<table>
<thead>
<tr>
<th>Title</th>
<th>Inequalities Involving Unitarily Invariant Norms and Operator Monotone Functions (Current topics on operator theory and operator inequalities)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Zhan, Xingzhi; Hiai, F.</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2002), 1259: 66-70</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41967">http://hdl.handle.net/2433/41967</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Inequalities Involving Unitarily Invariant Norms and Operator Monotone Functions

X. Zhan (Peking Univ. & Tohoku Univ.)

Joint work with F. Hiai (Tohoku Univ.)

We consider square complex matrices. A norm $\| \cdot \|$ on the space of $n \times n$ matrices is called unitarily invariant if

$$\|UAV\| = \|A\| \quad \forall A, \forall \text{unitary } U, V.$$ 

Such a norm is determined by a symmetric gauge function $\Phi$ on $\mathbb{R}^n$:

$$\|A\| = \Phi(s_1(A), \ldots, s_n(A))$$

where $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of $A$, that is, the eigenvalues of $|A| \equiv (A^*A)^{1/2}$.

Examples of unitarily invariant norms are:
- Schatten $p$-norm $\| \cdot \|_p$ ($1 \leq p \leq \infty$):
  $$\|A\|_p \equiv \left\{ \sum_{j=1}^{n} s_j(A)^p \right\}^{1/p}.$$ 

Then $\|A\|_\infty = s_1(A)$ is the spectral norm and $\|A\|_2 = \left\{ \sum_{i,j=1}^{n} |a_{ij}|^2 \right\}^{1/2}$ is the Frobenius norm.

- Fan $k$-norm $\| \cdot \|_{(k)}$ ($k = 1, 2, \ldots, n$):
  $$\|A\|_{(k)} \equiv \sum_{j=1}^{k} s_j(A).$$

For Hermitian matrices $A, B$, we write $A \geq B$ to mean that $A - B$ is positive semidefinite. In particular, $A \geq 0$ means that $A$ is positive semidefinite.

We consider only continuous nonnegative functions on $[0, \infty)$. $f(t)$ is called operator monotone if

$$A \geq 0 \Rightarrow f(A) \geq f(B).$$

\footnote{This paper appeared in Linear Algebra Appl. 341(2002) 151-169.}
Here $f(A)$ is defined by the usual functional calculus via the spectral decomposition of $A$.

Examples of operator monotone functions are:

- $t^p$ \((0 < p \leq 1)\), \quad \log(t + 1)

1. Convexity of certain functions involving unitarily invariant norms

**Theorem 1.** Given matrices $A, B \geq 0$, $\forall X$, real number $r > 0$, and any unitarily invariant norm, the function

$$ \phi(t) = \| A^t X B^{1-t} \|^r \cdot \| A^{1-t} X B^t \|^r $$

is convex on the interval [0, 1] and attains its minimum at $t = 1/2$. Consequently, it is decreasing on $[0, 1/2]$ and increasing on $[1/2, 1]$.

**Corollary 2.** For $0 \leq t \leq 1$,

$$ \| A^{1/2} X B^{1/2} \|^r \leq \| A^t X B^{1-t} \|^r \cdot \| A^{1-t} X B^t \|^r \leq \| AX \|^r \cdot \| XB \|^r $$

Note that this interpolates the known matrix Cauchy-Schwarz inequality

$$ \| A^{1/2} X B^{1/2} \|^r \leq \| AX \|^r \cdot \| XB \|^r. $$

**Corollary 3.** Let $A, B$ be positive definite and $X$ be arbitrary. For every $r > 0$ and every unitarily invariant norm, the function

$$ g(s) = \| A^s X B^s \|^r \cdot \| A^{-s} X B^{-s} \|^r $$

is convex on $(-\infty, \infty)$, attains its minimum at $s = 0$, and hence it is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

The case $r = 1, X = B = I$ (the identity matrix) of this result says that the condition number

$$ c(A^s) \equiv \| A^s \| \cdot \| A^{-s} \| $$

is increasing in $s > 0$, which is due to A. W. Marshall and I. Olkin (1965).
2. Norm inequalities for operator monotone functions with applications

A norm on $n \times n$ matrices is said to be normalized if $\|\text{diag}(1, 0, \ldots, 0)\| = 1$.

All the Fan $k$-norms ($k = 1, \ldots, n$) and Schatten $p$-norms ($1 \leq p \leq \infty$) are normalized.

**Theorem 4.** Let $f(t)$ be a nonnegative operator monotone function on $[0, \infty)$ and $\| \cdot \|$ be a normalized unitarily invariant norm. Then for every matrix $A$,
\[
\|f(A)\| \leq \|f(|A|)\|.
\]

This inequality is reversed when the norm is normalized in another way.

**Theorem 5.** Let $f(t)$ be a nonnegative operator monotone function on $[0, \infty)$ and $\| \cdot \|$ be a unitarily invariant norm with $\|I\| = 1$. Then for every matrix $A$,
\[
\|f(A)\| \geq \|f(|A|)\|.
\]

Given a unitarily invariant norm $\| \cdot \|$, for $p > 0$ define
\[
\|X\|^{(p)} \equiv \|X^p\|^{1/p}.
\]

Then it is known that when $p \geq 1$, $\| \cdot \|^{(p)}$ is also a unitarily invariant norm.

**Corollary 6.** Let $\| \cdot \|$ be a normalized unitarily invariant norm. Then for any matrix $A$, the function $p \mapsto \|A\|^{(p)}$ is decreasing on $(0, \infty)$ and
\[
\lim_{p \to \infty} \|A\|^{(p)} = \|A\|_\infty.
\]

The above limit formula remains valid without the normalization condition on $\| \cdot \|$.

We denote by $A \vee B$ the supremum of $A, B \geq 0 : A \vee B = \lim_{p \to \infty} ((A^p + B^p)/2)^{1/p}$. 
**Theorem 7.** Let $A, B$ be positive semidefinite. For every unitarily invariant norm, the function $p \mapsto \|(A^p + B^p)^{1/p}\|$ is decreasing on $(0, 1]$. For every normalized unitarily invariant norm, the function $p \mapsto \|A^p + B^p\|^{1/p}$ is decreasing on $(0, \infty)$ and 

$$\lim_{p \to \infty} \|A^p + B^p\|^{1/p} = \|A \vee B\|_{\infty}.$$

The above limit formula remains valid without the normalization condition.

**3. Norm inequalities of Hölder and Minkowski types**

**Theorem 8.** Let $1 \leq p, q \leq \infty$ with $p^{-1} + q^{-1} = 1$. For all matrices $A, B, C, D$ and every unitarily invariant norm,

$$2^{-1/p-1/2} \|C^*A + D^*B\| \leq \|A^p + |B|^p\|^{1/p} \cdot \|C^p + |D|^p\|^{1/q}.$$

Moreover, the constant $2^{-1/p-1/2}$ is best possible.

**Theorem 9.** Let $1 \leq p < \infty$. For any $A_i, B_i$ $(i = 1, 2)$ and every unitarily invariant norm,

$$2^{-1/p-1/2} \|A_1 + A_2|^p + |B_1 + B_2|^p\|^{1/p}
\leq \|A_1^p + |B_1|^p\|^{1/p} + \|A_2^p + |B_2|^p\|^{1/p}.$$

**Main Ingredients of the Proofs**

- **Integral representation:** A nonnegative operator monotone function $f(t)$ on $[0, \infty)$ is represented as

$$f(t) = \alpha + \beta t + \int_0^\infty \frac{st}{s+t}d\mu(s)$$

where $\alpha, \beta \geq 0$ and $\mu(\cdot)$ is a positive measure on $[0, \infty)$.

- **Dual norm:** Given a norm $\| \cdot \|$ on $n \times n$ matrices, the dual norm of $\| \cdot \|$ with respect to the Frobenius inner product is

$$\|A\|^D \equiv \max \{ |\text{tr} AX^*| : \|X\| = 1 \}.$$

If \( \| \cdot \| \) is a unitarily invariant norm and \( A \geq 0 \), then by the duality theorem we have
\[
\|A\| = \max \{ \text{tr} AB : B \geq 0, \|B\|^D = 1 \}.
\]

- **Theorem** [conjectured by F. Hiai and proved by T. Ando and X. Zhan, Math. Ann. 315 (1999)]: Let \( A, B \geq 0 \), and \( \| \cdot \| \) be a unitarily invariant norm. If \( f(t) \) nonnegative operator monotone on \([0, \infty)\), then
\[
\|f(A + B)\| \leq \|f(A) + f(B)\|.
\]
If \( g(t) \) is strictly increasing on \([0, \infty)\) with \( g(0) = 0, g(\infty) = \infty \) and the inverse function \( g^{-1} \) on \([0, \infty)\) is operator monotone, then
\[
\|g(A + B)\| \geq \|g(A) + g(B)\|.
\]