数理解析研究所講究録 1259 巻 2002 年 110-118

On numerical range and polar decomposition of the Aluthge transformation

神奈川大学 山崎 丈明 (Takeaki Yamazaki)
Department of Mathematics, Kanagawa University
東京理科大学 柳田 昌宏 (Masahiro Yanagida)
Department of Applied Mathematics, Tokyo University of Science

This report is based on the following paper:

[Y] T.Yamazaki, *On numerical range of the Aluthge transformation*, Linear Algebra Appl., **341** (2002), 111–117.

Abstract

Let $T = U|T|$ be the polar decomposition of an operator T. Aluthge defined an operator transformation $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ of T which is called Aluthge transformation. In this report, firstly, we shall show $w(T) \geq w(\widetilde{T})$ for any operator T, where $w(T)$ means numerical radius of T.

Secondly, we shall show the following relations: (i) If T is an $n \times n$ matrix, then $W(T) \supset W(\widetilde{T})$. (ii) If $N(T^*) \supset N(T)$, then $\overline{W(T)} \supset \overline{W(\widetilde{T})}$, where $W(T)$ means numerical range of T.

Lastly, we shall show that $\widetilde{T} = VU|\widetilde{T}|$ is also the polar decomposition, where $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = V\left||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right|$ is polar decomposition.

1. Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive if $(Tx, x) \geq 0$ holds for all $x \in H$. For each $T \in B(H)$, we write $W(T)$ for the *numerical range* of T, that is,

$$W(T) = \{(Tx, x) : \|x\| = 1\}.$$

$\overline{W(T)}$ and $w(T)$ mean the closure of $W(T)$ and the numerical radius of T, respectively. An operator T is said to be *spectraloid* if $w(T) = r(T)$, where $r(T)$ is the spectral radius of T.

Let $T = U|T|$ be the polar decomposition of T. Aluthge defined an operator transformation $\widetilde{T}$ of T by $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ in [1]. We call this transformation *Aluthge transformation.* Many authors have studied Aluthge transformation and there are many result on properties of this transformation. For example *"$\sigma(T) = \sigma(\widetilde{T})$ holds for all $T \in B(H)$"* in [2, 4, 9], and as a nice application of Furuta inequality [6], *"if $|T| \geq |T^*|$, then $|\widetilde{T}|^2 \geq |\widetilde{T}^*|^2$ holds"* in [1, 9].

As fundamental properties of Aluthge transformation, $\|T\| \geq \|\widetilde{T}\|$ and $r(T) = r(\widetilde{T})$ hold, obviously. And a result on the spectrum of Aluthge transformation stated above was shown in [2, 4, 9]. Moreover as a result on the numerical range of Aluthge transformation, I.B.Jung, E.Ko and C. Pearcy showed that *"if T is a 2×2 matrix, then $W(T) \supset W(\widetilde{T})$"* in [10].

On the other hand, the polar decomposition of Aluthge transformation has been discussed in [1], but the concrete form has not been obtained, yet.

In this report, firstly, we shall show a property of Aluthge transformation on the numerical radius, that is, $w(T) \geq w(\widetilde{T})$ for all $T \in B(H)$. And as an application of this result, we shall show a characterization of spectraloid operators via Aluthge transformation.

Secondly, we shall show the following inclusion relations: (i) If T is an $n \times n$ matrix, then $W(T) \supset W(\widetilde{T})$. (ii) If $N(T^*) \supset N(T)$, then $\overline{W(T)} \supset \overline{W(\widetilde{T})}$. (i) is an extension of above result by I.B.Jung, E.Ko and C.Pearcy in [10].

Lastly, we shall obtain a concrete form of the polar decomposition of $\widetilde{T}$.

2. Numerical radius

By considering the definition of Aluthge transformation, we can obtain the relations $\|T\| \geq \|\widetilde{T}\| \geq r(\widetilde{T}) = r(T)$, easily. In this section we shall show the following result:

Theorem 1 ([Y]). *Let $T \in B(H)$. Then $w(T) \geq w(\widetilde{T})$.*

To prove Theorem 1, we prepare the following results:

Lemma 2 ([Y]). *Let $T = U|T|$ be the polar decomposition of T. If there exists another decomposition $T = V|T|$, then $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}$.*

Proof. Let $H = N(|T|^{\frac{1}{2}}) \oplus N(|T|^{\frac{1}{2}})^{\perp}$.
In case $x \in N(|T|^{\frac{1}{2}})$. $\widetilde{T}x = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0 = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}x$.

In case $x \in N(|T|^{\frac{1}{2}})^{\perp} = \overline{R(|T|^{\frac{1}{2}})}$. There exists $y \in H$ such that $x = |T|^{\frac{1}{2}}y$. Then we have

$$\begin{aligned}\widetilde{T}x = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = |T|^{\frac{1}{2}}U|T|y = |T|^{\frac{1}{2}}Ty \\ = |T|^{\frac{1}{2}}V|T|y = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}x.\end{aligned}$$

Hence we have $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}$ on $H = N(|T|^{\frac{1}{2}}) \oplus N(|T|^{\frac{1}{2}})^{\perp}$. □

Lemma 3 ([Y]). *Let A be a positive operator, and $X \in B(H)$. Then the following inequality holds:*

$$\|A^rXA^{1-r} - zI\| \leq \|AX - zI\|^r\|XA - zI\|^{1-r} \quad \textit{for all } r \in [0,1] \textit{ and } z \in \mathbb{C}.$$

To prove Lemma 3, the following result is very important:

Theorem A ([7]). *Let A and B be positive operators, and $X \in B(H)$. Then the following inequalities hold:*

(i) $\|A^rXB^r\| \leq \|AXB\|^r\|X\|^{1-r}$ *for* $r \in [0,1]$.
(ii) $\|A^rXB^r\| \geq \|AXB\|^r\|X\|^{1-r}$ *for* $r > 1$.

Proof of Lemma 3. We may assume that A is invertible in this proof. Hence we have

$$\begin{aligned}\|A^rXA^{1-r} - zI\| &= \|A^r(XA - zI)A^{-r}\| \\ &\leq \|A(XA - zI)A^{-1}\|^r\|XA - zI\|^{1-r} \quad \text{by (i) of Theorem A} \\ &= \|AX - zI\|^r\|XA - zI\|^{1-r} \quad \text{for all } r \in [0,1] \text{ and } z \in \mathbb{C}.\end{aligned}$$

□

Let $T = U|T|$ be a decomposition of T. Then by Lemma 2, we have $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. So that we obtain the following inequality by putting $A = |T|$, $X = U$ and $r = \frac{1}{2}$ in Lemma 3.

$$\|\widetilde{T} - zI\| \leq \||T|U - zI\|^{\frac{1}{2}}\|T - zI\|^{\frac{1}{2}} \quad \text{for all } z \in \mathbb{C}. \tag{2.1}$$

Theorem B ([3]). *Let $T \in B(H)$. Then $w(T) \leq 1$ is equivalent to*

$$\|T - zI\| \leq 1 + \{1 + |z|^2\}^{\frac{1}{2}} \quad \textit{for all } z \in \mathbb{C}.$$

Proof of Theorem 1. Firstly, we shall show that for each $S \in B(H)$,

$$w(S) \leq 1 \quad \Longrightarrow \quad w(\widetilde{S}) \leq 1. \tag{2.2}$$

Let $S = U|S|$ be the polar decomposition of S. Then we have

$$(|S|Ux, x) = (U^*U|S|Ux, x) = \left(S\frac{Ux}{\|Ux\|}, \frac{Ux}{\|Ux\|}\right)(U^*Ux, x).$$

Then $W(|S|U) \subset W(S)W(U^*U)$, and we obtain

$$w(|S|U) \leq w(S). \tag{2.3}$$

If $w(S) \leq 1$, then $w(|S|U) \leq w(S) \leq 1$ by (2.3). By Theorem B, we have

$$\|S - zI\| \leq 1 + \{1 + |z|^2\}^{\frac{1}{2}} \quad \text{for all } z \in \mathbb{C}$$

and

$$\||S|U - zI\| \leq 1 + \{1 + |z|^2\}^{\frac{1}{2}} \quad \text{for all } z \in \mathbb{C}.$$

Hence by (2.1) we obtain

$$\|\widetilde{S} - zI\| \leq \||S|U - zI\|^{\frac{1}{2}}\|S - zI\|^{\frac{1}{2}} \leq 1 + \{1 + |z|^2\}^{\frac{1}{2}} \quad \text{for all } z \in \mathbb{C}.$$

And $w(\widetilde{S}) \leq 1$ follows from Theorem B.

Secondly, put $S = \dfrac{T}{w(T)}$. Then $\widetilde{S} = \dfrac{\widetilde{T}}{w(T)}$ and $w(S) \leq 1$. Then by (2.2), we have

$$w(\widetilde{S}) = \frac{w(\widetilde{T})}{w(T)} \leq 1.$$

Hence the proof of Theorem 1 is complete. □

Related to Aluthge transformation, for each natural number n, I.B.Jung, E.Ko and C.Pearcy defined *n-th Aluthge transformation* $\widetilde{T_n}$ of T by $\widetilde{T_n} = \widetilde{(\widetilde{T_{n-1}})}$ and $\widetilde{T_0} = T$ in [10]. By using n-th Aluthge transformation, we showed some properties of Aluthge transformation on operator norms, and pointed out that Aluthge transformation has similar properties to powers of operators in [13, 14, 15]. An operator T is said to be *normaloid* if $\|T\| = r(T)$ which is equivalent to $\|T\| = \|T^n\|^{\frac{1}{n}}$ for all natural number n. It is well known that *"every normaloid operator is spectraloid."* Related to normaloid operators and spectral radius, we obtained the following results:

Theorem C ([14]). *Let $T \in B(H)$. Then the following assertions are equivalent:*

(i) *T is normaloid.*
(ii) *$\|T\| = \|\widetilde{T_n}\|$ for all natural number n.*

Theorem D ([15]). *Let $T \in B(H)$. Then $\lim_{n\to\infty} \|\widetilde{T_n}\| = r(T)$.*

As an application of these results, we obtain a characterization of spectraloid operators which is a parallel result to Theorem C as follows:

Corollary 4 ([Y]). *Let $T \in B(H)$. Then the following assertions are equivalent:*

(i) *T is spectraloid.*
(ii) *$w(T) = w(\widetilde{T_n})$ for all natural number n.*

Proof of Corollary 4. Since $\|T\| \geq w(T) \geq r(T)$, we have

$$\lim_{n\to\infty} w(\widetilde{T_n}) = r(T)$$

by Theorem D. And by Theorem 1, we obtain the following inequalities:

$$w(T) \geq w(\widetilde{T}) \geq \cdots \geq w(\widetilde{T_n}).$$

Hence the proof is complete. □

3. Numerical range

In the previous section, we showed a relation between $w(T)$ and $w(\widetilde{T})$, and obtained a characterization of spectraloid operators. On the other hand, I.B.Jung, E.Ko and C.Pearcy obtained an extension of Theorem 1 in case T is 2×2 matrix, and have conjectured a relation between $W(\widetilde{T})$ and $W(T)$ as follows:

Theorem E ([10]). *Let T be a 2×2 matrix. Then $W(T) \supset W(\widetilde{T})$.*

Conjecture ([10]). *For every $T \in B(H)$, $W(T) \supset W(\widetilde{T})$.*

In this section, we shall show a relation between $W(T)$ and $W(\widetilde{T})$ as a partial solution of above conjecture:

Theorem 5 ([Y]). *Let $T = U|T|$ be a decomposition. If U is isometry, then $\overline{W(T)} \supset \overline{W(\widetilde{T})}$.*

To prove Theorem 5, we cite the following result:

Theorem F ([5]). *Let $T \in B(H)$. Then*

$$\overline{W(T)} = \bigcap_{\mu \in \mathbb{C}} \{\lambda : |\lambda - \mu| \leq w(T - \mu I)\}.$$

Proof of Theorem 5. First, we shall show the following assertion: If $S = V|S|$ is a decomposition such that V is isometry, then for each $\lambda \in \mathbb{C}$,

$$w(S - \lambda I) \leq 1 \quad \Longrightarrow \quad w(\widetilde{S} - \lambda I) \leq 1. \tag{3.1}$$

By (2.1), we have the following inequalities:

$$
\begin{aligned}
\|\widetilde{S} - zI\| &\le \||S|V - zI\|^{\frac{1}{2}}\|S - zI\|^{\frac{1}{2}} \\
&= \|V^*(S - zI)V\|^{\frac{1}{2}}\|S - zI\|^{\frac{1}{2}} \quad \text{by } V^*V = I \\
&\le \|S - zI\| \quad \text{for all } z \in \mathbb{C}.
\end{aligned} \tag{3.2}
$$

Assume that $w(S - \lambda I) \le 1$. Then by (3.2) and Theorem B, we have

$$\|\widetilde{S} - \lambda I - zI\| \le \|S - \lambda I - zI\| \le 1 + \{1 + |z|^2\}^{\frac{1}{2}} \quad \text{for all } z \in \mathbb{C}.$$

Hence we obtain $w(\widetilde{S} - \lambda I) \le 1$ by Theorem B.

Next, for each $\mu \in \mathbb{C}$, put $S = \dfrac{T}{w(T - \mu I)}$ and $\lambda = \dfrac{\mu}{w(T - \mu I)}$. Then $|S| = \dfrac{|T|}{w(T - \mu I)}$ holds, and $S = U\dfrac{|T|}{w(T - \mu I)}$ is a decomposition such that U is isometry, and also $\widetilde{S} = \dfrac{\widetilde{T}}{w(T - \mu I)}$. Moreover $w(S - \lambda I) \le 1$. Then by (3.1), we obtain

$$w(\widetilde{S} - \lambda I) = \frac{w(\widetilde{T} - \mu I)}{w(T - \mu I)} \le 1.$$

It is equivalent to

$$w(\widetilde{T} - \mu I) \le w(T - \mu I) \quad \text{for all } \mu \in \mathbb{C}.$$

Hence the proof is complete by Theorem F. □

By Theorem 5, we obtain the following extension of Theorem E.

Corollary 6 ([Y]). *If T is an $n \times n$ matrix, then $W(T) \supset W(\widetilde{T})$.*

Proof. Since T is an $n \times n$ matrix, there exists a unitary matrix U such that $T = U|T|$. Since it is in the finite dimensional case, $W(T)$ and $W(\widetilde{T})$ are both closed, and the proof is complete by Theorem 5. □

Corollary 7 ([Y]). *Let $T \in B(H)$ with $N(T^*) \supset N(T)$. Then*

$$\overline{W(T)} \supset \overline{W(\widetilde{T})} \supset \overline{W(\widetilde{T}_2)} \supset \cdots \supset \overline{W(\widetilde{T}_n)} \quad \textit{hold for all natural number } n.$$

Proof. Since $N(T^*) \supset N(T)$, we can choose an isometry U such that $T = U|T|$. Then we have $\overline{W(T)} \supset \overline{W(\widetilde{T})}$ by Theorem 5. So we have only to prove $N(\widetilde{T}^*) \supset N(\widetilde{T})$ if $N(T^*) \supset N(T)$.

By the definition of Aluthge transformation, $N(\widetilde{T}) \supset N(T)$ and $N(\widetilde{T}^*) \supset N(T)$ hold, easily. So we shall show $N(T) \supset N(\widetilde{T})$.

Let $x \in N(\widetilde{T})$. Then by $N(|T|^{\frac{1}{2}}) = N(T) \subset N(T^*) = N(|T^*|^{\frac{1}{2}})$, we have

$$\widetilde{T}x = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0 \Longrightarrow Tx = |T^*|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0.$$

Hence we obtain $N(T) \supset N(\widetilde{T})$, and $N(\widetilde{T}^*) \supset N(\widetilde{T}) = N(T)$. So that the proof is complete by Theorem 5. □

At the end of this section, we would like to summarize some good properties of Aluthge transformation. We remark that in case an operator T is invertible, $\widetilde{T}$ can be rewritten as $\widetilde{T} = |T|^{\frac{1}{2}}T|T|^{\frac{-1}{2}}$, in other words, $\widetilde{T}$ is similar to T. So we would like to summarize some properties of Aluthge transformation, powers of operators and operators which is similar to T as follows:

Results on $\widetilde{T_n}$	Results on T^n	Results on $S^{-1}TS$
$\sigma(\widetilde{T_n}) = \sigma(T)$ [1, 4, 9] $r(\widetilde{T_n}) = r(T)$	$\sigma(T^n) = \sigma(T)^n$ $r(T^n)^{\frac{1}{n}} = r(T)$	$\sigma(S^{-1}TS) = \sigma(T)$ $r(S^{-1}TS) = r(T)$
$w(\widetilde{T_n}) \leq w(T)$ Conjecture	$w(T^n)^{\frac{1}{n}} \leq w(T)$	$\bigcap_S \overline{W(S^{-1}TS)} = co\sigma(T)$ [8]
$\|\widetilde{T_n}\| \leq \|T\|$ $\lim_{n\to\infty} \|\widetilde{T_n}\| = r(T)$ [15]	$\|T^n\|^{\frac{1}{n}} \leq \|T\|$ $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T)$	$\inf_S \|S^{-1}TS\| = r(T)$ [12]

Results on $\widetilde{T_n}$	There exists a counterexample
$w(\widetilde{T_n}) \leq \cdots \leq w(\widetilde{T}) \leq w(T)$ $\overline{W(\widetilde{T_n})} \subset \cdots \subset \overline{W(\widetilde{T})} \subset \overline{W(T)}$ if $N(T) \subset N(T^*)$	$w(T^n)^{\frac{1}{n}} \leq w(T^{n-1})^{\frac{1}{n-1}} \leq \cdots \leq w(T)$
$\|\widetilde{T_n}\| \leq \cdots \leq \|\widetilde{T}\| \leq \|T\|$	$\|T^n\|^{\frac{1}{n}} \leq \|T^{n-1}\|^{\frac{1}{n-1}} \leq \cdots \leq \|T\|$

In above tables, $co\sigma(T)$ means convex hull of $\sigma(T)$.

Counterexample ([11]). Let

$$T = \begin{pmatrix} 0 & a^2 \\ b^2 & 0 \end{pmatrix}, \qquad a > b > 0.$$

Then we have $T^{2n} = a^{2n}b^{2n}I$, $T^{2n+1} = a^{2n}b^{2n}T$.

(i) Counterexample of $\|T^n\|^{\frac{1}{n}} \leq \|T^{n-1}\|^{\frac{1}{n-1}} \leq \cdots \leq \|T\|$.

By the above matrix T, we have

$$\|T\| = a^2, \quad \|T^{2n}\|^{\frac{1}{2n}} = ab \quad \text{and} \quad \|T^{2n+1}\|^{\frac{1}{2n+1}} = \{a^{2n}b^{2n}\|T\|\}^{\frac{1}{2n+1}} = ab\left(\frac{a}{b}\right)^{\frac{1}{2n+1}}.$$

Hence we obtain $\|T^{2n+1}\|^{\frac{1}{2n+1}} > \|T^{2n}\|^{\frac{1}{2n}}$ by $a > b > 0$.

(ii) Counterexample of $w(T^n)^{\frac{1}{n}} \leq w(T^{n-1})^{\frac{1}{n-1}} \leq \cdots \leq w(T)$.

By the same way to (i), we have

$$w(T) = \frac{a^2+b^2}{2}, \quad w(T^{2n})^{\frac{1}{2n}} = ab$$

and

$$w(T^{2n+1})^{\frac{1}{2n+1}} = \{a^{2n}b^{2n}w(T)\}^{\frac{1}{2n+1}} = ab\left(\frac{a^2+b^2}{2ab}\right)^{\frac{1}{2n+1}}.$$

Hence we obtain $w(T^{2n+1})^{\frac{1}{2n+1}} > w(T^{2n})^{\frac{1}{2n}}$ by $a > b > 0$ and $\dfrac{a^2+b^2}{2} > ab$.

We can understand that $\widetilde{T}$ has some good properties related to T^n and $S^{-1}TS$, and has some better properties than T^n. Moreover we conjecture the following assertion.

Conjecture. *For any operator T, does $\bigcap_n \overline{W(\widetilde{T_n})} = co\sigma(T)$ hold?*

4. Polar decomposition

In this section, we shall obtain the polar decomposition of $\widetilde{T}$ as follows:

Theorem 8. *Let $T = U|T|$ and*

(4.1) $$|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = V\left||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right|$$

be the polar decompositions. Then $\widetilde{T} = VU|\widetilde{T}|$ is also the polar decomposition.

By Theorem 8, we can obtain the polar decomposition of n-th Aluthge transformation for any natural number n, because the partial isometry which appears in the polar decomposition of $\widetilde{T}$ is a product of two partial isometries.

Proof. (i) Proof of $\widetilde{T} = VU|\widetilde{T}|$.

$$\begin{aligned} VU|\widetilde{T}| &= VU(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}}U^*U \\ &= V(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}}U \\ &= V\left||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right|U \\ &= |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U \quad \text{by (4.1)} \\ &= |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \widetilde{T}. \end{aligned}$$

(ii) We shall show $N(\widetilde{T}) = N(VU)$. Since $N(|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}) = N(V)$, we have

$$N(\widetilde{T}) = N(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) = N(|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U) = N(VU).$$

(iii) We shall prove that VU is a partial isometry. Since $N(VU)^{\perp} = N(|\widetilde{T}|)^{\perp} = \overline{R(|\widetilde{T}|)}$ hold by (ii), for any $x \in N(VU)^{\perp} = \overline{R(|\widetilde{T}|)}$, thete exists $\{y_n\} \subset H$ such that $x = \lim_{n\to\infty} |\widetilde{T}|y_n$. Then we have

$$\begin{aligned} \|VUx\| &= \|VU \lim_{n\to\infty} |\widetilde{T}|y_n\| = \| \lim_{n\to\infty} VU|\widetilde{T}|y_n\| = \| \lim_{n\to\infty} \widetilde{T}y_n\| \quad \text{by (i)} \\ &= \lim_{n\to\infty} \|\widetilde{T}y_n\| = \lim_{n\to\infty} \||\widetilde{T}|y_n\| = \| \lim_{n\to\infty} |\widetilde{T}|y_n\| = \|x\|, \end{aligned}$$

that is, VU is partial isometry.

Therefore the proof is complete. □

References

[1] A.Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory, **13** (1990), 307–315.

[2] A.Aluthge and D.Wang, *w-Hyponormal operators*, Integral Equations Operator Theory, **36** (2000), 1–10.

[3] C.S.Berger and J.G.Stampfli, *Mapping theorems for the numerical range*, Amer. J. Math., **89** (1967), 1047–1055.

[4] M.Chō, *On spectra of* AB *and* BA, Proc. KOTAC, **3** (2000), 15–19.

[5] T.Furuta, *Some characterizations of convexoid operators*, Rev. Roumaine Math. Pures Appl., **18** (1973), 893–900.

[6] T.Furuta, $A \geq B \geq 0$ *assures* $(B^rA^pB^r)^{1/q} \geq B^{(p+2r)/q}$ *for* $r \geq 0$, $p \geq 0$, $q \geq 1$ *with* $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc., **101** (1987), 85–88.

[7] E.Heinz, *Beiträge zur Störungstheoric der Spektralzerlegung*, Math. Ann., **123** (1951), 415–438.

[8] S.Hildebrandt, *Über den numerischen Wertebereich eines operators*, Math. Annal., **163** (1966), 230–247.

[9] T.Huruya, *A note on p-hyponormal operators*, Proc. Amer. Math. Soc., **125** (1997), 3617–3624.

[10] I.B.Jung, E.Ko and C.Pearcy, *Aluthge transforms of operators*, Integral Equations Operator Theory, **37** (2000), 437–448.

[11] T.Kato, *A short introduction to perturbation theory for linear operators*, Springer Verlag, New York, 1982.

[12] G.-C.Rota, *On models for linear operators*, Comm. Pure. Appl. Math., **13** (1960), 469–472.

[13] T.Yamazaki, *Parallelisms between Aluthge transformation and powers of operators*, Acta Sci. Math. (Szeged), **67** (2001), 809–820.

[14] T.Yamazaki, *Characterization of* $\log A \geq \log B$ *and normaloid operators via Heinz inequality*, to appear in Integral Equations Operator Theory.

[15] T.Yamazaki, *An expression of spectral radius via Aluthge transformation*, Proc. Amer. Math. Soc., **130** (2002), 1131–1137.

(Takeaki Yamazaki) Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan

E-mail address: `yamazt26@kanagawa-u.ac.jp`

(Masahiro Yanagida) Department of Applied Mathematics, Faculty of Science, Tokyo University of Science, Tokyo 162-8601, Japan

E-mail address: `yanagida@rs.kagu.sut.ac.jp`