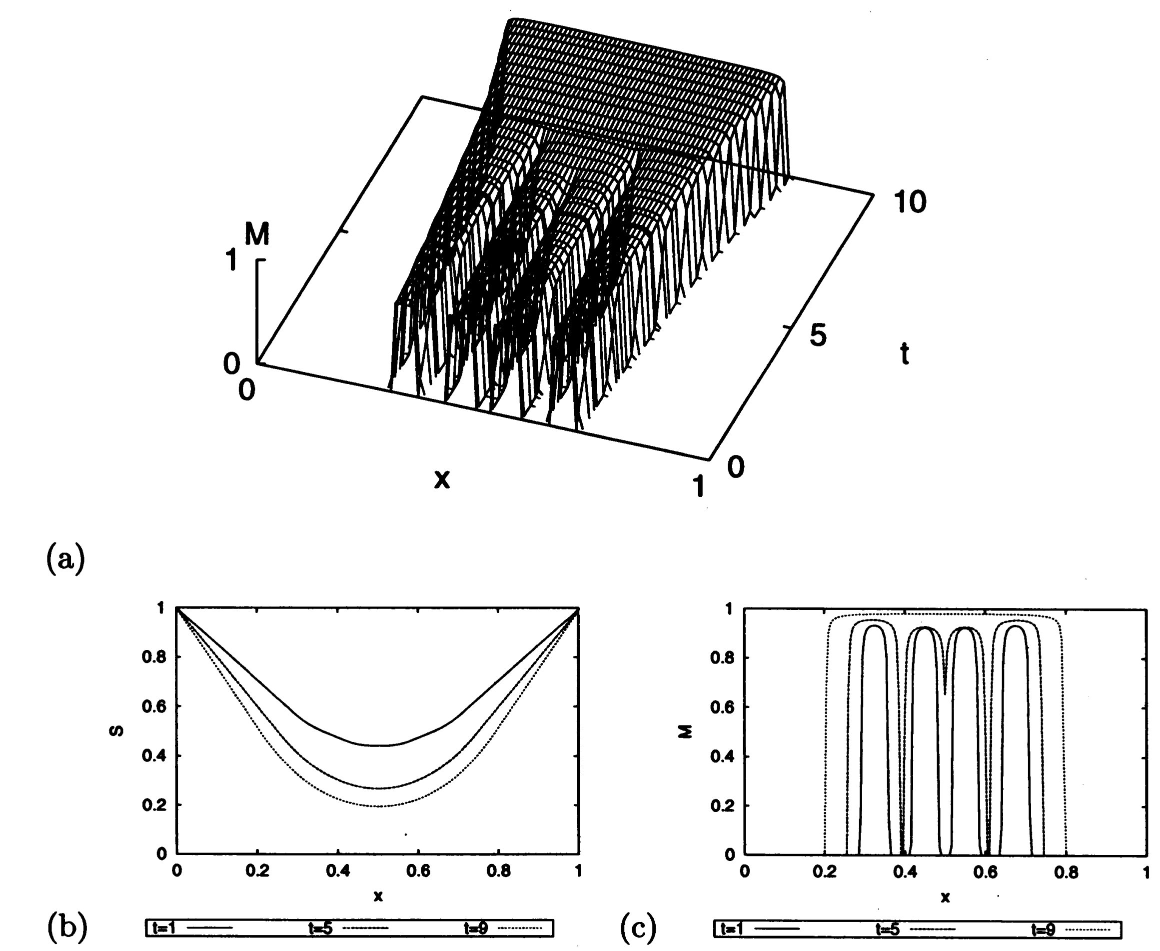

Fig. 2: Development of a symmetric solution under symmetric initial bioma seed and symmetric boundary conditions for nutrients: (a) evolution of M in tim and cut through the system for three different values of t for (b) S and (c) M.

Simulations of this scenario are shown in Figure 2. It can be seen that the developing biofilm structure is symmetric. Nutrients are nowhere limited in the system and all colonies grow. Eventually, all the colonies merge and form a homogeneous structure. The observed quick decline of C is due to the different characteristic time-scales of nutrient transport/conversion and biomass production in biofilm systems (cf. [KMS84]).

(b) Development of a spatially irregular biofilm structure due to spatially heterogeneous initial biomass seed: In the second case, we disturb the symmetry in the initial seed and consider

$$\Omega_2(0) = [0.25L, 0.3L] \cup [0.4L, 0.45L] \cup [0.5L, 0.55L] \cup [0.65L, 0.7L] \tag{4.2a'}$$

instead. The initial and boundary conditions (4.2b,c,d) are chosen as in (a).

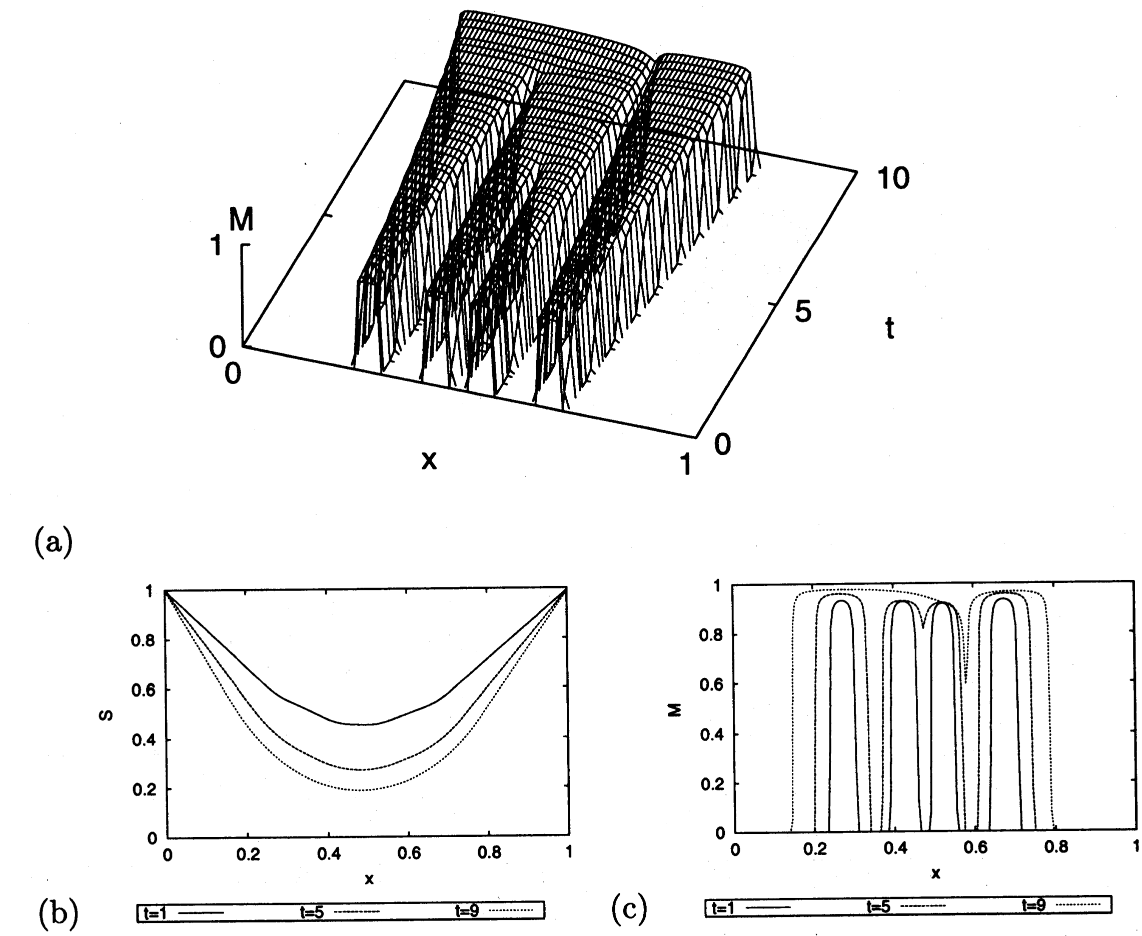

Fig. 3: Development of a spatially heterogeneous solution under asymmetric initial biomass seed and symmetric boundary conditions for nutrients: (a) evolution of M in time and cut through the system for three different values t for (b) S and (c) M.

The simulation results are shown in Figure 3. The symmetry in the solution of (1) around the center of the domain $\Omega = [0, L]$ is now disturbed due to the irregularity of the initial seed. Colonies merge the earlier the closer they have been to eachother at initial time.

c) Development of a spatially irregular biofilm structure due to spatially heterogeneous nutrient supply: As a last example we show the development of a biofilm under non-symmetric boundary conditions for S, mimicking spatially hetreogeneous nutrient availability. The initial seed and initial conditions (4.2.a'), (4.2.b), as well as boundary conditions (4.2d) for M are chosen as in the previous example. The boundary conditions for S are now

(4.2c") $$\frac{\partial S}{\partial x}\Big|_{x=0} = 0, \quad S(t, L) = 0$$

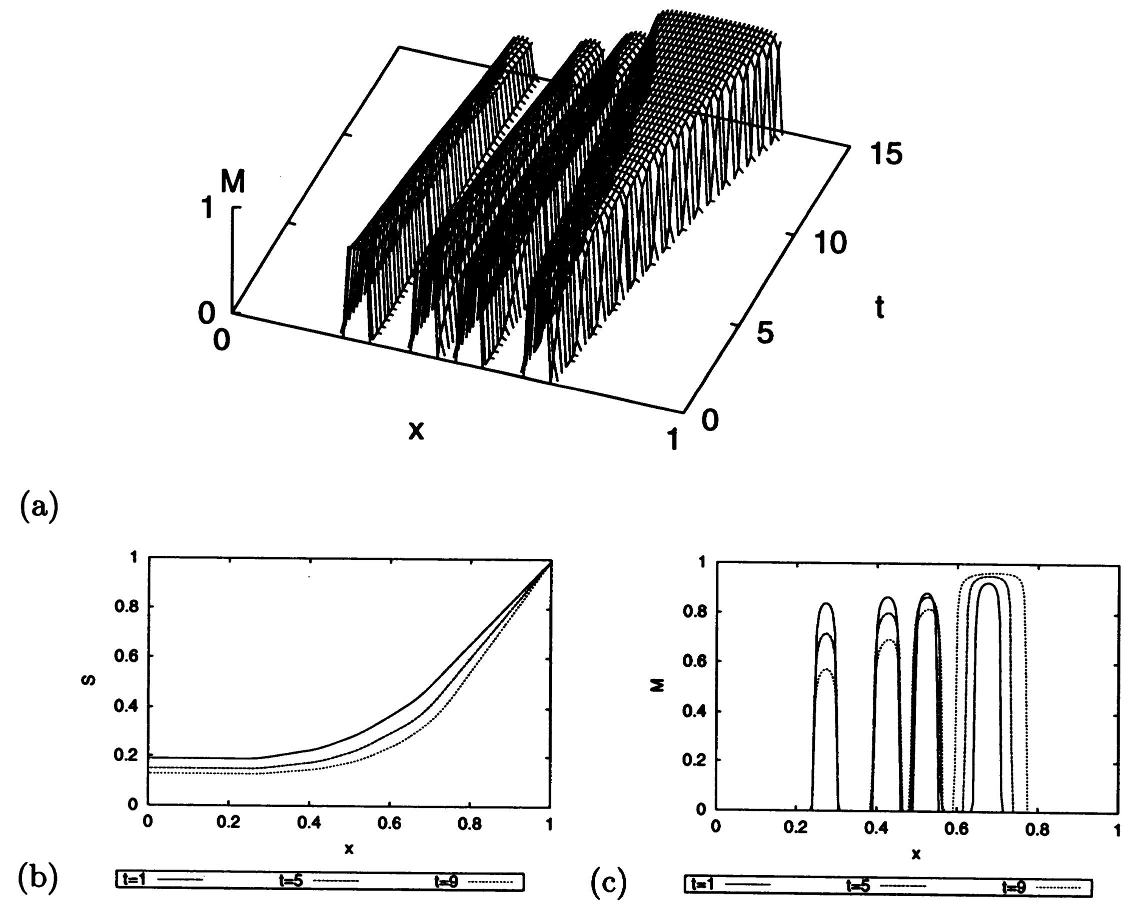

Fig. 4: Development of a spatially heterogeneous solution under asymmetric initial biomass seed and asymmetric boundary conditions for nutrients: (a) evolution of M in time and cut through the system for three different values t for (b) S and (c) M.

Simulation results in Figure 4 show a very different qualitative behaviour of biofilm development compared to the previous cases: The colony closest to the nutrient source at $x = L$ grows fastest and does not leave enough nutrients for

S leeward colonies, where S becomes rate limiting. These colonies grow slower or decay due to $\frac{K_3 S}{K_2+S} - K_4 < 0$. The biggest colony grows into the direction of the source and no merging takes place. The evolving biofilm structure remains heterogeneous throughout all t. In fully three-dimensional simulations this can lead eventually to the formation of mushroom-shaped biofilm architectures.

§5 Conclusion

In this paper existence and longtime behavior of solutions of a highly nonlinear reaction-diffusion system arising in biofilm modelling have been studied. The results obtained here confirm by rigorous mathematical analysis important model features that have been so far investigated only by some *ad hoc* numerical simulations. In particular, it could be shown that the global existence (in time) of the model solution depends on the boundary conditions specified for biomass in the same way as expected from laboratory experiments. The most important model property that could be proved is that the local biomass density obeys an upper bound. Based on the mathematical analysis of the prototype biofilm model presented here, further biofilm processes can be studied. This will be published in future articles.

References

[BaV92] A. Babin and M. Vishik, *Attractors of Evolutionary Equations*, North-Holland, Amsterdam, 1992.

[BeG95] P. Benilan and R. Gariepy, *Strong Solutions in L^1 of Degenerate Parabolic Equations*, J. Diff. Eqns. **199** (1995), 473–502.

[BrC79] H. Brezis and M. Crandal, *Uniqueness of Solutions of the Initial Data Problem for $u_t - \Delta\phi(u) = 0$*, J. Math. Pure Appl. **58** (1979), 153–163.

[CHA87] Ph. Clement, H. Heijmans, S. Angenent, C. van Duijn and B. de Paqter, *One Parametrical Semigroups*, North-Holland, Amsterdam, 1987.

[Dub65] Dubinskij Yu., *Weak convergence for nonlinear elliptic and parabolic problems*, Mat. Sb. **67(109)** (1965), 609–642.

[EPL01] H. Eberl, D. Parker, and M. van Loosdrecht, *A New Determinstic Spatio-Temporal Continuum Model For Biofilm Development*, J. Theor. Medicine **3(3)** (2001), 161–176.

[FLS96] E. Feireisl, P. Laurencot and F. Simodon, *Global Attractors for Degenerate Parabolic Equations in Unbounded Domains*, J. Diff. Eqns. **129(2)** (1996), 239–261.

[KMS84] J.C. Kissel, P.L. McCarty, and R.L. Street, *Numerical Simulation of Mixed-Culture Biofilms*, J. Env. Eng. **110(2)** (1984), 393–411.

[LSU67] O. Ladyzhenskaya, O. Solonnikov and N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, Translated from Russian, AMS, Providence, RI, 1967.

[Lio69] J. Lions, *Quelques methodes de resolution des problemes aux limites non linearies*, Dunod, Paris, 1969.

[Tem88] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New-York, 1988.

[Tri78] H. Triebel, *Interpolation theory, functional spaces, differential operators*, North-Holland, Amsterdam, 1978.

[WG86] O. Wanner and W. Gujer, A Multispecies Biofilm Model **28** (1986), 314–328.