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Kyoto University
REMARKS ON POSITIVE MAPS ON SELF-DUAL CONES

岩手大・人文社会科学部 三浦 康秀 (YASUHIDE MIURA)

ここではヒルベルト空間における selfdual cone を保存する意味での正値写像および作用素の順序 (⊆) に関する基本的な性質を考える。内容は [MI] を部分的に含む。

§1. INTRODUCTION

Let $\mathcal{H}$ be a separable complex Hilbert space with an inner product $(\cdot, \cdot)$. A convex cone $\mathcal{H}^+$ in $\mathcal{H}$ is said to be selfdual if $\mathcal{H}^+ = \{ \xi \in \mathcal{H} | (\xi, \eta) \geq 0 \forall \eta \in \mathcal{H}^+ \}$. The set of all bounded operators is denoted by $L(\mathcal{H})$. For a fixed selfdual cone $\mathcal{H}^+$, we shall write

$$A \preceq B \quad \text{if} \quad (B - A)(\mathcal{H}^+) \subset \mathcal{H}^+, \ A, B \in L(\mathcal{H}).$$

Since $\mathcal{H}$ is algebraically spanned by $\mathcal{H}^+$, the relation '⊆' defines the partial order on $L(\mathcal{H})$.

Recall a selfdual cone associated with a standard von Neumann algebra in the sense of Haagerup [H], which appears in the form $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ where $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$ and $J$ is an isometric involution related to a selfdual cone $\mathcal{H}^+$ in $\mathcal{H}$. For example, $\ell^2^+ = \{ \xi = \{ \lambda_n \} | \lambda_n \geq 0 \}$ is a selfdual cone associated with an abelian standard von Neumann algebra $\ell^\infty$. Then, for $A = (\lambda_{ij}) \in L(\ell^2)$, $A \succeq O$ if and only if $\lambda_{ij} \geq 0$ for $i, j = 1, 2, \cdots$.

Moreover, suppose that $(\mathcal{H}, \mathcal{H}_n^+, n \in \mathbb{N})$ and $(\tilde{\mathcal{H}}, \tilde{\mathcal{H}}_n^+, n \in \mathbb{N})$ are matrix ordered Hilbert spaces. Here $\mathcal{H}_n^+$ denotes a selfdual cone in $\mathcal{H}_n = M_n(\mathcal{H})$. A linear map $A$ of $\mathcal{H}$ into $\tilde{\mathcal{H}}$ is said to be $n$-positive (resp. $n$-co-positive) when the multiplicity map $A_n (= A \otimes \text{id}_n)$ satisfies $A_n \mathcal{H}_n^+ \subset \tilde{\mathcal{H}}_n^+$ (resp. $(A_n \mathcal{H}_n^+) \subset (\tilde{\mathcal{H}}_n^+)^\ast$). Here $\ast(\cdot)$ denotes a set of all transposed matrices. When $A$ is $n$-positive (resp. $n$-co-positive) for all
$n \in \mathbb{N}$, $A$ is said to be completely positive (resp. completely co-positive). Put, for $A \in L(\mathcal{H})$

$$\hat{A}\xi = AJAJ\xi, \quad \xi \in \mathcal{H}.$$ 

It is known that if, in a matrix ordered standard form $(\mathcal{M}, \mathcal{H}, \mathcal{H}_+)$ as introduced in [SW2], $A \in \mathcal{M}$ then $\hat{A}$ is completely positive, and we shall write $\hat{A} \succ_{cp} O$.

§2. Positive maps associated with selfdual cones

We obtain the following proposition for a general selfdual cone in a finite dimensional Hilbert space. In particular, when $\mathcal{H}^+$ is associated with an abelian von Neumann algebra, that is, a matrix is entrywise positive, it is known as the Peron theorem (see, example [HJ, Corollary 8.2.6]).

**Proposition 2.1.** Let $\mathcal{H}$ be an $n$-dimensional Hilbert space with a selfdual cone $\mathcal{H}^+$. If $A$ is an injective linear operator on $\mathcal{H}$ satisfying $A \succ O$, then there exist a number $\lambda > 0$ and a non-zero element $\xi_0 \in \mathcal{H}^+$ such that $A\xi_0 = \lambda\xi_0$.

*Proof.* Put

$$\mathcal{V} = \text{co}\{\xi \in \mathcal{H}^+| \|\xi\| = 1\},$$

where co denotes the convex hull. Consider the map $r$ defined by

$$r(\xi) = \frac{A\xi}{\|A\xi\|}, \xi \in \mathcal{V}.$$  

By assumption $r$ maps $\mathcal{V}$ to itself. Note that $0 \notin \mathcal{V}$. Because, by the Carathéodory theorem (see, for example [La, Theorem 2.23]) any element $\xi \in \mathcal{V}$ can be expressed as

$$\xi = \lambda_1\xi_1 + \cdots + \lambda_s\xi_s,$$

where $\lambda_1, \cdots, \lambda_s > 0, \xi_1, \cdots, \xi_s \in \mathcal{H}^+$ with $\|\xi_1\| = \cdots = \|\xi_s\| = 1$ and $1 \leq s \leq n + 1$. It follows that $\xi \succeq \lambda_1\xi_1(\mathcal{H}^+)$, and so $\|\xi\| \geq \|\lambda_1\xi_1\| = |\lambda_1| > 0$. Since a convex hull of a compact set is compact [La, Theorem 2.30], it follows from Schauder's fixed point theorem [Sd, Satz I] that there exists an element $\xi_0 \in \mathcal{V}$ satisfying $r(\xi_0) = \xi_0$. Hence $A\xi_0 = \|A\xi_0\| \xi_0$. □

The following fundamental proposition is valid for a general selfdual cone. It says that the order $'\succeq'$ is different from the usual order $'\leq'$ based on positivity of hermitian operators in point of compatibility with product.
(2.2). (cf. [IM, Proposition 1]) Let $\mathcal{H}$ be a Hilbert space with a selfdual cone $\mathcal{H}^+$. Then for bounded operators on $\mathcal{H}$ we have the following properties:

1. If $O \leq A_1 \leq B_1$ and $O \leq A_2 \leq B_2$, then $O \leq A_1 A_2 \leq B_1 B_2$. In particular, if $O \leq A \leq B$, then $A^n \leq B^n$ for every natural number $n$.

2. If $O \leq A \leq B$, then $O \leq A^* \leq B^*$.

3. If $A, A^{-1}, B, B^{-1} \geq O$ and $A \leq B$, then $B^{-1} \leq A^{-1}$.

4. If $O \leq A \leq B$, then $||A|| \leq ||B||$.

Proof. We sketch a proof which is similar to [IM].

1. By assumption $A_i(\mathcal{H}^+) \subset \mathcal{H}^+$ and $(B_i - A_i)(\mathcal{H}^+) \subset \mathcal{H}^+$ hold for $i = 1, 2$. Since $B_1 B_2 - A_1 A_2 = B_1(B_2 - A_2) + (B_1 - A_1)A_2$, we obtain the desired inequality.

2. Let $A(\mathcal{H}^+) \subset \mathcal{H}^+$. Then we have $(A^* \xi, \eta) = (\xi, A \eta) \geq 0$ for all $\xi, \eta \in \mathcal{H}^+$. The selfduality of $\mathcal{H}^+$ shows that $A^* \geq O$. Exchanging the role of $A$ and $B - A$ we obtain the desired property.

3. If $A \leq B$, then $B^{-1} = A^{-1}AB^{-1} \leq A^{-1}BB^{-1} = A^{-1}$ from (1).

4. For $A \geq O$, put $||A||_+ = \sup\{||A \xi||; \xi \in \mathcal{H}^+, ||\xi|| \leq 1\}$. Suppose $O \leq A \leq B$. Note that if $\eta - \xi \in \mathcal{H}^+$ for $\xi, \eta \in \mathcal{H}^+$, then $||\xi|| \leq ||\eta||$. Since $||A||_+ \leq ||B||_+$, it suffices to show $||\cdot||_+ = ||\cdot||$. It is known that any element $\xi \in \mathcal{H}$ can be written as $\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4), \xi_1 \perp \xi_2, \xi_3 \perp \xi_4$, for some $\xi_i \in \mathcal{H}^+$. Then $||\xi||^2 = \sum_{i=1}^4 ||\xi_i||^2$. Noticing that $A \geq O$, we see that

$$||A \xi||^2 = \sum_{i=1}^4 ||A \xi_i||^2 - 2(A \xi_1, A \xi_2) - 2(A \xi_3, A \xi_4) \leq ||A(\xi_1 + \xi_2)||^2 + ||A(\xi_3 + \xi_4)||^2 \leq ||A||^2_+ ||\xi||^2.$$  

It follows that $||A|| \leq ||A||_+$. The converse inequality is trivial. \qed

(2.3). Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ be a standard form of a von Neumann algebra. For a selfadjoint element $A \in \mathcal{M} \cup \mathcal{M}'$, the following conditions are equivalent:

1. $A \geq O$.

2. $A \in \mathcal{Z}(\mathcal{M})$ and $A \geq O$.

Proof. (1) $\Rightarrow$ (2): Since $A \geq O$ if and only if $JAJ \geq O$, it suffices to investigate the case $A \in \mathcal{M}$. Suppose $A \geq O, A \in \mathcal{M}$. Since any element of $\mathcal{H}$ can be written

$$||A \xi||^2 = \sum_{i=1}^4 ||A \xi_i||^2 - 2(A \xi_1, A \xi_2) - 2(A \xi_3, A \xi_4) \leq ||A(\xi_1 + \xi_2)||^2 + ||A(\xi_3 + \xi_4)||^2 \leq ||A||^2_+ ||\xi||^2.$$  

It follows that $||A|| \leq ||A||_+$. The converse inequality is trivial. \qed
as \( \xi + i\eta \) with \( J\xi = \xi, J\eta = \eta \), it follows that for such elements \( \xi, \eta \)

\[
JAJ(\xi + i\eta) = JA(\xi - i\eta) = JA\xi + iJA\eta = A(\xi + i\eta).
\]

Hence \( A \in Z(M) \) and \( A^* = JAJ = A \). Choose an arbitrary element \( \xi \in \mathcal{H} \). Then one can write as \( \xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4) \), \( \xi_i \in \mathcal{H}^+ \) such that \( \mathcal{M}\xi_1 \perp \mathcal{M}\xi_2, \mathcal{M}\xi_3 \perp \mathcal{M}\xi_4 \). We then have

\[
(A\xi, \xi) = (A\xi_1 - A\xi_2 + i(A\xi_3 - A\xi_4), \xi_1 - \xi_2 + i(\xi_3 - \xi_4))
\]

\[
= \sum_{i=1}^{4} (A\xi_i, \xi_i) \geq 0
\]

because \( (A\xi_1, \xi_2) = (A\xi_3, \xi_4) = 0 \) and \( ((A(\xi_1 - \xi_2), \xi_3 - \xi_4) \) is a real number. Hence \( A \geq O \).

(2) \( \Rightarrow \) (1): It is immediate. \( \square \)

(2.4). Suppose that \( A \in L(H)^+ \) has a closed range in which \( A\mathcal{H}^+ \) is a selfdual cone. Then we obtain the following properties:

1. Under the condition that \( \mathcal{H}^+ \) is a facially homogeneous selfdual cone in \( \mathcal{H} \), if \( A \geq O \), then for all \( \lambda \in \mathbb{R} \), \( A^\lambda \geq O \).

2. For a matrix ordered standard form \( (\mathcal{M}, \mathcal{H}, \mathcal{H}_{n}^+) \), if \( A \geq O \) and the support projection of \( A \) is completely positive, then for all \( \lambda \in \mathbb{R} \), \( A^\lambda \geq_{cp} O \).

Here the inverse for a not invertible \( A \) is taken as reduced by the support projection of \( A \).

Proof. (1) Let \( P \) denote the support projection of \( A \). By assumption we obtain that \( P \geq O \) and \( PA^+ = A\mathcal{H}^+ \). Hence, by [I, Proposition II.1.6], \( P\mathcal{H}^+ \) is facially homogeneous. Since \( A = PA = AP \) and \( PA \) maps \( P\mathcal{H}^+ \) onto itself, it follows from [I, Corollary II.3.2] that there exists a derivation \( \delta \in D(P\mathcal{H}^+) \) such that \( PA|_{P\mathcal{H}} = e^\delta \). Hence

\[
A^\lambda = Pe^\lambda \delta P \geq O
\]

for every real number \( \lambda \).

(2) Put \( \mathcal{N} = PM|_{P\mathcal{H}} \). Since \( P \) is completely positive, we see from [MN, Lemma 3] that \( (\mathcal{N}, P\mathcal{H}, P\mathcal{H}_{n}^+) \) is a matrix ordered standard form. It follows
from [C, Theorem 3.3] that there exists an element $B \in \mathcal{N}^+$ such that $PA = BJ_{P\mathcal{H}^+}BJ_{P\mathcal{H}^+}P$. Hence

$$A^\lambda = B^\lambda J_{P\mathcal{H}^+}B^\lambda J_{P\mathcal{H}^+}P \succeq_{cp} O$$

for every real number $\lambda$. □

A simple counter-example can show that it is essential in the above proposition for $A\mathcal{H}^+$ to be selfdual. In fact, we obtain the following remark:

**Remark.** In the case $\mathbb{C}^n^+$ (non-negative entries), a necessary and sufficient condition for $A \in M_n^+$ to enjoy $AC^n^+ = \mathbb{C}^n^+$ is that $A$ is a non-singular positive definite diagonal matrix. We obtain the following facts:

1. In the case $\mathbb{C}^n^+$, if $A \in M_n^+$ and $A \succeq O$, then there exists a real number $s \geq 1$ such that $A^\lambda \succeq O$ for all $\lambda \in [s, +\infty)$.

2. In the case $\mathbb{C}^n^+$, if $A \in M_n^+, A \succeq O, \det A \neq 0$ and $AC^n^+ \subsetneq \mathbb{C}^n^+$, then there exists a real number $s' < 0$ such that $A^\lambda \nsubseteq O$ for all $\lambda \in (-\infty, s']$.

Indeed, let $A \in M_n$ be entrywise positive and positive semi-definite. We may assume $\| A \| = 1$. Let $1, a_1, \cdots, a_m, 0 \leq m \leq n - 1$, be distinct eigenvalues of $A$. Since $A$ can be diagonalized by a real orthogonal matrix, each entry of $A^\lambda$ is written in the form

$$f(\lambda) = \alpha_0 + \alpha_1 a_1^\lambda + \cdots + \alpha_m a_m^\lambda$$

for some real numbers $\alpha_k$. Then $\alpha_0$ must be positive, since $A^n \succeq O$ for all $n \in \mathbb{N}$ by (2.2) (1) and $0 \leq a_k < 1, 1 \leq k \leq m$. From the continuity of the function we can find a number $s \geq 1$ such that $f(\lambda) > 0$ for all $\lambda \geq s$. So (1) holds. Suppose, in addition, that $A$ is non-singular and $A\mathbb{C}^n^+ \subsetneq \mathbb{C}^n^+$. If $A^{-\lambda_0} \succeq O$ for some $\lambda_0 > 0$, then $A^{-\ell\lambda_0} \succeq O$ for all $\ell \in \mathbb{N}$. From (1), $A^{\ell\lambda_0} \succeq O$ for a large $\ell \in \mathbb{N}$. This implies that $A^{\ell\lambda_0}$ is diagonal, and so is $A$, a contradiction. Therefore, (2) holds.

(2.5). For a matrix ordered standard form $(M, \mathcal{H}, \mathcal{H}_n^+)$, suppose that $A \in L(\mathcal{H})$, and $B \in M$ is an injective operator with a dense range. Then, $O \preceq A \preceq \hat{B}$ if and only if there exists an element $C \in Z(M)$ with $O \preceq C \preceq I$ such that $A = C\hat{B}$. In particular, if $M$ is a factor, then one can choose a scalar $\lambda$ with $0 \leq \lambda \leq 1$ such that $A = \lambda\hat{B}$. 
Proof. Consider the polar decomposition $B = U|B|$ of $B$. By assumption $U$ is a unitary element of $\mathcal{M}$, and so $\hat{U} \succeq O$ and $\hat{U}^* \succeq O$ by (2.2). Hence we may assume $B$ to be positive semi-definite. Let $B = \int_0^\|B\| \lambda dE_\lambda$ be a spectral decomposition of $B$. Put $P_n = \int_n^\|B\| \mathbf{1} dE_\lambda$ for $n \in \mathbb{N}$. Then one sees that $\hat{P}_n \not\subset I$ and $\hat{P}_n A\hat{P}_n \not\subset \hat{P}_n \hat{B} \hat{P}_n$ by (2.2). Since $\hat{P}_n \hat{B} \hat{P}_n$ is invertible on $\hat{P}_n \mathcal{H}$, where the inverse shall be denoted by $(\hat{P}_n \hat{B} \hat{P}_n)^{-1}$, we have

$$O \not\subset \hat{P}_n A\hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \not\subset \hat{P}_n.$$ 

There then exists an element $c_n$ in an order ideal $Z_{\hat{P}_n \mathcal{H}^+}$ of a selfdual cone $\hat{P} \mathcal{H}^+$ with $\|c_n\| \leq 1$ such that $\hat{P}_n A\hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \xi = c_n \xi$ for all $\xi \in \hat{P}_n \mathcal{H}$. By [I, Theorem VI.1.2 3)] we obtain that $c_n \in Z(\hat{P}_n \mathcal{M}|_{\hat{P}_n \mathcal{H}})^+$. Since $\hat{P}_n Z(\mathcal{M})\hat{P}_n = Z(\hat{P}_n \mathcal{M}\hat{P}_n)$, we can find an element $C_n \in Z(\mathcal{M})$ such that $c_n \xi = \hat{P}_n C_n \hat{P}_n \xi$ for all $\xi \in \hat{P}_n \mathcal{H}$. Since $P_n B = BP_n$, $n \in \mathbb{N}$, we have

$$\hat{P}_{n+1} C_{n+1} \hat{P}_{n+1} \xi = \hat{P}_{n+1} A\hat{P}_{n+1} (\hat{P}_{n+1} \hat{B} \hat{P}_{n+1})^{-1} \hat{P}_n \xi = \hat{P}_n A\hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \xi = \hat{P}_n C_n \hat{P}_n \xi$$

for all $\xi \in \hat{P}_n \mathcal{H}$. Since $\{\hat{P}_n C_n \hat{P}_n\}$ is a bounded sequence, one can define

$$C \xi = \lim_{n \to \infty} \hat{P}_n C_n \hat{P}_n \xi, \ \xi \in \mathcal{H}.$$ 

Thus $C \in Z(\mathcal{M})$, $O \leq C \leq I$ and we get

$$A = \operatorname{s-lim}_{n \to \infty} \hat{P}_n A\hat{P}_n = \operatorname{s-lim}_{n \to \infty} \hat{P}_n C_n \hat{P}_n A\hat{P}_n = C \hat{B}.$$ 

The converse implication is immediate. Indeed, if $C \in Z(\mathcal{M})$ with $O \leq C \leq I$, then $I - C \succeq O$, and so$I - C \succeq O$. Hence $\hat{B} - C \hat{B} = (I - C)\hat{B} \succeq O$. This completes the proof. \ }
§3. COMPLETE ORDER OF OPERATORS

Consider two matrix ordered standard forms \((\mathcal{M}^{(1)}, \mathcal{H}^{(1)}, \mathcal{H}_{n}^{(1)+})\) and \((\mathcal{M}^{(2)}, \mathcal{H}^{(2)}, \mathcal{H}_{n}^{(2)+})\) with respective canonical involutions \(J^{(1)}\) and \(J^{(2)}\). For an arbitrary element \(\xi \in \mathcal{H}^{(1)}\), let \(R_{\xi}\) be a right slice map of \(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}\) into \(\mathcal{H}^{(2)}\) such that

\[
R_{\xi}(\xi' \otimes \eta') = (\xi', \xi)\eta', \xi' \in \mathcal{H}^{(1)}, \eta' \in \mathcal{H}^{(2)}.
\]

For any element \(x \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}\), we put

\[
r(x)\xi = R_{J^{(1)}}(\epsilon x), \xi \in \mathcal{H}^{(1)}.
\]

Then \(r(x)\) is a map of Hilbert-Schmidt class of \(\mathcal{H}^{(1)}\) to \(\mathcal{H}^{(2)}\). A set of all maps of Hilbert-Schmidt class of \(\mathcal{H}^{(1)}\) to \(\mathcal{H}^{(2)}\) is denoted by \(HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})\). A set of all completely positive maps of \((\mathcal{H}^{(1)}, \mathcal{H}_{n}^{(1)+})\) to \((\mathcal{H}^{(2)}, \mathcal{H}_{n}^{(2)+})\) in \(HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})\) is denoted by \(CPHS(\mathcal{H}^{(1)}+\times \mathcal{H}^{(2)}+\))\). Here \(\mathcal{H}_{n}^{(1)+}\), \(n \in \mathbb{N}\), means a family of the self-dual cones associated with \(\mathcal{M}^{(1)}\), that is \(\mathcal{H}_{n}^{(1)+} = \{[\xi_{ij}]_{i,j=1}^{n} | [\xi_{ij}]_{i,j=1}^{n} \in \mathcal{H}_{n}^{(1)+}\}\). We shall write \(\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+}\) for a selfdual cone associated with a von Neumann tensor product \(\mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)}\). It was shown in [MT, SW1] that

\[
\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+} = \{x \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} | r(x) \in CPHS(\mathcal{H}^{(1)}+\times \mathcal{H}^{(2)}+\))\}.
\]

Thus

\[r : \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \rightarrow HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})\]

is an isometry mapping \(\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+}\) onto \(CPHS(\mathcal{H}^{(1)}+\times \mathcal{H}^{(2)}+\))\).

Indeed, \(r\) is isometric. Suppose that \(HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})\) has an inner product

\[
\langle A, B \rangle = \sum_{k=1}^{\infty} (A e_{k}, B e_{k}),
\]

where \(\{e_{k}\}\) is a complete orthogonal basis of \(\mathcal{H}^{(1)}\). Noticing that \(\{J^{(1)} e_{k}\}\) is a complete orthogonal basis of \(\mathcal{H}^{(1)}\), we obtain for a complete orthogonal basis \(\{f_{k}\}\)
\[ \langle r(J^{(1)}e_{i} \otimes f_{j}), r(J^{(1)}e_{i'} \otimes f_{j'}) \rangle = \sum_{k=1}^{\infty} (r(J^{(1)}e_{i} \otimes f_{j})(e_{k}), r(J^{(1)}e_{i'} \otimes f_{j'})(e_{k})) = \sum_{k=1}^{\infty} (R_{J^{(1)_{Ck}}}(J^{(1)}e_{i} \otimes f_{j}', J^{(1)}e_{i'} \otimes f_{j'})) \]
\[ = \sum_{k=1}^{\infty} (J^{(1)}e_{i}, J^{(1)}e_{k})f_{j}, (J^{(1)}e_{i'}, J^{(1)}e_{k})f_{j'}) = \delta_{i'i'} \delta_{jj'} \]

for \( i, j, i', j' = 1, 2, \ldots \).

Therefore, \( (r(\mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)})r^{-1}, HS(H^{(1)}, H^{(2)}), r(J^{(1)} \otimes J^{(2)})r^{-1}, CPHS(H^{(1)+}, H^{(2)+})) \) is a standard form. Using the Radon-Nikodym theorem for \( L^{2} \)-spaces [S, Theorem 1.2], we obtain the following theorem:

(3.1). Let \( (\mathcal{M}, H, H^{+}) \) be a matrix ordered standard form. Then \( (r(\mathcal{M}' \otimes \mathcal{M})r^{-1}, HS(H, H), r(J \otimes J)r^{-1}, CPHS(H^{+}, H^{+})) \) is a standard form which is isomorphic to \( (\mathcal{M}' \otimes \mathcal{M}, H \otimes H, J \otimes J, H^{+} \otimes H^{+}) \) by the identification \( r : H \otimes H \rightarrow HS(H, H) \) defined as above. If \( A, B \in HS(H, H) \) satisfies \( O \leq_{cp} A \leq_{cp} B \), then there exists an element \( C \in \mathcal{M}' \otimes \mathcal{M} \) with \( O \leq C \leq I \) such that \( A = rC r^{-1} B \).

(3.2). If in (3.1) \( \mathcal{M} \) is an injective factor (or semi-finite injective von Neumann algebra) on a separable Hilbert space \( H \), then the above statement is valid for \( A \in L(H) \) instead of \( A \in HS(H, H) \).

Proof. Suppose that \( \mathcal{M} \) is the von Neumann algebra in the statement. There then exists an increasing net \( \{ E_{i} \} \) of completely positive projections of finite rank on \( H \) which converges strongly to 1 by [M1, Theorem 1.4]. It follows that \( O \leq_{cp} E_{i} A \leq_{cp} E_{i} B \). Hence
\[ \text{Tr}(A^{*}E_{i}A) \leq \text{Tr}(B^{*}E_{i}B) \leq \text{Tr}(B^{*}B). \]
Considering a limit with respect to \( i \), we have \( \text{Tr}(A^{*}A) < +\infty \). Using (3.1) we obtain the desired result. \( \square \)
(3.3). For a matrix ordered standard form $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$, any element $A \in HS(\mathcal{H})$ can be uniquely decomposed into the following:

$$A = A_1 - A_2 + i(A_3 - A_4)$$

where $A_1 \perp A_2, A_3 \perp A_4, A_i \in CPHS(\mathcal{H}^+)$. The proof of the above proposition is immediate from a decomposition theorem of vectors in the ordered Hilbert space.

§4. DECOMPOSITION OF POSITIVE MAPS

The purpose of this section is to show that any order isomorphism between non-commutative $L^2$-spaces associated with von Neumann algebras is decomposed into a sum of a completely positive and a completely co-positive maps. The result is an $L^2$ version of a theorem of Kadison [K] for a Jordan isomorphism on operator algebras.

We first generalize a theorem of A. Connes [C] for the polar decomposition of an order isomorphism, to the case where a von Neumann algebra is non-$\sigma$-finite.

(4.1). Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \tilde{J}, \tilde{\mathcal{H}}^+)$ be standard forms, and $A$ be a linear bijection of $\mathcal{H}$ onto $\tilde{\mathcal{H}}$ satisfying $A\mathcal{H}^+ = \tilde{\mathcal{H}}^+$. Then for a polar decomposition $A = U|A|$ of $A$ we obtain the following properties:

1. There exists a unique invertible operator $B$ in $\mathcal{M}^+$ such that $|A| = BJBJ$.
   (cf. [I, Corollary II.3.2])

2. There exists a unique Jordan $*$-isomorphism $\alpha$ of $\mathcal{M}$ onto $\tilde{\mathcal{M}}$ such that

$$\langle \alpha(X)\xi, \xi \rangle = \langle XU^{-1}\xi, U^{-1}\xi \rangle$$

for all $X \in \mathcal{M}, \xi \in \tilde{\mathcal{H}}^+$.

Proof. (1) Let $\mathcal{M}$ be non-$\sigma$-finite. Choose an increasing net $\{p_i\}_{i \in I}$ of $\sigma$-finite projections in $\mathcal{M}$ converging strongly to 1. Put $q_i = p_i J p_i J$. By [[C, Theorem 4.2]] $q_i \mathcal{H}^+$ is a closed face of $\tilde{\mathcal{H}}^+$. Since $A$ is an order isomorphism, $A(q_i \mathcal{H}^+)$ is a closed face of $\tilde{\mathcal{H}}^+$. There then exists a $\sigma$-finite projection $p'_i \in \tilde{\mathcal{M}}$ such that $A(q_i \mathcal{H}^+) = q'_i \mathcal{H}^+$ where $q'_i$ denotes $p'_i J p'_i J$. Hence $q'_i A q_i$ is an order isomorphism.
of $q_i\mathcal{H}^+$ onto $q_i'\tilde{\mathcal{H}}^+$. These cones appear respectively in the reduced standard forms $(q_i\mathcal{M}q_i, q_i\mathcal{H}, q_iJq_i, q_i\mathcal{H}^+)$ and $(q_i'\tilde{\mathcal{M}}q_i', q_i'\tilde{\mathcal{H}}, q_i'Jq_i', q_i'\tilde{\mathcal{H}}^+)$. Put $A_i = (q_i'Aq_i)^*q_i'Aq_i$. Then $A_i \in q_i\mathcal{M}^+q_i$ is an order automorphism on $q_i\mathcal{H}^+$. By [C, Theorem 3.3] there exists a unique invertible operator $B_i \in q_i\mathcal{M}^+q_i$ such that $A_i = B_iJ_iB_iJ_i$, where $J_i$ denotes $q_iJq_i$. Taking a logarithm of both sides, we have $\log A_i = \log B_i + J_i(\log B_i)J_i$. Since $\{A_i\}$ is a bounded net, $\{\log B_i\}$ is bounded. Indeed, we have in a standard form that a map

$$X \mapsto \delta_X = \frac{1}{2}(X + JXJ)$$

is a Jordan isomorphism of a selfadjoint part of $\mathcal{M}$ into a selfadjoint part of a set of all order derivations $D(\mathcal{H}^+)$ by [I, Corollary VI.2.3]. It is known that any isomorphism of a JB-algebra into another JB-algebra is isometry(see [HS, Proposition 3,4,3]). Hence

$$\|\delta_X\| = \|X\|, \quad X \in \mathcal{M}_{\text{s.a.}}.$$

Thus $\{\log B_i\}$ is bounded. It follows that $\{p_i(\log B_i)p_i\}$ is bounded because $p_i\mathcal{M}p_i$ and $q_i\mathcal{M}q_i$ are $*$-isomorphic. Therefore, one can find a subnet of $\{p_i \log B_i p_i\}$ which converges to some element $C \in \mathcal{M}^+$ in the $\sigma$-weak topology. We may index the subnet as the same $i \in I$. We then have for $\xi, \eta \in \mathcal{H}$

$$(C + JJC)q_i\xi, q_i\eta = \lim_i((p_i(\log B_i)p_i + Jp_i(\log B_i)p_iJ)q_i\xi, q_i\eta)$$

$$= ((\log B_i + J_i(\log B_i)J_i)q_i\xi, q_i\eta)$$

$$= \lim_i(\log A_iq_i\xi, q_i\eta)$$

$$= (\log A^*Aq_i\xi, q_i\eta),$$

using the facts that $q_iXq_iJq_iXq_iJq_i = p_iXp_iJp_iXp_iJq_i$ for all $X \in \mathcal{M}$, and under the strong topology $\{A_i\}$ converges to $A^*A$; hence $\{q_i(\log A_i)q_i\}$ converges to $\log A^*A$. Since $\bigcup_{i \in I} q_i\mathcal{H}$ is dense in $\mathcal{H}$, we obtain the equality $C + JJC = \log A^*A$. Therefore, $e^{C}Je^{C}J = A^*A$. Thus there exists an element $B \in \mathcal{M}^+$ such that $|A| = BJBJ$. Since, in addition, $q_iBq_iJq_iBq_iJq_i = q_i|A|q_i$, one easily sees the invertibility and the unicity of $B$ using the same properties as in the $\sigma$-finite case.
From (1) we have $U = AB^{-1}JB^{-1}J$. It follows that $U$ is an isometry satisfying $U\mathcal{H}^+ = \tilde{\mathcal{H}}^+$. Let $p_i$ and $q_i$ be as in (1). There then exists a $\sigma$-finite projection $p_i' \in \tilde{\mathcal{M}}$ such that $U(q_i\mathcal{H}^+) = q_i'\tilde{\mathcal{H}}^+$ with $q_i' = p_i'\tilde{J}p_i'\tilde{J}$. Using also [C, Theorem 3.3], one can find a unique Jordan $*$-isomorphism $\alpha_i$ of $q_i\mathcal{M}q_i$ onto $q_i'\tilde{\mathcal{M}}q_i'$ such that

$$(\alpha_i(q_iXq_i)\xi, \xi) = (q_iXq_iU^{-1}\xi, U^{-1}\xi)$$

for all $X \in \mathcal{M}, \xi \in q_i'\tilde{\mathcal{H}}^+$. Fixed now $X \in \mathcal{M}_{s.a.}$. Since $p_i'\tilde{\mathcal{M}}p_i'$ and $q_i'\tilde{\mathcal{M}}q_i'$ are $*$-isomorphic, there exists a unique operator $Y_i \in p_i'\tilde{\mathcal{M}}_{s.a.}p_i'$ such that $Y_i|_{q_i'\tilde{\mathcal{H}}} = \alpha_i(q_iXq_i)$. Using an isometry between the Jordan algebras, one sees that $\{\alpha_i(q_iXq_i)\}$ is a bounded net, because $||\alpha_i(q_iXq_i)|| = ||q_iXq_i|| \leq ||X||, i \in I$. Thus $\{Y_i\}$ is bounded. We may then say that $\{Y_i\}$ converges to some operator $Y \in \tilde{\mathcal{M}}_{s.a.}$ in the $\sigma$-weak topology. We then have for $\xi \in \tilde{\mathcal{H}}^+$

$$(Yq_j'\xi, q_j'\xi) = \lim_i(Yq_j'\xi, q_j'\xi) = \lim_i(\alpha(q_iXq_i)q_j'\xi, q_j'\xi)
\quad = \lim_i(q_iXq_iU^{-1}q_j'\xi, U^{-1}q_j'\xi)
\quad = (XU^{-1}q_j'\xi, U^{-1}q_j'\xi).$$

Taking a limit with respect to $j$, we obtain

$$(Y\xi, \xi) = (XU^{-1}\xi, U^{-1}\xi)$$

for all $\xi \in \tilde{\mathcal{H}}^+$. It is known that any normal state on the von Neumann algebra $\tilde{\mathcal{M}}$ is represented by a vector state with respect to an element of $\tilde{\mathcal{H}}^+$ (see [H, Lemma 2.10 (1)]). Therefore, the above element $Y$ is uniquely determined. Moreover, we have $q_i'Yq_i' = \alpha_i(q_iXq_i)$. It follows that $\{\alpha_i(q_iXq_i)\}$ converges to $Y$ in the strong topology. Hence one can define $\alpha(X) = Y$ for all $X \in \mathcal{M}$. It is now immediate that $\alpha(X^2) = \alpha(X)^2$ for all $X \in \mathcal{M}_{s.a.}$. Considering the inverse order isomorphism $U^{-1}$, we have $\alpha(\mathcal{M}) = \tilde{\mathcal{M}}$. This completes the proof. \(\square\)

In the following proposition we deal with a reduced matrix ordered standard form by a completely positive projection.

(4.2). With $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ a matrix ordered standard form, let $E$ be a completely positive projection on $\mathcal{H}$. Then $(EM\mathcal{E}, E\mathcal{H}, E\mathcal{H}_n)$ is a matrix ordered standard form.
Proof. The statement was shown in [MN, Lemma 3] where $M$ is $\sigma$-finite. In the case where $M$ is not $\sigma$-finite, since $E$ is a completely positive projection, there exists a von Neumann algebra $N$ such that $(N, E\mathcal{H}, E_n\mathcal{H}_n^+)$ is a matrix ordered standard form by [M2, Lemma 3]. Hence $EM|_{E\mathcal{H}} = N$ and $(EME, E\mathcal{H}, E_n\mathcal{H}_n^+)$ is a matrix ordered standard form by using the same discussion as in the proof in [M3].

Now, we shall state the decomposition theorem for an order isomorphism between non-commutative $L^2$-spaces.

(4.3). Let $(M, H, H_+^n)$ and $(\tilde{M}, \tilde{H}, \tilde{H}_n^+)$ be matrix ordered standard forms. Suppose that $A$ is a $1$-positive map of $H$ into $\mathcal{H}$ such that $AH^+$ is a selfdual cone in the closed range of $A$. If both the support projection $E$ and the range projection $F$ of $A$ are completely positive, then there exists a central projection $P$ of $EME$ such that $AP$ is completely positive and $A(E - P)$ is completely co-positive.

In particular, if $A$ is an order isomorphism of $H$ onto $\tilde{H}$, then there exists a central projection $P$ of $M$ such that $AP$ is completely positive and $A(1 - P)$ is completely co-positive.

Proof. We first consider the case where $A$ is an order isomorphism. Let $U, B$ and $\alpha$ be as in (4.1). It follows from a theorem of Kadison [K] that there exists a central projection $P$ of $M$ satisfying

$$\alpha : M_P \rightarrow \tilde{M}_{\alpha(P)}, \text{ onto } *\text{-isomorphism}$$

and

$$\alpha : M_{1-P} \rightarrow \tilde{M}_{\alpha(1-P)}, \text{ onto } *\text{-anti-isomorphism}.$$ 

Indeed, $\alpha(P)$ is a central projection of $\tilde{M}$. Since $\alpha$ preserves a $*$-operation and power, $\alpha(P)$ is a projection. Suppose that $Q$ is an arbitrary projection in $M$. Since $\alpha$ is order preserving, we have $\alpha(QP) \leq \alpha(P)$ and $\alpha(Q(1 - P)) \leq \alpha(1 - P)$. It follows that two projections $\alpha(P)$ and $\alpha(QP)$ are commutative, and so are $\alpha(1-P)$ and $\alpha(Q(1 - P))$. Hence $\alpha(Q) = \alpha(QP + Q(1 - P))$ and $\alpha(P)$ commute. Since $\alpha$ is bijective, a set $\alpha(Q)$ generates a von Neumann algebra $\tilde{M}$. Therefore, $\alpha(P)$ belongs to a center of $\tilde{M}$. Now, there then exists a unique completely positive
isometry $u : \mathcal{H} \to \alpha(P)\tilde{\mathcal{H}}$ such that

$$u(\mathcal{H}^+) = \alpha(P)\tilde{\mathcal{H}}^+$$
and
$$\alpha(x) = uxu^{-1}, \quad x \in \mathcal{M}_P$$

by [M3, Proposition 2.4] which is also valid for the non-σfinite case. Hence

$$(UXU^{-1}\xi, \xi) = (ux^{-1}\xi, \xi), \quad x \in \mathcal{M}_P, \quad \xi \in \alpha(P)\tilde{\mathcal{H}}^+.$$ We have from the unicity of a completely positive isometry $UP = u$. Note that $\alpha(P)UP = UP$. Indeed, we have for $\xi, \in \alpha(1 - P)\tilde{\mathcal{H}}^+$ the equality

$$\|PU^{-1}\xi\|^2 = (UPU^{-1}\xi, \xi) = (\alpha(P)\xi, \xi) = 0.$$ This yields $PU^{-1}\alpha(1 - P) = O$, and so $PU^{-1} = PU^{-1}\alpha(P)$. Therefore, we obtain $AP = UB\tilde{J}BJP = uBJBJP$ and $AP$ is completely positive.

We next consider a $*$-isomorphism $\alpha' : \mathcal{M}_1-P \to \tilde{\mathcal{M}}_1-P$ defined by $\alpha'(X) = \tilde{J}\alpha(X)^*\tilde{J}, X \in \mathcal{M}_1-P$. There then exists a unique completely positive isometry $v : (1 - P)\mathcal{H} \to \alpha(1 - P)\tilde{\mathcal{H}}$ such that

$$v(1 - P)\mathcal{H}^+ = (1 - \alpha(P))\tilde{\mathcal{H}}^+$$
and
$$\alpha'(x) = vxv^{-1}, \quad x \in \mathcal{M}_1-P.$$ Then we have $\alpha(x) = \tilde{J}vx^{*}v^{-1}\tilde{J}, x \in \mathcal{M}_1-P$. Note that the complete positivity above means $v(1 - P)\mathcal{H}^+ = (1 - \alpha(P))\tilde{\mathcal{H}}^+$, where $\tilde{\mathcal{H}}^+$ denotes the selfdual cones associated with $\tilde{\mathcal{M}}'$. Hence $v$ is a completely co-positive map under the setting $(\mathcal{M}, \mathcal{H}, \mathcal{H}^+)$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \tilde{\mathcal{H}}^+)$. Hence

$$(UXU^{-1}\xi, \xi) = (\tilde{J}vx^{*}v^{-1}\tilde{J}\xi, \xi)$$
$$= (\tilde{J}\xi, vx^{*}v^{-1}\tilde{J}\xi)$$
$$= (uxu^{-1}\xi, \xi)$$
for all $x \in \mathcal{M}_1-P, \xi, \in (1 - P)\mathcal{H}^+$. It follows that $U(1 - P) = v$. We conclude by the equality $A(1 - P) = vBJBJ(1 - P)$ that $A(1 - P)$ is completely co-positive.

We now consider a general $A$. Since $A\mathcal{H}^+ \subset \tilde{\mathcal{H}}^+$, we have $A\mathcal{H}^+ \subset F\tilde{\mathcal{H}}^+$. Since $F$ is a projection, $F\tilde{\mathcal{H}}^+$ is a selfdual cone in $F\tilde{\mathcal{H}}$. It follows from the selfduality of $A\mathcal{H}^+$ that $A\mathcal{H}^+ = F\tilde{\mathcal{H}}^+$. This yields from (4.2) that $FAE$ is an order isomorphism of $E\mathcal{H}$ onto $F\tilde{\mathcal{H}}$ in the sense of matrix ordered standard forms $(E\mathcal{M}E, E\mathcal{H}, E_n\alpha^+_n)$.
and \((F\tilde{M}F, F\tilde{H}, F_{n}\tilde{H}_{n}^{+})\). Using the first part of the proof, we obtain the desired result. Indeed, there exists a central projection \(P \in EME\) such that \(FAP\) is completely positive and \(FA(E - P)\) is completely co-positive under the reduced matrix ordered standard forms. We obtain the inclusion

\[ t(A_n(E_n - P_n)\mathcal{H}_n^+) = t(F_nA_n(E_n - P_n)\mathcal{H}_n^+) \subseteq F_n\tilde{H}_n^+ \subseteq \tilde{H}_n^+. \]

This completes the proof. \(\square\)

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DEPARTMENT OF MATHEMATICS, FACULTY OF HUMANITIES AND SOCIAL SCIENCES, UNIVERSITY, MORIOKA, 020-8550, JAPAN

*E-mail address:* ymiura@iwate-u.ac.jp