Inequalities Involving Unitarily Invariant Norms and Operator Monotone Functions

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We consider square complex matrices. A norm $\| \cdot \|$ on the space of $n \times n$ matrices is called unitarily invariant if

$$\| UAV \| = \| A \| \quad \forall A, \forall \text{unitary } U, V.$$ 

Such a norm is determined by a symmetric gauge function $\Phi$ on $\mathbb{R}^n$:

$$\| A \| = \Phi(s_1(A), \ldots, s_n(A))$$

where $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of $A$, that is, the eigenvalues of $|A| \equiv (A^*A)^{1/2}$.

Examples of unitarily invariant norms are:

Schatten $p$-norm $\| \cdot \|_p$ ($1 \leq p \leq \infty$):

$$\| A \|_p \equiv \left\{ \sum_{j=1}^{n} s_j(A)^p \right\}^{1/p}.$$  

Then $\| A \|_\infty = s_1(A)$ is the spectral norm and $\| A \|_2 = \{\sum_{i,j=1}^{n} |a_{ij}|^2\}^{1/2}$ is the Frobenius norm.

Fan $k$-norm $\| \cdot \|_{(k)}$ ($k = 1, 2, \ldots, n$):

$$\| A \|_{(k)} \equiv \sum_{j=1}^{k} s_j(A).$$

For Hermitian matrices $A, B$, we write $A \geq B$ to mean that $A - B$ is positive semidefinite. In particular, $A \geq 0$ means that $A$ is positive semidefinite.

We consider only continuous nonnegative functions on $[0, \infty)$. $f(t)$ is called operator monotone if

$$A \geq B \geq 0 \implies f(A) \geq f(B).$$

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1This paper appeared in Linear Algebra Appl. 341(2002) 151-169.
Here $f(A)$ is defined by the usual functional calculus via the spectral decomposition of $A$.

Examples of operator monotone functions are:

$$t^p \ (0 < p \leq 1), \ \log(t + 1)$$

1. Convexity of certain functions involving unitarily invariant norms

**Theorem 1.** Given matrices $A, B \geq 0, \forall X$, real number $r > 0$, and any unitarily invariant norm, the function

$$\phi(t) = \| A^t X B^{1-t} \|^r \cdot \| A^{1-t} X B^t \|^r$$

is convex on the interval $[0, 1]$ and attains its minimum at $t = 1/2$. Consequently, it is decreasing on $[0, 1/2]$ and increasing on $[1/2, 1]$.

**Corollary 2.** For $0 \leq t \leq 1$,

$$\| A^{1/2} X B^{1/2} \|^r \leq \| A^t X B^{1-t} \|^r \cdot \| A^{1-t} X B^t \|^r \leq \| AX \|^r \cdot \| XB \|^r$$

Note that this interpolates the known matrix Cauchy-Schwarz inequality

$$\| A^{1/2} X B^{1/2} \|^r \leq \| AX \|^r \cdot \| XB \|^r.$$ 

**Corollary 3.** Let $A, B$ be positive definite and $X$ be arbitrary. For every $r > 0$ and every unitarily invariant norm, the function

$$g(s) = \| A^s X B^s \|^r \cdot \| A^{-s} X B^{-s} \|^r$$

is convex on $(-\infty, \infty)$, attains its minimum at $s = 0$, and hence it is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

The case $r = 1, X = B = I$ (the identity matrix) of this result says that the condition number

$$c(A^s) \equiv \| A^s \cdot \| A^{-s} \|$$

is increasing in $s > 0$, which is due to A. W. Marshall and I. Olkin (1965).
2. Norm inequalities for operator monotone functions with applications

A norm on $n \times n$ matrices is said to be normalized if $\| \text{diag}(1, 0, \ldots, 0) \| = 1$.

All the Fan $k$-norms ($k = 1, \ldots, n$) and Schatten $p$-norms ($1 \leq p \leq \infty$) are normalized.

**Theorem 4.** Let $f(t)$ be a nonnegative operator monotone function on $[0, \infty)$ and $\| \cdot \|$ be a normalized unitarily invariant norm. Then for every matrix $A$,

$$f(\|A\|) \leq \|f(|A|)\|.$$

This inequality is reversed when the norm is normalized in another way.

**Theorem 5.** Let $f(t)$ be a nonnegative operator monotone function on $[0, \infty)$ and $\| \cdot \|$ be a unitarily invariant norm with $\|I\| = 1$. Then for every matrix $A$,

$$f(\|A\|) \geq \|f(|A|)\|.$$

Given a unitarily invariant norm $\| \cdot \|$, for $p > 0$ define

$$\|X\|^{(p)} \equiv \| |X|^p \|^{1/p}.$$

Then it is known that when $p \geq 1$, $\| \cdot \|^{(p)}$ is also a unitarily invariant norm.

**Corollary 6.** Let $\| \cdot \|$ be a normalized unitarily invariant norm. Then for any matrix $A$, the function $p \mapsto \|A\|^{(p)}$ is decreasing on $(0, \infty)$ and

$$\lim_{p \to \infty} \|A\|^{(p)} = \|A\|_{\infty}.$$

The above limit formula remains valid without the normalization condition on $\| \cdot \|$.

We denote by $A \vee B$ the supremum of $A, B \geq 0: A \vee B = \lim_{p \to \infty} [(A^p + B^p)^{1/p}]^{1/p}$. 


Theorem 7. Let $A, B$ be positive semidefinite. For every unitarily invariant norm, the function $p \mapsto \|(A^p + B^p)^{1/p}\|$ is decreasing on $(0, 1]$. For every normalized unitarily invariant norm, the function $p \mapsto \|A^p + B^p\|^{1/p}$ is decreasing on $(0, \infty)$ and
\[
\lim_{p \to \infty} \|A^p + B^p\|^{1/p} = \|A \lor B\|_{\infty}.
\]

The above limit formula remains valid without the normalization condition.

3. Norm inequalities of Hölder and Minkowski types

Theorem 8. Let $1 \leq p, q \leq \infty$ with $p^{-1} + q^{-1} = 1$. For all matrices $A, B, C, D$ and every unitarily invariant norm,
\[
2^{-|\frac{1}{p} - \frac{1}{2}|} \|C^* A + D^* B\| \leq \| |A|^p + |B|^p \|^{1/p} \cdot \| |C|^q + |D|^q \|^{1/q}.
\]
Moreover, the constant $2^{-|\frac{1}{p} - \frac{1}{2}|}$ is best possible.

Theorem 9. Let $1 \leq p < \infty$. For any $A_i, B_i (i = 1, 2)$ and every unitarily invariant norm,
\[
2^{-|\frac{1}{p} - \frac{1}{2}|} \| |A_1 + A_2|^p + |B_1 + B_2|^p \|^{1/p} \\
\leq \| |A_1|^p + |B_1|^p \|^{1/p} + \| |A_2|^p + |B_2|^p \|^{1/p}.
\]

Main Ingredients of the Proofs

- **Integral representation:** A nonnegative operator monotone function $f(t)$ on $[0, \infty)$ is represented as
\[
f(t) = \alpha + \beta t + \int_0^\infty \frac{st}{s + t} d\mu(s)
\]
where $\alpha, \beta \geq 0$ and $\mu(\cdot)$ is a positive measure on $[0, \infty)$.

- **Dual norm:** Given a norm $\| \cdot \|$ on $n \times n$ matrices, the dual norm of $\| \cdot \|$ with respect to the Frobenius inner product is
\[
\|A\|^D \equiv \max \{|\text{tr} AX^*| : \|X\| = 1\}.
\]
If \( \| \cdot \| \) is a unitarily invariant norm and \( A \geq 0 \), then by the duality theorem we have
\[
\|A\| = \max \{ \operatorname{tr} AB : B \geq 0, \|B\|^D = 1 \}.
\]

- **Theorem** [conjectured by F. Hiai and proved by T. Ando and X. Zhan, Math. Ann. 315 (1999)]: Let \( A, B \geq 0 \), and \( \| \cdot \| \) be a unitarily invariant norm. If \( f(t) \) nonnegative operator monotone on \([0, \infty)\), then
\[
\|f(A + B)\| \leq \|f(A) + f(B)\|.
\]

If \( g(t) \) is strictly increasing on \([0, \infty)\) with \( g(0) = 0, g(\infty) = \infty \) and the inverse function \( g^{-1} \) on \([0, \infty)\) is operator monotone, then
\[
\|g(A + B)\| \geq \|g(A) + g(B)\|.
\]