Inequalities Involving Unitarily Invariant Norms
and Operator Monotone Functions\(^1\)

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We consider square complex matrices. A norm \(\| \cdot \|\) on the space of \(n \times n\) matrices is called *unitarily invariant* if

\[
\|UAV\| = \|A\| \quad \forall A, \forall \text{unitary } U, V.
\]

Such a norm is determined by a symmetric gauge function \(\Phi\) on \(\mathbb{R}^n\):

\[
\|A\| = \Phi(s_1(A), \ldots, s_n(A))
\]

where \(s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)\) are the singular values of \(A\), that is, the eigenvalues of \(|A| \equiv (A^*A)^{1/2}\).

Examples of unitarily invariant norms are:

*Schatten p-norm* \(\| \cdot \|_p\) (1 \(\leq p \leq \infty\):

\[
\|A\|_p \equiv \left\{\sum_{j=1}^{n} s_j(A)^p\right\}^{1/p}.
\]

Then \(\|A\|_\infty = s_1(A)\) is the *spectral norm* and \(\|A\|_2 = \left\{\sum_{i,j=1}^{n} |a_{ij}|^2\right\}^{1/2}\) is the *Frobenius norm*.

*Fan k-norm* \(\| \cdot \|_{(k)}\) (\(k = 1, 2, \ldots, n\)):

\[
\|A\|_{(k)} \equiv \sum_{j=1}^{k} s_j(A).
\]

For Hermitian matrices \(A, B\), we write \(A \geq B\) to mean that \(A - B\) is positive semidefinite. In particular, \(A \geq 0\) means that \(A\) is positive semidefinite.

We consider only continuous nonnegative functions on \([0, \infty)\). \(f(t)\) is called *operator monotone* if

\[
A \geq B \geq 0 \implies f(A) \geq f(B).
\]

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Here \( f(A) \) is defined by the usual functional calculus via the spectral decomposition of \( A \).

Examples of operator monotone functions are:

\[
\cdot \quad t^p \ (0 < p \leq 1), \quad \log(t + 1)
\]

1. Convexity of certain functions involving unitarily invariant norms

**Theorem 1.** Given matrices \( A, B \geq 0, \forall X \), real number \( r > 0 \), and any unitarily invariant norm, the function

\[
\phi(t) = \left\| A^t X B^{1-t} \right\| r \cdot \left\| A^{1-t} X B^t \right\| r
\]

is convex on the interval \([0, 1]\) and attains its minimum at \( t = 1/2 \). Consequently, it is decreasing on \([0, 1/2]\) and increasing on \([1/2, 1]\).

**Corollary 2.** For \( 0 \leq t \leq 1 \),

\[
\left\| A^{1/2} X B^{1/2} \right\|^r \leq \left\| A^t X B^{1-t} \right\|^r \cdot \left\| A^{1-t} X B^t \right\|^r \leq \left\| AX \right\|^r \cdot \left\| XB \right\|^r
\]

Note that this interpolates the known matrix Cauchy-Schwarz inequality

\[
\left\| A^{1/2} X B^{1/2} \right\|^r \leq \left\| AX \right\|^r \cdot \left\| XB \right\|^r.
\]

**Corollary 3.** Let \( A, B \) be positive definite and \( X \) be arbitrary. For every \( r > 0 \) and every unitarily invariant norm, the function

\[
g(s) = \left\| A^s X B^s \right\|^r \cdot \left\| A^{-s} X B^{-s} \right\|^r
\]

is convex on \(( -\infty, \infty )\), attains its minimum at \( s = 0 \), and hence it is decreasing on \(( -\infty, 0)\) and increasing on \((0, \infty)\).

The case \( r = 1, X = B = I \) (the identity matrix) of this result says that the condition number

\[
c(A^s) \equiv \left\| A^s \right\| \cdot \left\| A^{-s} \right\|
\]

is increasing in \( s > 0 \), which is due to A. W. Marshall and I. Olkin (1965).
2. Norm inequalities for operator monotone functions with applications

A norm on $n \times n$ matrices is said to be normalized if $\|\text{diag}(1,0, \ldots, 0)\| = 1$.

All the Fan $k$-norms ($k = 1, \ldots, n$) and Schatten $p$-norms ($1 \leq p \leq \infty$) are normalized.

**Theorem 4.** Let $f(t)$ be a nonnegative operator monotone function on $[0, \infty)$ and $\| \cdot \|$ be a normalized unitarily invariant norm. Then for every matrix $A$,

$$f(\|A\|) \leq \|f(|A|)\|.$$

This inequality is reversed when the norm is normalized in another way.

**Theorem 5.** Let $f(t)$ be a nonnegative operator monotone function on $[0, \infty)$ and $\| \cdot \|$ be a unitarily invariant norm with $\|I\| = 1$. Then for every matrix $A$,

$$f(\|A\|) \geq \|f(|A|)\|.$$

Given a unitarily invariant norm $\| \cdot \|$, for $p > 0$ define

$$\|X\|^{(p)} \equiv \|X^p\|^{1/p}.$$

Then it is known that when $p \geq 1$, $\| \cdot \|^{(p)}$ is also a unitarily invariant norm.

**Corollary 6.** Let $\| \cdot \|$ be a normalized unitarily invariant norm. Then for any matrix $A$, the function $p \mapsto \|A\|^{(p)}$ is decreasing on $(0, \infty)$ and

$$\lim_{p \to \infty} \|A\|^{(p)} = \|A\|_{\infty}.$$  

The above limit formula remains valid without the normalization condition on $\| \cdot \|$.

We denote by $A \vee B$ the supremum of $A, B \geq 0 : A \vee B = \lim_{p \to \infty}\{(A^p + B^p)^{1/p}\}$.  


Theorem 7. Let $A, B$ be positive semidefinite. For every unitarily invariant norm, the function $p \mapsto \|(A^p + B^p)^{1/p}\|$ is decreasing on $(0, 1]$. For every normalized unitarily invariant norm, the function $p \mapsto \|A^p + B^p\|^{1/p}$ is decreasing on $(0, \infty)$ and
\[
\lim_{p \to \infty} \|A^p + B^p\|^{1/p} = \|A \vee B\|_{\infty}.
\]
The above limit formula remains valid without the normalization condition.

3. Norm inequalities of Hölder and Minkowski types

Theorem 8. Let $1 \leq p, q \leq \infty$ with $p^{-1} + q^{-1} = 1$. For all matrices $A, B, C, D$ and every unitarily invariant norm,
\[
2^{-\frac{1}{2p} - \frac{1}{2q}} \|C^* A + D^* B\| \leq \| |A|^p + |B|^p \|^{1/p} \cdot \| |C|^q + |D|^q \|^{1/q}.
\]
Moreover, the constant $2^{-\frac{1}{2p} - \frac{1}{2q}}$ is best possible.

Theorem 9. Let $1 \leq p < \infty$. For any $A_i, B_i$ ($i = 1, 2$) and every unitarily invariant norm,
\[
2^{-\frac{1}{2p} - \frac{1}{2q}} \| |A_1 + A_2|^p + |B_1 + B_2|^p \|^{1/p}
\leq \| |A_1|^p + |B_1|^p \|^{1/p} + \| |A_2|^p + |B_2|^p \|^{1/p}.
\]

Main Ingredients of the Proofs

- Integral representation: A nonnegative operator monotone function $f(t)$ on $[0, \infty)$ is represented as
  \[
f(t) = \alpha + \beta t + \int_0^\infty \frac{st}{s+t} d\mu(s)
\]
where $\alpha, \beta \geq 0$ and $\mu(\cdot)$ is a positive measure on $[0, \infty)$.

- Dual norm: Given a norm $\| \cdot \|$ on $n \times n$ matrices, the dual norm of $\| \cdot \|$ with respect to the Frobenius inner product is
  \[
  \|A\|^D \equiv \max \{|\operatorname{tr} AX^*| : \|X\| = 1\}.
  \]
If $\| \cdot \|$ is a unitarily invariant norm and $A \geq 0$, then by the duality theorem we have

$$\|A\| = \max \{ \text{tr} \ AB : B \geq 0, \|B\|^D = 1 \}.$$ 

- **Theorem** [conjectured by F. Hiai and proved by T. Ando and X. Zhan, Math. Ann. 315 (1999)]: Let $A, B \geq 0$, and $\| \cdot \|$ be a unitarily invariant norm. If $f(t)$ nonnegative operator monotone on $[0, \infty)$, then

$$\|f(A + B)\| \leq \|f(A) + f(B)\|.$$ 

If $g(t)$ is strictly increasing on $[0, \infty)$ with $g(0) = 0$, $g(\infty) = \infty$ and the inverse function $g^{-1}$ on $[0, \infty)$ is operator monotone, then

$$\|g(A + B)\| \geq \|g(A) + g(B)\|.$$