Relations between two inequalities

$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ and their applications (Current topics on operator theory and operator inequalities)

Author(s)

Ito, Masatoshi; Yamazaki, Takeaki

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and their applications

東京理科大学 伊藤公智 (Masatoshi Ito)
(Department of Applied Mathematics, Tokyo University of Science)

神奈川大工 山崎文明 (Takeaki Yamazaki)
(Department of Mathematics, Kanagawa University)

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**Abstract**

Let $A$ and $B$ be positive invertible operators. Then for each $p \geq 0$ and $r \geq 0$, two inequalities

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \text{ and } A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$$

are equivalent. In this report, we shall show relations between these inequalities in case $A$ and $B$ are not invertible. And we shall show some applications of this result to operator classes.

1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$.

As a recent development on order preserving operator inequalities, it is known the following Theorem F.

**Theorem F (Furuta inequality [9]).**

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) \hspace{1cm} $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) \hspace{1cm} $(A^{\frac{p}{2}}A^pA^{\frac{p}{2}})^{\frac{1}{q}} \geq (A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$. 

\[ (1, 1) \rightarrow (0, -r) \rightarrow (1, 0) \rightarrow (p, q) \]

\[ p = q, \quad q = 1, \quad p = q \]

\[ (1+r)q = p+r \]

**Figure 1**
Theorem F yields the famous Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$" by putting $r = 0$ in (i) or (ii) of Theorem F. Alternative proofs of Theorem F are given in [6] and [18] and also an elementary one page proof in [10]. It was shown by Tanahashi [19] that the domain drawn for $p, q$ and $r$ in the Figure 1 is the best possible one for Theorem F.

As an application of Theorem F, the following result was shown in [7] and [11].

**Theorem FC ([7][11])**. Let $A, B > 0$. Then the following assertions are mutually equivalent:

(i) $\log A \geq \log B$.

(ii) $(B^\frac{f}{2}A^pB^\frac{1}{2})^\frac{r}{p+r} \geq B^r$ for all $p \geq 0$ and $r \geq 0$.

(iii) $A^r \geq (A^\frac{f}{2}B^pA^\frac{1}{2})^\frac{r}{p+r}$ for all $p \geq 0$ and $r \geq 0$.

We remark that this result is an extension of [4] in case $p = r$, and an excellent proof of this result which used only Theorem F was shown in [22].

On the other hand, the following assertions are well known: Let $A$ and $B$ be positive invertible operators. Then

(1) $A \geq B \implies \log A \geq \log B$.

(2) $\log A \geq \log B \implies (B^\frac{f}{2}A^pB^\frac{1}{2})^\frac{r}{p+r} \geq B^r$ and $A^p \geq (A^\frac{f}{2}B^pA^\frac{1}{2})^\frac{r}{p+r}$ for all $p \geq 0$ and $r \geq 0$.

(3) For each $p \geq 0$ and $r \geq 0$, $(B^\frac{f}{2}A^pB^\frac{1}{2})^\frac{r}{p+r} \geq B^r \iff A^p \geq (A^\frac{f}{2}B^pA^\frac{1}{2})^\frac{r}{p+r}$.

(1) holds since $\log t$ is an operator monotone function. (2) is an immediate consequence of Theorem FC. (3) was shown in [11].

Related to these results, it is known in [23] that invertibility of (1) and (2) can be replaced with the condition $N(A) = N(B) = \{0\}$, that is, (1) and (2) hold for some non-invertible operators $A$ and $B$. But we have not known whether invertibility of $A$ and $B$ in (3) can be replaced with looser condition or not. In this report, we shall show relations between

$$(B^\frac{f}{2}A^pB^\frac{1}{2})^\frac{r}{p+r} \geq B^r \quad \text{and} \quad A^p \geq (A^\frac{f}{2}B^pA^\frac{1}{2})^\frac{r}{p+r}$$

when $A$ and $B$ are not invertible.

Next, An operator $T$ is said to be hyponormal if $T^*T \geq TT^*$. An operator $T$ is invertible log-hyponormal (defined in [20]) if $\log T^*T \geq \log TT^*$. For each $s > 0$ and $t > 0$, an operator $T$ belongs to class $A(s, t)$ (defined in [8]) if $(|T^*|^s|T|^2|T^*|^t)^\frac{1}{2} \geq |T|^2$, where $|T| = (T^*T)^{\frac{1}{2}}$. Class $A(s, t)$ is introduced as a generalization of class $A$
Every invertible hyponormal operator is log-hyponormal.

(2) Every invertible log-hyponormal operator belongs to class $wA(s, t)$ for all $s > 0$ and $t > 0$.

(3) For each $s > 0$ and $t > 0$, invertible class $wA(s, t)$ equals invertible class $A(s, t)$.

There are many papers on these classes in case of invertible operators, for example [8], [20] and [24].

On the other hand, even if an operator is non-invertible, log-hyponormality can be defined by $N(T^*) \supset N(T)$ and $\log A \geq \log B$, where $A$ and $B$ are the compressions of $T^*T$ and $TT^*$ to $\overline{R(T)}$, respectively. This definition implicitly appeared in [3] and it was pointed out in [23] that it is the general form of log-hyponormality. Ando [3] showed that every hyponormal operator is log-hyponormal and every log-hyponormal operator is paranormal. Moreover, Uchiyama [23] showed that every log-hyponormal operator is also included in class A (even if an operator is non-invertible). In this report, we shall show that for each $s > 0$ and $t > 0$, class $A(s, t)$ coincides with class $wA(s, t)$, that is, we shall show (3) without invertibility of operators, and show some properties of class $A(s, t)$ operators. Lastly, we shall show a normality of class $A(s, t)$ operators for $s > 0$ and $t > 0$.

2 Relations between

\[(B^\frac{p}{2}A^p B^\frac{r}{2})^\frac{r}{p+r} \geq B^r \quad \text{and} \quad A^p \geq (A^\frac{p}{2} B^r A^\frac{r}{2})^\frac{p}{p+r}\]

In this section, we shall show the following result:

**Theorem 1.** Let $A$ and $B$ be positive operators. Then for each $p \geq 0$ and $r \geq 0$, the following assertions hold:

(i) If $(B^\frac{p}{2}A^p B^\frac{r}{2})^\frac{r}{p+r} \geq B^r$, then $A^p \geq (A^\frac{p}{2} B^r A^\frac{r}{2})^\frac{p}{p+r}$.

(ii) If $A^p \geq (A^\frac{p}{2} B^r A^\frac{r}{2})^\frac{p}{p+r}$ and $N(A) \subset N(B)$, then $(B^\frac{p}{2}A^p B^\frac{r}{2})^\frac{r}{p+r} \geq B^r$. 
We remark on Theorem 1 that the assumption of (ii) has a kernel condition $N(A) \subset N(B)$, but the assumption of (i) does not have any kernel conditions. If $A$ and $B$ are invertible, then $N(A) = N(B) = \{0\}$ holds, and the kernel condition of (ii) in Theorem 1 is satisfied. Hence we know that Theorem 1 is a generalization of (3) in the previous section.

To prove Theorem 1, we prepare the following lemma.

**Lemma 2.** Let $A$ and $B$ be positive operators. Then the following assertions hold:

(i) \[ \lim_{\epsilon \to +0} A^{\frac{1}{2}}(A + \epsilon I)^{-1}A^{\frac{1}{2}} = \lim_{\epsilon \to +0} (A + \epsilon I)^{-1}A = P_{N(A)^\perp}, \]
where $P_M$ is the projection onto a closed subspace $M$.

(ii) \[ \lim_{\epsilon \to +0} A^{\frac{1}{2}}B^{\frac{1}{2}}((B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\alpha} + \epsilon I)^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha} \] for $\alpha \in (0, 1)$.

We remark that if $A$ and $B$ are both positive invertible, then

\[ A^{\frac{1}{2}}B^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{-\alpha}B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha} \] for $\alpha \in (0, 1)$

by the following Lemma F. Therefore we can regard (ii) of Lemma 2 as a non-invertible version of Lemma F for $\lambda \in (0, 1)$.

**Lemma F ([12]).** Let $A$ be a positive invertible operator and $B$ be an invertible operator. Then

\[ (BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^* \]

holds for any real number $\lambda$.

**Proof of Lemma 2.**

**Proof of (i).** We give a proof which is a slightly modification of the proof of [5, Lemma].

Let $A = \int_0^{||A||} t dF(t)$ be the spectral decomposition of $A$. Then

\[ \lim_{\epsilon \to +0} (A + \epsilon I)^{-1}A = \lim_{\epsilon \to +0} \int_0^{||A||} \frac{t}{t + \epsilon} dF(t) = \int_0^{||A||} \chi_{(0, ||A||]}(t) dF(t) = I - F(0), \]

where $\chi_{(0, ||A||]}(t)$ is the characteristic function on $(0, ||A||]$. Since $I - F(0) = P_{N(A)\perp}$, we have

\[ \lim_{\epsilon \to +0} A^{\frac{1}{2}}(A + \epsilon I)^{-1}A^{\frac{1}{2}} = \lim_{\epsilon \to +0} (A + \epsilon I)^{-1}A = P_{N(A)^\perp}. \]

**Proof of (ii).** Let $A^{\frac{1}{2}}B^{\frac{1}{2}} = U|A^{\frac{1}{2}}B^{\frac{1}{2}}|$ be the polar decomposition of $A^{\frac{1}{2}}B^{\frac{1}{2}}$. 

Then we have
\[
\lim_{\epsilon \to +0} A^{\frac{1}{2}}B^{\frac{1}{2}}\{(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\alpha} + \epsilon I\}^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}
\]
\[
= \lim_{\epsilon \to +0} U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2\alpha + \epsilon I})^{-1}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{\alpha}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}U^*
\]
\[
= U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}P_{N(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{\perp})}P_{N(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{\perp})}U^*
\]
\[
= U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2(1-\alpha)}U^* = |B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2(1-\alpha)} = (A^{\frac{1}{2}}B^{\frac{1}{2}})^{1-\alpha}.
\]

Hence the proof is complete. 

\[\square\]

Proof of Theorem 1. Let \(\epsilon > 0\). And also we may assume \(p > 0\) and \(r > 0\).

Proof of (i). Since \((B^{\frac{1}{2}}A^{p}B^{\frac{2}{2}})^{\frac{1}{r+p}} \geq B^r\), we obtain
\[
A^{\frac{1}{2}}B^{\frac{1}{2}}(B^{\frac{1}{2}} + \epsilon I)^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} \geq A^{\frac{1}{2}}B^{\frac{1}{2}}\{(B^{\frac{1}{2}}A^{p}B^{\frac{2}{2}})^{\frac{1}{r+p}} + \epsilon I\}^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}. \tag{2.1}
\]

In (2.1), by tending \(\epsilon \to +0\) and Lemma 2, we obtain
\[
A^{\frac{1}{2}}P_{N(B)^{\perp}}A^{\frac{1}{2}} \geq (A^{\frac{1}{2}}B^rA^{\frac{1}{2}})^{\frac{1}{r+p}}.
\]

Hence we have
\[
A^{p} \geq A^{\frac{1}{2}}P_{N(B)^{\perp}}A^{\frac{1}{2}} \geq (A^{\frac{1}{2}}B^rA^{\frac{1}{2}})^{\frac{1}{r+p}}.
\]

Proof of (ii). Since \(A^{p} \geq (A^{\frac{1}{2}}B^rA^{\frac{1}{2}})^{\frac{1}{r+p}}\), we obtain
\[
B^{\frac{1}{2}}A^{\frac{1}{2}}(A^{p} + \epsilon I)^{-1}A^{\frac{1}{2}}B^{\frac{1}{2}} \leq B^{\frac{1}{2}}A^{\frac{1}{2}}\{(A^{\frac{1}{2}}B^rA^{\frac{1}{2}})^{\frac{1}{r+p}} + \epsilon I\}^{-1}A^{\frac{1}{2}}B^{\frac{1}{2}}. \tag{2.2}
\]

In (2.2), by tending \(\epsilon \to +0\) and Lemma 2, we obtain
\[
B^{\frac{1}{2}}P_{N(A)^{\perp}}B^{\frac{1}{2}} \leq (B^{\frac{1}{2}}A^{p}B^{\frac{1}{2}})^{\frac{1}{r+p}}. \tag{2.3}
\]

On the other hand,
\[
N(A) \subset N(B) \iff P_{N(A)^{\perp}} \geq P_{N(B)^{\perp}}. \tag{2.4}
\]

By (2.3) and (2.4), we have
\[
(B^{\frac{1}{2}}A^{p}B^{\frac{1}{2}})^{\frac{1}{r+p}} \geq B^{\frac{1}{2}}P_{N(A)^{\perp}}B^{\frac{1}{2}} \geq B^{\frac{1}{2}}P_{N(B)^{\perp}}B^{\frac{1}{2}} = B^r.
\]

Therefore the proof is complete. \[\square\]
Remark. We recall the assumptions of (i) and (ii) in Theorem 1. Here we assume $p = r = 1$ in Theorem 1.

(i-a) $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$.

(ii-a) $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ and $N(A) \subset N(B)$.

We proved that (i-a) ensures $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ and (ii-a) ensures (i-a) in Theorem 1, so we might expect that (i-a) and (ii-a) are equivalent. But we have the following counterexample.

Example 1. $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$ and $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$, but $N(A) \not\subset N(B)$.

Let $A = 2\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $A^{\frac{1}{2}} = \sqrt{\frac{2}{5}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = B$. Hence

$$\sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$$

and

$$A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}} = \sqrt{\frac{2}{5}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

But $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \in N(A)$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \not\in N(B)$, so that $N(A) \not\subset N(B)$.

Moreover, we have the following example in [16].

Example 2 ([16]). $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$, but $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B$ and $N(A) \not\subset N(B)$.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then we can check $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$, $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B$ and $N(A) \not\subset N(B)$, easily.

Therefore we recognize that $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ requires some condition to be equivalent to $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$. So we consider the following condition.

(ii-a') $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ and $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) \subset N(B)$.

We can easily check that $N(A) \subset N(B)$ ensures $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) \subset N(B)$. And also $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$ ensures $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) \subset N(B)$ since $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) = N(B^{\frac{1}{2}}AB^{\frac{1}{2}}) = N((B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}}) \subset N(B)$, so that (i-a) ensures (ii-a') by (i) in Theorem 1.

But, unfortunately, we understand that (ii-a') does not ensure (i-a) by the following example.
Example 3. $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ and $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) \subset N(B)$, but $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B$.

Let $A = \frac{1}{2} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $A^{\frac{1}{2}} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = B$. Hence

$$A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}} = \frac{1}{\sqrt{50}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

and $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) = N(B) = \left\{ t \begin{pmatrix} 0 \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\}$ since $A^{\frac{1}{2}}B^{\frac{1}{2}} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. But

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B.$$

At the end of this remark, we note that Cho-Huruya-Kim [5] gave an example such that $N(T) \not\subset N(T^{*})$, $N(T) \not\supset N(T^{*})$ and $|\overline{T}| \geq |T| \geq |(\overline{T})^{s}|$ (i.e., $T$ is $w$-hyponormal) by using $A$ and $B$ in Example 1 stated above, where $\overline{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ and $T = U|T|$ is the polar decomposition of $T$.

3 Applications

In this section, we shall show some applications of Theorem 1 to operator classes. In section 1, we introduced definitions of some operator classes, here we recall definitions of these classes as follows:

**Definition 1.** Let $s > 0$, $t > 0$ and $T = U|T|$ be the polar decomposition of $T$.

(i) $T$ belongs to class $A(s, t) \iff (|T^{*}|^{s}|T|^{2s}|T^{*}|^{t})^{\frac{s}{s+t}} \geq |T^{*}|^{2t}$.

(ii) $T$ belongs to class $wA(s, t)$

$$\iff (|T^{*}|^{s}|T|^{2s}|T^{*}|^{t})^{\frac{s}{s+t}} \geq |T^{*}|^{2t} \text{ and } |T|^{2s} \geq (|T^{*}|^{s}|T^{*}|^{2t}|T|^{s})^{\frac{s}{s+t}}$$

$$\iff |\tilde{T}_{s,t}|^{\frac{2s}{s+t}} \geq |T|^{2t} \text{ and } |T|^{2s} \geq (|\tilde{T}_{s,t}|)^{\frac{2s}{s+t}},$$

where $\tilde{T}_{s,t} = |T|^{s}U|T|^{t}$ (generalized Aluthge transformation).

(iii) $T$ belongs to class $A \iff |T^{2}| \geq |T|^{2}$, that is, $T$ belongs to class $A(1, 1)$.

(iv) $T$ is $w$-hyponormal $\iff |\tilde{T}| \geq |T| \geq |(\tilde{T})^{s}|$, that is, $T$ belongs to class $wA(\frac{1}{2}, \frac{1}{2})$, where $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (Aluthge transformation).

(i), (ii), (iii) and (iv) of Definition 1 were defined in [8], [16], [14] and [2], respectively. We remark that Aluthge transformation has many interesting properties, and many authors study this transformation, for instance, [1], [13], [15] and [17]. These classes
include invertible log-hyponormal operators, and are included in normaloid (i.e., $\|T\| = r(T)$, where $r(T)$ is the spectral radius of $T$). It has been known that for each $s > 0$ and $t > 0$, class $A(s, t)$ includes class $wA(s, t)$ by the definitions (i) and (ii). And also for each $s > 0$ and $t > 0$, every invertible class $A(s, t)$ operator is an invertible class $wA(s, t)$ operator, which was shown in [8] and [16]. More precise inclusion relations among class $wA(s, t)$, and powers of class $wA(s, t)$ operators were already shown as follows:

**Theorem A ([16], [26]).**

(i) For each $s > 0$ and $t > 0$, every class $wA(s, t)$ operator is a class $wA(\alpha, \beta)$ operator for any $\alpha \geq s$ and $\beta \geq t$.

(ii) Let $T$ be a class $wA(s, t)$ operator for $s \in (0, 1]$ and $t \in (0, 1]$. Then for each natural number $n$, $T^n$ belongs to class $wA(\frac{s}{n}, \frac{t}{n})$.

We remark that Theorem A holds for classes of invertible class $A(s, t)$ operators instead of class $wA(s, t)$, which were shown in [8] and [25]. We can summarize inclusion relations among these classes as the following Figure 2. Dotted lines in the diagram mean that we need invertibility of operators to prove the relations.

![Figure 2](image-url)

Here, in general, we can obtain that class $A(s, t)$ coincides with class $wA(s, t)$ by (i) of Theorem 1 as follows:
**Theorem 3.** For each $s > 0$ and $t > 0$, the following assertions hold:

(i) Class $A(s, t)$ coincides with class $wA(s, t)$.

(ii) Class $A$ coincides with class $wA(1, 1)$.

(iii) Class $A(\frac{1}{2}, \frac{1}{2})$ coincides with the class of $w$-hyponormal operators, i.e., class $wA(\frac{1}{2}, \frac{1}{2})$.

We can prove Theorem 3 by only applying (i) of Theorem 1 to definitions of these classes, so we omit to prove. By (iii) of Theorem 3, we have

\[ |\tilde{T}| \geq |T| \iff (|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*| \iff T : w$-hyponormal \iff $|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|.

Hence

\[ |\tilde{T}| \geq |T| \implies |T| \geq |(\tilde{T})^*|,

that is, we may as well define $w$-hyponormality by only $|\tilde{T}| \geq |T|$.

Next, we shall show some properties of class $A(s, t)$ operators without the assumption of invertibility, which are known as properties of invertible class $A(s, t)$ operators and class $wA(s, t)$ operators.

**Theorem 4.**

(i) For each $s > 0$ and $t > 0$, every class $A(s, t)$ operator is a class $A(\alpha, \beta)$ operator for any $\alpha \geq s$ and $\beta \geq t$.

(ii) Let $T$ be a class $A(s, t)$ operator for $s \in (0, 1]$ and $t \in (0, 1]$. Then for each natural number $n$, $T^n$ belongs to class $A(\frac{s}{n}, \frac{t}{n})$.

Proof is very easy by (i) of Theorem 3 and Theorem A, so we omit the proof, too. By putting $s = t = 1$ in (ii) of Theorem 4 and noting that class $A(\frac{1}{2}, \frac{1}{2})$ equals $w$-hyponormality by (iii) of Theorem 3, we obtain the following result on powers of class A operators without the assumption of invertibility.

**Corollary 5.** Let $T$ be a class $A$ operator. Then for each natural number $n$, $T^n$ belongs to class $A(\frac{1}{n}, \frac{1}{n})$. Especially $T^2$ is $w$-hyponormal.

At the end of this section, we shall summarize relations among these classes which are obtained in this section as follows: Please compare Figure 3 with Figure 2 stated.
4 Normality

In this section, we shall show a normality of some non-normal operators. It is well known that if $T$ and $T^*$ are hyponormal, then $T$ is normal. But in the case $T$ and $T^*$ belong to weaker class than hyponormal, this assertion is not obvious. Many authors obtained many results on this problem, and the following results were known until now.

**Theorem B ([21])**. *If $T$ is a class A operator and $T^*$ is a $w$-hyponormal operator, then $T$ is normal.*

**Theorem C ([3])**. *If $T$ and $T^*$ are paranormal operators satisfying $N(T) = N(T^*)$, then $T$ is normal.*

Here, we shall generalize Theorem B as follows:

**Theorem 6**. *Let $s_1 > 0$, $s_2 > 0$, $t_1 > 0$ and $t_2 > 0$. If $T$ belongs to class $A(s_1, t_1)$ and $T^*$ belongs to class $A(s_2, t_2)$, then $T$ is normal.*

Put $s_1 = t_1 = 1$ and $s_2 = t_2 = \frac{1}{2}$ in Theorem 6, we have Theorem B by Theorem 3, put $s_1 = t_1 = s_2 = t_2 = 1$ in Theorem 6, we obtain the following result on class A:

**Corollary 7**. *If $T$ and $T^*$ belong to class A, then $T$ is normal.*

To prove Theorem 6, we need the following results:
Lemma 8. Let $A$ and $B$ be self-adjoint operators, and $X \in B(H)$ satisfying

\[ X^*AX \geq X^*BX. \]

If $N(A) \supset N(X^*)$ and $N(B) \supset N(X^*)$, then $A \geq B$.

**Proof.** Let $H = \overline{R(X)} \oplus N(X^*)$ and $x = Xy + z$, where $y \in H$ and $z \in N(X^*)$. Put $T = A - B$. Then $T = T^*$ and $N(T) \supset N(X^*)$. Hence we have

\[ (Tx, x) = (TXy, Xy) + (TXy, z) + (Tz, Xy) + (Tz, z) = (X^*TXy, y) \geq 0, \]

that is, $A \geq B$.

Proposition 9. Let $A \geq 0$ and $B \geq 0$. If

\[ B^{\frac{1}{2}}AB^{\frac{1}{2}} \geq B^2 \]

and

\[ A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^2, \]

then $A = B$.

**Proof.** Put $E = P_{N(A)\perp}$ and $F = P_{N(B)\perp}$. (4.1) is equivalent to $B^{\frac{1}{2}}FAFB^{\frac{1}{2}} \geq B^2 = B^{\frac{1}{2}}BB^{\frac{1}{2}}$. By Lemma 8, we have $FAF \geq B$ since $N(FAF) \supset N(B^{\frac{1}{2}})$ and $N(B) = N(B^{\frac{1}{2}})$. Then we have

\[ (A^{\frac{1}{2}}FA^{\frac{1}{2}})^2 = A^{\frac{1}{2}}FAFA^{\frac{1}{2}} \geq A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^2 \text{ by (4.2)}, \]

and we obtain the following (4.3) by L"{o}wner-Heinz theorem.

\[ A^{\frac{1}{2}}FA^{\frac{1}{2}} \geq A. \]

(4.3) is equivalent to $A^{\frac{1}{2}}EFEA^{\frac{1}{2}} \geq A^{\frac{1}{2}}EA^{\frac{1}{2}}$. By Lemma 8, we have $EFE \geq E$ since $N(EFE) \supset N(A^{\frac{1}{2}})$ and $N(E) = N(A^{\frac{1}{2}})$. Therefore we obtain $EFE = E$, and then $F \geq E$, so that $N(A) \supset N(B)$. Hence

\[ A \geq B \]

by applying Lemma 8 to (4.1).

By the same way, we also get $B \geq A$, so that $A = B$.

**Proof of Theorem 6.** Let $p = \max\{s_1, s_2, t_1, t_2\}$. Firstly, if $T$ belongs to class $A(s_1, t_1)$, then $T$ belongs to class $A(p, p)$ by (i) of Theorem 4. This class coincides with class $wA(p, p)$ by (i) of Theorem 3. Hence we have

\[ (|T^*|^p|T|^{2p}|T^*|^p)^{\frac{1}{2}} \geq |T^*|^{2p} \text{ and } |T|^{2p} \geq (|T|^p|T^*|^2p|T|^p)^{\frac{1}{2}}. \]

(4.4)
Secondly, if $T^*$ belongs to class $A(s_2, t_2)$, then $T^*$ belongs to class $A(p, p)$ by (i) of Theorem 4. This class coincides with class $wA(p, p)$ by (i) of Theorem 3. Hence we have

$$
(|T|^p|T^*|^{2p}|T|^p)^{\frac{1}{2}} \geq |T|^{2p} \quad \text{and} \quad |T^*|^{2p} \geq (|T^*|^p|T|^{2p}|T^*|^p)^{\frac{1}{2}}.
$$

Therefore

$$
|T|^p|T^*|^{2p}|T|^p = |T|^{4p} \quad \text{and} \quad |T^*|^{4p} = |T^*|^p|T|^{2p}|T^*|^p
$$

hold by (4.4) and (4.5), and then $|T| = |T^*|$ by Proposition 9. \square

References


[9] T.Furuta, $A \geq B \geq 0$ assures $(B^r A^p B')^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1 + 2r)q \geq p + 2r$, Proc. Amer. Math. Soc., 101 (1987), 85–88.


