Relations between two inequalities

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and their applications (Current topics on operator theory and operator inequalities)

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and their applications

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M. Ito and T. Yamazaki, Relations between two inequalities \((B^{\frac{r}{2}} A^{p} B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{r} \text{ and } A^{p} \geq (A^{\frac{r}{2}} B^{r} A^{\frac{r}{2}})^{\frac{p}{p+r}}\) and their applications, to appear in Integral Equations Operator Theory.

Abstract

Let \(A\) and \(B\) be positive invertible operators. Then for each \(p \geq 0\) and \(r \geq 0\), two inequalities

\[(B^{\frac{r}{2}} A^{p} B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{r} \text{ and } A^{p} \geq (A^{\frac{r}{2}} B^{r} A^{\frac{r}{2}})^{\frac{p}{p+r}}\]

are equivalent. In this report, we shall show relations between these inequalities in case \(A\) and \(B\) are not invertible. And we shall show some applications of this result to operator classes.

1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space \(H\). An operator \(T\) is said to be positive (denoted by \(T \geq 0\)) if \((Tx, x) \geq 0\) for all \(x \in H\).

As a recent development on order preserving operator inequalities, it is known the following Theorem F.

Theorem F (Furuta inequality [9]).

If \(A \geq B \geq 0\), then for each \(r \geq 0\),

\[(i) \quad (B^{\frac{r}{2}} A^{p} B^{\frac{r}{2}})^{\frac{1}{r}} \geq (B^{\frac{r}{2}} B^{p} B^{\frac{r}{2}})^{\frac{1}{r}}\]

and

\[(ii) \quad (A^{\frac{r}{2}} A^{p} A^{\frac{r}{2}})^{\frac{1}{r}} \geq (A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}})^{\frac{1}{r}}\]

hold for \(p \geq 0\) and \(q \geq 1\) with \((1+r)q \geq p+r\).
Theorem F yields the famous Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$" by putting $r = 0$ in (i) or (ii) of Theorem F. Alternative proofs of Theorem F are given in [6] and [18] and also an elementary one page proof in [10]. It was shown by Tanahashi [19] that the domain drawn for $p, q$ and $r$ in the Figure 1 is the best possible one for Theorem F.

As an application of Theorem F, the following result was shown in [7] and [11].

**Theorem FC ([7][11]).** Let $A, B > 0$. Then the following assertions are mutually equivalent:

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{p}{p+r}} \geq B^r$ for all $p \geq 0$ and $r \geq 0$.

(iii) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{p}{p+r}}$ for all $p \geq 0$ and $r \geq 0$.

We remark that this result is an extension of [4] in case $p = r$, and an excellent proof of this result which used only Theorem F was shown in [22].

On the other hand, the following assertions are well known: Let $A$ and $B$ be positive invertible operators. Then

(1) $A \geq B \implies \log A \geq \log B$.

(2) $\log A \geq \log B \implies (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{p}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{p}{p+r}}$ for all $p \geq 0$ and $r \geq 0$.

(3) For each $p \geq 0$ and $r \geq 0$, $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{p}{p+r}} \geq B^r$ \iff $A^p \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{p}{p+r}}$.

(1) holds since $\log t$ is an operator monotone function. (2) is an immediate consequence of Theorem FC. (3) was shown in [11].

Related to these results, it is known in [23] that invertibility of (1) and (2) can be replaced with the condition $N(A) = N(B) = \{0\}$, that is, (1) and (2) hold for some non-invertible operators $A$ and $B$. But we have not known whether invertibility of $A$ and $B$ in (3) can be replaced with looser condition or not. In this report, we shall show relations between

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{p}{p+r}} \geq B^r \text{ and } A^p \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{p}{p+r}}$$

when $A$ and $B$ are not invertible.

Next, An operator $T$ is said to be hyponormal if $T^* T \geq TT^*$. An operator $T$ is invertible log-hyponormal (defined in [20]) if $\log T^* T \geq \log TT^*$. For each $s > 0$ and $t > 0$, an operator $T$ belongs to class $A(s, t)$ (defined in [8]) if $(|T^*|^s |T|^2 |T^*|^t)^{\frac{1}{s+t}} \geq |T|^2t$, where $|T| = (T^* T)^{\frac{1}{2}}$. Class $A(s, t)$ is introduced as a generalization of class $A$.
([T^2| \geq |T|^2]) defined in [14]. We remark that class A equals class A(1, 1) and class A
is introduced as a class of operators including invertible log-hyponormal operators and
included in the class of paranormal operators ([|T^2| \geq |T|^2| for all unit vectors x \in H]).
Moreover, for each s > 0 and t > 0, an operator T belongs to class \( wA(s, t) \) (defined in
[16]) if ([|T^s|T^2|T^s|t|)^{1/t} \geq |T^s|^2t and [T^2s] \geq ([|T^s|T^s|t|T^s|)^{1/t}. Obviously, for each
s > 0 and t > 0, every class \( wA(s, t) \) operator belongs to class A(s, t). As inclusion
relations among these classes, the following assertions hold by (1), (2) and (3):

(1') Every invertible hyponormal operator is log-hyponormal.
(2') Every invertible log-hyponormal operator belongs to class \( wA(s, t) \) for all s > 0 and
t > 0.
(3') For each s > 0 and t > 0, invertible class \( wA(s, t) \) equals invertible class \( A(s, t) \).

There are many papers on these classes in case of invertible operators, for example
[8], [20] and [24].

On the other hand, even if an operator is non-invertible, log-hyponormality can be
defined by \( N(T^s) \supset N(T) \) and \( \log A \geq \log B \), where A and B are the compressions of
\( T^sT \) and \( TT^s \) to \( \overline{R(T)} \), respectively. This definition implicitly appeared in [3] and it was
pointed out in [23] that it is the general form of log-hyponormality. Ando [3] showed
that every hyponormal operator is log-hyponormal and every log-hyponormal operator
is paranormal. Moreover, Uchiyama [23] showed that every log-hyponormal operator is
also included in class A (even if an operator is non-invertible). In this report, we shall
show that for each s > 0 and t > 0, class \( A(s, t) \) coincides with class \( wA(s, t) \), that is,
we shall show (3') without invertibility of operators, and show some properties of class
\( A(s, t) \) operators. Lastly, we shall show a normality of class \( A(s, t) \) operators for s > 0
and t > 0.

2 Relations between
\[
(B^p A^p B^p)^{\frac{r}{p+r}} \geq B^r \quad \text{and} \quad A^p \geq (A^p B^r A^p)^{\frac{p}{p+r}}
\]

In this section, we shall show the following result:

Theorem 1. Let A and B be positive operators. Then for each \( p \geq 0 \) and \( r \geq 0 \), the
following assertions hold:

(i) If \( (B^p A^p B^p)^{\frac{r}{p+r}} \geq B^r \), then \( A^p \geq (A^p B^r A^p)^{\frac{p}{p+r}} \).
(ii) If \( A^p \geq (A^p B^r A^p)^{\frac{p}{p+r}} \) and \( N(A) \subset N(B) \), then \( (B^p A^p B^p)^{\frac{r}{p+r}} \geq B^r \).
We remark on Theorem 1 that the assumption of (ii) has a kernel condition $N(A) \subset N(B)$, but the assumption of (i) does not have any kernel conditions. If $A$ and $B$ are invertible, then $N(A) = N(B) = \{0\}$ holds, and the kernel condition of (ii) in Theorem 1 is satisfied. Hence we know that Theorem 1 is a generalization of (3) in the previous section.

To prove Theorem 1, we prepare the following lemma.

**Lemma 2.** Let $A$ and $B$ be positive operators. Then the following assertions hold:

(i) $\lim_{\varepsilon \to +0} A^{\frac{1}{2}}(A + \varepsilon I)^{-1}A^{\frac{1}{2}} = \lim_{\varepsilon \to +0} (A + \varepsilon I)^{-1}A = P_{N(A)^\perp}$, where $P_{N(A)^\perp}$ is the projection onto a closed subspace $N(A)^\perp$.

(ii) $\lim_{\varepsilon \to +0} A^{\frac{1}{2}}B^{\frac{1}{2}}\{(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\alpha} + \varepsilon I\}^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha}$ for $\alpha \in (0, 1)$.

We remark that if $A$ and $B$ are both positive invertible, then

$$A^{\frac{1}{2}}B^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{-\alpha}B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha}$$

for $\alpha \in (0, 1)$ by the following Lemma F. Therefore we can regard (ii) of Lemma 2 as a non-invertible version of Lemma F for $\lambda \in (0, 1)$.

**Lemma F ([12]).** Let $A$ be a positive invertible operator and $B$ be an invertible operator. Then

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda^2-1}A^{\frac{1}{2}}B^*$$

holds for any real number $\lambda$.

**Proof of Lemma 2.**

**Proof of (i).** We give a proof which is a slightly modification of the proof of [5, Lemma].

Let $A = \int_0^{||A||} t dF(t)$ be the spectral decomposition of $A$. Then

$$\lim_{\varepsilon \to +0} (A + \varepsilon I)^{-1}A = \lim_{\varepsilon \to +0} \int_0^{||A||} \frac{t}{t + \varepsilon} dF(t) = \int_0^{||A||} x(0, ||A||) dF(t) = I - F(0),$$

where $x(0, ||A||) dF(t)$ is the characteristic function on $(0, ||A||)$. Since $I - F(0) = P_{N(A)^\perp}$, we have

$$\lim_{\varepsilon \to +0} A^{\frac{1}{2}}(A + \varepsilon I)^{-1}A^{\frac{1}{2}} = \lim_{\varepsilon \to +0} (A + \varepsilon I)^{-1}A = P_{N(A)^\perp}.$$

**Proof of (ii).** Let $A^{\frac{1}{2}}B^{\frac{1}{2}} = U |A^{\frac{1}{2}}B^{\frac{1}{2}}|$ be the polar decomposition of $A^{\frac{1}{2}}B^{\frac{1}{2}}$. 

Then we have

\[
\lim_{\epsilon \to +0} A^{\frac{1}{2}} B^{\frac{1}{2}} \{(B^{\frac{1}{2}} AB^{\frac{1}{2}})\} + \epsilon I\}^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}}
\]

\[
= \lim_{\epsilon \to +0} U|A^{\frac{1}{2}} B^{\frac{1}{2}}|^{1-\alpha} A^{\frac{1}{2}} B^{\frac{1}{2}} (|A^{\frac{1}{2}} B^{\frac{1}{2}}| \epsilon I)^{-1} U^{*}
\]

\[
= U|A^{\frac{1}{2}} B^{\frac{1}{2}}|^{1-\alpha} P_{N((A^{\frac{1}{2}} B^{\frac{1}{2}})\perp)} A^{\frac{1}{2}} B^{\frac{1}{2}} U^{*} \quad \text{by (i)}
\]

\[
= U|A^{\frac{1}{2}} B^{\frac{1}{2}}|^{2(1-\alpha) U^{*} = |B^{\frac{1}{2}} A^{\frac{1}{2}}|^{2(1-\alpha)}} = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{1-\alpha}.
\]

Hence the proof is complete.\(\square\)

**Proof of Theorem 1.** Let \(\epsilon > 0\). And also we may assume \(p > 0\) and \(r > 0\).

**Proof of (i).** Since \((B^{\frac{1}{2}} A^{p} B^{\frac{1}{2}})\perp \geq B^{r}\), we obtain

\[
\begin{align*}
A^{\frac{p}{2}} B^{\frac{1}{2}} (B^{r} + \epsilon I)^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}} & \geq A^{\frac{p}{2}} B^{\frac{1}{2}} \{(B^{\frac{1}{2}} A^{p} B^{\frac{1}{2}})\}^{\perp} + \epsilon I\}^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}}.
\end{align*}
\]

In (2.1), by tending \(\epsilon \to +0\) and Lemma 2, we obtain

\[
A^{\frac{p}{2}} P_{N(B)^{-\perp}} A^{\frac{1}{2}} \geq (A^{\frac{p}{2}} B^{r} A^{\frac{1}{2}})^{\perp}.
\]

Hence we have

\[
A^{p} \geq A^{\frac{p}{2}} P_{N(B)^{-\perp}} A^{\frac{1}{2}} \geq (A^{\frac{p}{2}} B^{r} A^{\frac{1}{2}})^{\perp}.
\]

**Proof of (ii).** Since \(A^{p} \geq (A^{\frac{p}{2}} B^{r} A^{\frac{1}{2}})^{\perp}\), we obtain

\[
\begin{align*}
B^{\frac{1}{2}} A^{\frac{1}{2}} (A^{p} + \epsilon I)^{-1} A^{\frac{1}{2}} B^{\frac{1}{2}} & \leq B^{\frac{1}{2}} A^{\frac{1}{2}} \{(A^{\frac{1}{2}} B^{r} A^{\frac{1}{2}})^{\perp} + \epsilon I\}^{-1} A^{\frac{1}{2}} B^{\frac{1}{2}}.
\end{align*}
\]

In (2.2), by tending \(\epsilon \to +0\) and Lemma 2, we obtain

\[
B^{\frac{1}{2}} P_{N(A)^{-\perp}} B^{\frac{1}{2}} \leq (B^{\frac{1}{2}} A^{p} B^{\frac{1}{2}})^{\perp}.
\]

On the other hand,

\[
N(A) \subset N(B) \iff P_{N(A)^{-\perp}} \geq P_{N(B)^{-\perp}}.
\]

By (2.3) and (2.4), we have

\[
(B^{\frac{1}{2}} A^{p} B^{\frac{1}{2}})^{\perp} \geq B^{\frac{1}{2}} P_{N(A)^{-\perp}} B^{\frac{1}{2}} \geq B^{\frac{1}{2}} P_{N(B)^{-\perp}} B^{\frac{1}{2}} = B^{r}.
\]

Therefore the proof is complete.\(\square\)
Remark. We recall the assumptions of (i) and (ii) in Theorem 1. Here we assume $p = r = 1$ in Theorem 1.

(i-a) $(B^\frac{1}{2}AB^\frac{1}{2})^\frac{1}{2} \geq B$.

(ii-a) $A \geq (A^\frac{1}{2}BA^\frac{1}{2})^\frac{1}{2}$ and $N(A) \subset N(B)$.

We proved that (i-a) ensures $A \geq (A^\frac{1}{2}BA^\frac{1}{2})^\frac{1}{2}$ and (ii-a) ensures (i-a) in Theorem 1, so we might expect that (i-a) and (ii-a) are equivalent. But we have the following counterexample.

Example 1. $(B^\frac{1}{2}AB^\frac{1}{2})^\frac{1}{2} \geq B$ and $A \geq (A^\frac{1}{2}BA^\frac{1}{2})^\frac{1}{2}$, but $N(A) \not\subset N(B)$.

Let $A = 2\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $A^\frac{1}{2} = \sqrt{\frac{2}{5}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B^\frac{1}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = B$. Hence

$$\sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (B^\frac{1}{2}AB^\frac{1}{2})^\frac{1}{2} \geq B$$

and

$$A \geq (A^\frac{1}{2}BA^\frac{1}{2})^\frac{1}{2} = \sqrt{\frac{2}{5}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$  

But $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \in N(A)$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \not\in N(B)$, so that $N(A) \not\subset N(B)$.

Moreover, we have the following example in [16].

Example 2 ([16]). $A \geq (A^\frac{1}{2}BA^\frac{1}{2})^\frac{1}{2}$, but $(B^\frac{1}{2}AB^\frac{1}{2})^\frac{1}{2} \not\geq B$ and $N(A) \not\subset N(B)$.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then we can check $A \geq (A^\frac{1}{2}BA^\frac{1}{2})^\frac{1}{2}$, $(B^\frac{1}{2}AB^\frac{1}{2})^\frac{1}{2} \not\geq B$ and $N(A) \not\subset N(B)$, easily.

Therefore we recognize that $A \geq (A^\frac{1}{2}BA^\frac{1}{2})^\frac{1}{2}$ requires some condition to be equivalent to $(B^\frac{1}{2}AB^\frac{1}{2})^\frac{1}{2} \geq B$. So we consider the following condition.

(ii-a') $A \geq (A^\frac{1}{2}BA^\frac{1}{2})^\frac{1}{2}$ and $N(A^\frac{1}{2}B^\frac{1}{2}) \subset N(B)$.

We can easily check that $N(A) \subset N(B)$ ensures $N(A^\frac{1}{2}B^\frac{1}{2}) \subset N(B)$. And also $(B^\frac{1}{2}AB^\frac{1}{2})^\frac{1}{2} \geq B$ ensures $N(A^\frac{1}{2}B^\frac{1}{2}) \subset N(B)$ since $N(A^\frac{1}{2}B^\frac{1}{2}) = N(B^\frac{1}{2}AB^\frac{1}{2}) = N((B^\frac{1}{2}AB^\frac{1}{2})^\frac{1}{2}) \subset N(B)$, so that (i-a) ensures (ii-a') by (i) in Theorem 1.

But, unfortunately, we understand that (ii-a') does not ensure (i-a) by the following example.
Example 3. $A \geq (A^{1/2}BA^{1/2})^{1/2}$ and $N(A^{1/2}B^{1/2}) \subset N(B)$, but $(B^{1/2}AB^{1/2})^{1/2} \not\geq B$.

Let $A = \frac{1}{2} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $A^{1/2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = B$. Hence $A \geq (A^{1/2}BA^{1/2})^{1/2} = \frac{1}{\sqrt{50}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

and $N(A^{1/2}B^{1/2}) = N(B) = \{ t \begin{pmatrix} 0 \\ 1 \end{pmatrix} : t \in \mathbb{C} \}$ since $A^{1/2}B^{1/2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. But $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (B^{1/2}AB^{1/2})^{1/2} \not\geq B$.

At the end of this remark, we note that Chô-Huruya-Kim [5] gave an example such that $N(T) \not\subset N(T^*)$, $N(T) \not\supset N(T^*)$ and $|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$ (i.e., $T$ is $w$-hyponormal) by using $A$ and $B$ in Example 1 stated above, where $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ and $T = U|T|$ is the polar decomposition of $T$.

3 Applications

In this section, we shall show some applications of Theorem 1 to operator classes. In section 1, we introduced definitions of some operator classes, here we recall definitions of these classes as follows:

Definition 1. Let $s > 0$, $t > 0$ and $T = U|T|$ be the polar decomposition of $T$.

(i) $T$ belongs to class $A(s, t) \iff (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$.

(ii) $T$ belongs to class $wA(s, t)$

\[ \iff (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t} \text{ and } |T|^{2s} \geq (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \]

\[ \iff |\tilde{T}_{s,t}|^{\frac{2s}{s+t}} \geq |T|^{2t} \text{ and } |T|^{2s} \geq (|\tilde{T}_{s,t}|^{\frac{2s}{s+t}})^{\frac{s}{s+t}} \]

where $\tilde{T}_{s,t} = |T^*|^tU|T|^{t}$ (generalized Aluthge transformation).

(iii) $T$ belongs to class $A \iff |T^2| \geq |T|^2$, that is, $T$ belongs to class $A(1, 1)$.

(iv) $T$ is $w$-hyponormal \( \iff |\tilde{T}| \geq |T| \geq |(\tilde{T})^*| \), that is, $T$ belongs to class $wA(\frac{1}{2}, \frac{1}{2})$,

where $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ (Aluthge transformation).

(i), (ii), (iii) and (iv) of Definition 1 were defined in [8], [16], [14] and [2], respectively. We remark that Aluthge transformation has many interesting properties, and many authors study this transformation, for instance, [1], [13], [15] and [17]. These classes
include invertible log-hyponormal operators, and are included in normaloid (i.e., \( ||T|| = r(T) \), where \( r(T) \) is the spectral radius of \( T \)). It has been known that for each \( s > 0 \) and \( t > 0 \), class \( A(s, t) \) includes class \( wA(s, t) \) by the definitions (i) and (ii). And also for each \( s > 0 \) and \( t > 0 \), every invertible class \( A(s, t) \) operator is an invertible class \( wA(s, t) \) operator, which was shown in [8] and [16]. More precise inclusion relations among class \( wA(s, t) \), and powers of class \( wA(s, t) \) operators were already shown as follows:

**Theorem A ([16], [26]).**

(i) For each \( s > 0 \) and \( t > 0 \), every class \( wA(s, t) \) operator is a class \( wA(\alpha, \beta) \) operator for any \( \alpha \geq s \) and \( \beta \geq t \).

(ii) Let \( T \) be a class \( wA(s, t) \) operator for \( s \in (0, 1] \) and \( t \in (0, 1] \). Then for each natural number \( n, T^n \) belongs to class \( wA(s, t) \).

We remark that Theorem A holds for classes of invertible class \( A(s, t) \) operators instead of class \( wA(s, t) \), which were shown in [8] and [25]. We can summarize inclusion relations among these classes as the following Figure 2. Dotted lines in the diagram mean that we need invertibility of operators to prove the relations.

**Figure 2**

Here, in general, we can obtain that class \( A(s, t) \) coincides with class \( wA(s, t) \) by (i) of Theorem 1 as follows:
Theorem 3. For each $s > 0$ and $t > 0$, the following assertions hold:

(i) Class $A(s, t)$ coincides with class $wA(s, t)$.

(ii) Class $A$ coincides with class $wA(1, 1)$.

(iii) Class $A(\frac{1}{2}, \frac{1}{2})$ coincides with the class of $w$-hyponormal operators, i.e., class $wA(\frac{1}{2}, \frac{1}{2})$.

We can prove Theorem 3 by only applying (i) of Theorem 1 to definitions of these classes, so we omit to prove. By (iii) of Theorem 3, we have

$$|\tilde{T}| \geq |T| \iff (|T^*|^\frac{1}{2}|T||T^*|^\frac{1}{2})^\frac{1}{2} \geq |T^*| \iff T : \text{class } A(\frac{1}{2}, \frac{1}{2})$$

$$\iff T : \text{w-hyponormal} \iff |\tilde{T}| \geq |T| \geq |(\tilde{T})^*|.$$ 

Hence

$$|\tilde{T}| \geq |T| \implies |T| \geq |(\tilde{T})^*|,$$

that is, we may as well define $w$-hyponormality by only $|\tilde{T}| \geq |T|$.

Next, we shall show some properties of class $A(s, t)$ operators without the assumption of invertibility, which are known as properties of invertible class $A(s, t)$ operators and class $wA(s, t)$ operators.

Theorem 4.

(i) For each $s > 0$ and $t > 0$, every class $A(s, t)$ operator is a class $A(\alpha, \beta)$ operator for any $\alpha \geq s$ and $\beta \geq t$.

(ii) Let $T$ be a class $A(s, t)$ operator for $s \in (0, 1]$ and $t \in (0, 1]$. Then for each natural number $n$, $T^n$ belongs to class $A(\frac{s}{n}, \frac{t}{n})$.

Proof is very easy by (i) of Theorem 3 and Theorem A, so we omit the proof, too. By putting $s = t = 1$ in (ii) of Theorem 4 and noting that class $A(\frac{1}{2}, \frac{1}{2})$ equals $w$-hyponormality by (iii) of Theorem 3, we obtain the following result on powers of class A operators without the assumption of invertibility.

Corollary 5. Let $T$ be a class $A$ operator. Then for each natural number $n$, $T^n$ belongs to class $A(\frac{1}{n}, \frac{1}{n})$. Especially $T^2$ is $w$-hyponormal.

At the end of this section, we shall summarize relations among these classes which are obtained in this section as follows: Please compare Figure 3 with Figure 2 stated
4 normality

In this section, we shall show a normality of some non-normal operators. It is well known that if \( T \) and \( T^* \) are hyponormal, then \( T \) is normal. But in the case \( T \) and \( T^* \) belong to weaker class than hyponormal, this assertion is not obvious. Many authors obtained many results on this problem, and the following results were known until now.

**Theorem B ([21]).** If \( T \) is a class \( A \) operator and \( T^* \) is a \( w \)-hyponormal operator, then \( T \) is normal.

**Theorem C ([3]).** If \( T \) and \( T^* \) are paranormal operators satisfying \( N(T) = N(T^*) \), then \( T \) is normal.

Here, we shall generalize Theorem B as follows:

**Theorem 6.** Let \( s_1 > 0, s_2 > 0, t_1 > 0 \) and \( t_2 > 0 \). If \( T \) belongs to class \( A(s_1, t_1) \) and \( T^* \) belongs to class \( A(s_2, t_2) \), then \( T \) is normal.

Put \( s_1 = t_1 = 1 \) and \( s_2 = t_2 = \frac{1}{2} \) in Theorem 6, we have Theorem B by Theorem 3, put \( s_1 = t_1 = s_2 = t_2 = 1 \) in Theorem 6, we obtain the following result on class A:

**Corollary 7.** If \( T \) and \( T^* \) belong to class \( A \), then \( T \) is normal.

To prove Theorem 6, we need the following results:
Lemma 8. Let $A$ and $B$ be self-adjoint operators, and $X \in B(H)$ satisfying

\[ X^*AX \geq X^*BX. \]

If $N(A) \supset N(X^*)$ and $N(B) \supset N(X^*)$, then $A \geq B$.

Proof. Let $H = \overline{R(X)} \oplus N(X^*)$ and $x = Xy + z$, where $y \in H$ and $z \in N(X^*)$. Put $T = A - B$. Then $T = T^*$ and $N(T) \supset N(X^*)$. Hence we have

\[ (Tx, x) = (TXy, Xy) + (Tz, Xy) + (Tz, z) = (X^*TXy, y) \geq 0, \]

that is, $A \geq B$. \hfill $\square$

Proposition 9. Let $A \geq 0$ and $B \geq 0$. If

\[ B^{\frac{1}{2}}AB^{\frac{1}{2}} \geq B^2 \quad (4.1) \]

and

\[ A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^2, \quad (4.2) \]

then $A = B$.

Proof. Put $E = P_{N(A)^\perp}$ and $F = P_{N(B)^\perp}$. (4.1) is equivalent to $B^{\frac{1}{2}}FAF \geq B^2 = B^{\frac{1}{2}}BB^{\frac{1}{2}}$. By Lemma 8, we have $FAF \geq B$ since $N(FAF) \supset N(B^{\frac{1}{2}})$ and $N(B) = N(B^{\frac{1}{2}})$. Then we have

\[ (A^{\frac{1}{2}}FA^{\frac{1}{2}})^2 = A^{\frac{1}{2}}FAFA^{\frac{1}{2}} \geq A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^2 \quad \text{by (4.2),} \]

and we obtain the following (4.3) by Löwner-Heinz theorem.

\[ A^{\frac{1}{2}}FA^{\frac{1}{2}} \geq A. \quad (4.3) \]

(4.3) is equivalent to $A^{\frac{1}{2}}EFEA^{\frac{1}{2}} \geq A^{\frac{1}{2}}EA^{\frac{1}{2}}$. By Lemma 8, we have $EFE \geq E$ since $N(EFE) \supset N(A^{\frac{1}{2}})$ and $N(E) = N(A^{\frac{1}{2}})$. Therefore we obtain $EFE = E$, and then $F \geq E$, so that $N(A) \supset N(B)$. Hence

\[ A \geq B \]

by applying Lemma 8 to (4.1).

By the same way, we also get $B \geq A$, so that $A = B$. \hfill $\square$

Proof of Theorem 6. Let $p = \max\{s_1, s_2, t_1, t_2\}$.

Firstly, if $T$ belongs to class $A(s_1, t_1)$, then $T$ belongs to class $A(p, p)$ by (i) of Theorem 4. This class coincides with class $wA(p, p)$ by (i) of Theorem 3. Hence we have

\[ (|T^*|^p|T|^p|T^*|^p)^{\frac{1}{2}} \geq |T^*|^p \quad \text{and} \quad |T|^p \geq (|T|^p|T^*|^p|T|^p)^{\frac{1}{2}}. \quad (4.4) \]
Secondly, if $T^*$ belongs to class $\mathrm{A}(s_2, t_2)$, then $T^*$ belongs to class $A(p, p)$ by (i) of Theorem 4. This class coincides with class $wA(p, p)$ by (i) of Theorem 3. Hence we have

\[
(|T|^p|T^*|^{2p}|T|^p)^{\frac{1}{2}} \geq |T|^{2p} \quad \text{and} \quad |T^*|^{2p} \geq (|T^*|^p|T|^{2p}|T^*|^p)^{\frac{1}{2}}.
\]

Therefore

\[
|T|^p|T^*|^{2p}|T|^p = |T|^{4p} \quad \text{and} \quad |T^*|^{4p} = |T^*|^p|T|^{2p}|T^*|^p
\]

hold by (4.4) and (4.5), and then $|T| = |T^*|$ by Proposition 9.

\[\square\]

References


[9] T. Furuta, *A $\geq B \geq 0$ assures $(B^rA^pB^r)^{1/q} \geq B^{p+2r}/q$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$*, Proc. Amer. Math. Soc., 101 (1987), 85–88.


