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Necessary Conditions for the Gevrey-Well-Posedness of Schrödinger Type Equations

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1 Introduction

We study necessary conditions under which the following Cauchy problem of Schrödinger type,

\[ Lu = \left( i\partial_t + \Delta + \sum_{j=1}^{n} b_j(x)\partial_{x_j} + c(x) \right) u = f(t,x), \quad u(0,x) = \varphi(x), \]  

(1.1)
is well-posed in Gevrey spaces \( G^s, \ 1 < s < \infty \). Here \( G^s = \lim_{\rho \to 0} G^s_{\rho} \), and \( G^s_{\rho} \) is the Hilbert space \( G^s_{\rho} = \{ v \in L^2(\mathbb{R}^n) : \| v \|_{s,\rho} = \| \exp(\rho \langle \xi \rangle^{1/s}) \hat{v}(\xi) \|_{L^2} < \infty \} \), where \( \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \) and \( \hat{v} \) is the usual Fourier transform of \( v \).

**Definition 1.1.** We say that the Cauchy problem for the operator \( L \) is **forward** \( G^s \) **well-posed** if for every \( T > 0 \) and every \( \rho_0 > 0 \) there are constants \( C = C(T, \rho_0) \) and \( \rho > 0 \) such that for every \( \varphi \in G^s_{\rho_0}, f \in C([0, T], G^s_{\rho_0}) \) there is a unique solution \( u \in C([0, T], G^s_{\rho}) \) to (1.1) with

\[ \| u(t, \cdot) \|_{s, \rho} \leq C \| \varphi \|_{s, \rho_0} + C \int_0^t \| f(\tau, \cdot) \|_{s, \rho_0} d\tau, \quad 0 \leq t \leq T. \]

If the coefficients \( b_j \) are purely imaginary valued, then **a priori** estimates of a solution \( u \) to (1.1) in the spaces \( L^2, H^\infty, \) and \( G^s_{\rho} \) can be easily derived, and the well-posedness of this Cauchy problem follows by standard arguments. The situation is more delicate when \( \Re b_j \neq 0 \). For example, the Cauchy problem for the operator

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\( L = i \partial_t + \partial_x^2 + \partial_x \) is neither well-posed in \( L^2 \) nor in \( G^{s} \), \( 1 < s < \infty \), as can be shown by an explicit representation of the solution, see also [12]. Generally, well-posedness requires a certain decay of \( \Re b_j(x) \) at infinity.

Therefore, we propose the following condition:

**Condition 1.** There is a constant \( M = M(d_0) \) such that

\[
\sup_{x \in \mathbb{R}^{n}, \omega \in S^{n-1}} \left| \int_{0}^{\sigma} \sum_{j=1}^{n} \Re b_{j}(x + 2\theta\omega)\omega_{j} \, d\theta \right| \leq M(1 + |\sigma|)^{d_0}, \quad \forall \sigma \in \mathbb{R}.
\]

We assume that the coefficients \( b_j \) and \( c \) belong to Gevrey spaces \( G^{s_b} \), \( G^{s} \):

\[
\|\partial_{\alpha}^{\alpha} b_j(\cdot)\|_{L^\infty} \leq C^{1+|\alpha|}\alpha!^{s_b}, \quad \forall \alpha, \tag{1.2}
\]

\[
\|\partial_{\alpha}^{\alpha} c(\cdot)\|_{L^\infty} \leq C^{1+|\alpha|}\alpha!^{s_b}, \quad \forall \alpha.
\]

**Theorem 1.** Let (1.2) be satisfied, and let \( d_0 \) be a number with \( d_0 > 3/(s + 1) \) and \( d_0 > 2/(s + 1 - s_b) \). If Condition 1 is violated, then the Cauchy problem for the operator \( L \) is not \( G^{s} \) well-posed.

Sufficient conditions for the \( G^{s} \) well-posedness of the Cauchy problem for the operator \( L = i \partial_t + \Delta + \sum_{j=1}^{n} b_j(t, x)\partial_{x_{j}} + c(t, x) \) were given in [2], namely \( \Re b_j(t, x) = o((x)^{1/s-1}) \). In case of the model operator \( L = i \partial_t + \Delta + (x)^{d-1} \partial_{x} \) with \( x \in \mathbb{R}^{1} \), and \( 0 < d < 1 \), the Cauchy problem is therefore well-posed if \( d < 1/s \). On the other hand, Theorem 1 implies ill-posedness for \( d > 3/(s + 1) \).

This gap can be closed if we suppose that the coefficients \( b_j \) decay not too rapidly:

**Condition 2.** There are \( x_0 \in \mathbb{R}^{n}, \omega_0 \in S^{n-1} \), and \( \varepsilon_0 > 0, c_0 > 0 \) such that

\[
-\sum_{j=1}^{n} \Re b_{j}(x + \tau\omega')\omega_{j} \geq 2c_0 (\tau)^{d_0 - 1},
\]

for all \( \tau \geq 0, |x - x_0| < \varepsilon_0, \) and all \( \omega, \omega' \in S^{n-1} \) with \( |\omega - \omega_0| < \varepsilon_0, |\omega' - \omega_0| < \varepsilon_0 \).

**Theorem 2.** Suppose (1.2) with \( s_b < s \) and Condition 2. Then \( d_0 \leq 1/s \) is necessary for the \( G^{s} \) well-posedness.

A necessary condition for \( H^\infty \) well-posedness was given in [7]:

\[
\sup_{x \in \mathbb{R}^{n}, \omega \in S^{n-1}} \left| \int_{0}^{\sigma} \sum_{j=1}^{n} \Re b_{j}(x + 2\theta\omega)\omega_{j} \, d\theta \right| \leq M \log(1 + |\sigma|) + N, \quad \forall \sigma \in \mathbb{R}.
\]
This condition is sufficient in the case of one space dimension; and it is sufficient in the cases of two or more space dimensions if one supposes certain relations on derivatives of the coefficients $b_j$, see [8].

The investigation of an operator with variable coefficients in the principal part, 
$L = i\partial_t + \sum_{j,k} a_{jk}(x) \partial_{x_j} \partial_{x_k} + \sum_j b_j(x) \partial_{x_j} + c(x)$, where $a(x, \xi) = \sum_{j,k} a_{jk}(x) \xi_j \xi_k \geq c_0 |\xi|^2$, $c_0 > 0$, requires the introduction of the bicharacteristic strip $(X, P) = (X, P)(t, x, p)$, which is the solution to the Hamilton–Jacobi equations,

$$\partial_t X_j = \partial_{P_j} a(X, P), \quad \partial_t P_j = -\partial_{X_j} a(X, P), \quad (X, P)(0, x, p) = (x, p).$$

Then a necessary condition for the $H^\infty$ well-posedness is

$$\sup_{\|\sigma\|} \left| \int_0^\sigma \sum_{j=1}^n \Re b_j(X(t, x, \omega)) P_j(\theta, x, \omega) d\theta \right| \leq M \log(1 + |\sigma|) + N, \quad \forall \sigma \in \mathbb{R},$$

under some additional condition. For details, see [6].

Sufficient and necessary conditions for $H^s$ well-posedness were discussed in [3], [4] and [13]. These conditions are similar to the conditions for $H^\infty$ well-posedness if a loss of regularity is allowed, otherwise similar to the conditions of $L^2$ well-posedness.

In [9] and [11], the following necessary condition for $L^2$ well-posedness was shown:

$$\sup_{x \in \mathbb{R}^n, \omega \in S^{n-1}} \left| \int_0^\sigma \sum_{j=1}^n \Re b_j(X(t, x, \omega)) P_j(\theta, x, \omega) d\theta \right| \leq M, \quad \forall \sigma \in \mathbb{R}.$$ 

This condition is also sufficient, see [10].

Schrödinger type equations with a lower order term of order strictly less than 1 were investigated in [1]; and sufficient conditions for $G^s$ well-posedness were proved.

Theorem 1 and Theorem 2 will be discussed simultaneously; and the both cases will be called Case I and Case II, respectively. The following lemma, which gives us an integrated estimate of $\Re b_j$ from below, is quite essential.

**Lemma 1.1.** Assume that $0 < d_0 < 1$ and that Condition 1 is violated. Then, for each $k \in \mathbb{N}$, there are $x_k \in \mathbb{R}^n$, $\sigma_k \in \mathbb{R}_+$, $\omega_k \in S^{n-1}$ with the property that

$$- \int_0^\sigma \sum_{j=1}^n \Re b_j(x_k + 2\theta \omega_k) \omega_k, d\theta = k(1 + \sigma_k)^{d_0},$$

$$- \int_0^\sigma \sum_{j=1}^n \Re b_j(x_k + 2\theta \omega_k) \omega_k, d\theta \geq k d_0 \sigma(1 + \sigma_k)^{d_0 - 1}, \quad 0 \leq \sigma \leq \sigma_k,$$

where $\sigma_k$ tends to infinity for $k \to \infty$. 


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This lemma gives us a sequence \( \{ \sigma_k \}_k \) tending to infinity in Case I. In Case II, we choose this sequence arbitrarily, but still tending to infinity. Now we fix special initial data, \( \varphi_k(x) = \varphi(x - x_k) \) (in Case I), and \( \varphi_k(x) = \varphi(x - x_0) \) (in Case II), where \( \varphi \in G^{s_0}_{\rho_0} \) is given by \( \tilde{\varphi}(\xi) = \langle \xi \rangle^{-(n+1)/2} \exp(-\rho_0 \langle \xi \rangle^{1/s}) \). Assuming that (1.1) is \( G^s \) well-posed, there is a unique solution \( u_k \in C^1([0, T], G^s_\rho) \) of

\[
Lu_k = 0, \quad u_k(0, x) = \varphi_k(x).
\]

Next we define a seminorm \( E_k(t) \) for the function \( u_k(t, \cdot) \).

Let \( h = h(x) \in G^{s_0} \) (with \( s_0 > 1 \) very close to 1) be a function with

\[
h(x) = \begin{cases} 
0 & : |x| \geq 1, \\
1 & : |x| \leq 1/2, \quad 0 \leq h(x) \leq 1.
\end{cases}
\]

We choose the symbols

\[
w_k(t, x, \xi) = h \left( \frac{x - x_k - 2t \sigma_k^{\delta_3} \omega_k}{\sigma_k^{\delta_1}} \right) h \left( \frac{\xi - \sigma_k^{\delta_3} \omega_k}{\sigma_k^{\delta_2}} \right), \quad \text{(Case I)},
\]

\[
w_k(t, x, \xi) = h \left( \frac{x - x_0 - 2t \sigma_0 \omega_0}{\varepsilon \langle 2t \sigma_k \rangle} \right) h \left( \frac{\xi - \sigma_k \omega_0}{\sigma_k^{\delta_2}} \right), \quad \text{(Case II)},
\]

where \( 0 < \varepsilon \ll \varepsilon_0, \delta_1 = 1 - d_0, \) and \( \delta_2, \delta_3 \) are certain positive constants. For multiindices \( \alpha, \beta \in \mathbb{N}^n \), we specify

\[
w_k^{(\alpha \beta)}(t, x, \xi) = \partial^\alpha_y h(y) \partial^\beta_\eta (\eta) \bigg|_{y = \sigma_k^{\delta_1} (x - x_k - 2t \sigma_k^{\delta_3} \omega_k), \eta = \sigma_k^{-\delta_2} (\xi - \sigma_k^{\delta_3} \omega_k)},
\]

\[
w_k^{(\alpha \beta)}(t, x, \xi) = \partial^\alpha_y h(\varepsilon^{-1} y) \partial^\beta_\eta (\eta) \bigg|_{y = \langle 2t \sigma_k \rangle^{-1} (x - x_0 - 2t \sigma_0 \omega_0), \eta = \sigma_k^{-\delta_2} (\xi - \sigma_k \omega_0)},
\]

in Case I, Case II, respectively. These cut-off symbols are supported near the bicharacteristic strip. With some positive constant \( \kappa \), we set \( N \ni N_0 = \lfloor \sigma_k^{\kappa/s_1} \rfloor \), choose \( s_1 > s_0 \), and define the seminorm

\[
E_k(t) = \sum_{|\alpha| \leq N_0, |\beta| \leq N_0 - 2} (\alpha! \beta!)^{-s_1} \left\| W_k^{(\alpha \beta)}(t, x, D_x) u_k(t, x) \right\|_{L^2(\mathbb{R}^2)}.
\]

The ill-posedness of the Cauchy problem can be proved by estimates of \( E_k(t) \) from above and below which contradict for large \( \sigma_k \) if we choose \( \delta_1, \delta_2, \delta_3, \kappa, \varepsilon \) suitably. For reasons of space, we omit the tedious calculations, which can be found in [5], and only sketch the proof.
REFERENCES

It is easy to estimate $E_k$ from above: the symbols $w^{(\alpha\beta)}_k$ belong to the Hörmander class $S^0_{0,0}$, then the Calderon–Vaillancourt theorem and the presumed well-posedness of the Cauchy problem give

$$E_k(t) \leq C \sigma_k^C \|\varphi\|_{s,\rho_0}.$$ 

To get an estimate from below, we write

$$v^{(\alpha\beta)}_k(t, x) = W^{(\alpha\beta)}_k(t, x, D_x)u_k(t, x),$$

$$B(x, D_x) = -\sum_{j=1}^n \Re b_j(x)D_{x_j},$$

and can deduce that

$$\left\|v^{(\alpha\beta)}_k\right\|_{L^2} \left\|\partial_t v^{(\alpha\beta)}_k\right\|_{L^2} = \Re \left(\partial_t v^{(\alpha\beta)}_k, v^{(\alpha\beta)}_k\right)$$

$$= \Re \left(-i[L, W^{(\alpha\beta)}_k]u_k, v^{(\alpha\beta)}_k\right) + \Re \left(i \Delta v^{(\alpha\beta)}_k, v^{(\alpha\beta)}_k\right)$$

$$+ \sum_{j=1}^n \Re \left(ib_j \partial_{x_j} v^{(\alpha\beta)}_k, v^{(\alpha\beta)}_k\right) + \Re \left(ic v^{(\alpha\beta)}_k, v^{(\alpha\beta)}_k\right)$$

$$\geq -\left\|[L, W^{(\alpha\beta)}_k]u_k\right\|_{L^2} \left\|v^{(\alpha\beta)}_k\right\|_{L^2} + \Re \left(B(x, D_x) v^{(\alpha\beta)}_k, v^{(\alpha\beta)}_k\right) - C \left\|v^{(\alpha\beta)}_k\right\|_{L^2}^2.$$ 

Now we need an estimate of $\left\|[L, W^{(\alpha\beta)}_k]u_k\right\|_{L^2}$ from above, and an estimate of $\Re \left(B(x, D_x) v^{(\alpha\beta)}_k, v^{(\alpha\beta)}_k\right)$ from below.

The symbol of $[L, W^{(\alpha\beta)}_k]$ can be written as an asymptotic expansion, up to some remainder, and $\left\|[L, W^{(\alpha\beta)}_k]u_k\right\|_{L^2}$ can be estimated by certain norms $\left\|v^{(\alpha+\gamma,\beta+\delta)}_k\right\|_{L^2}$ plus some remainder which becomes negligible for $\sigma_k \to \infty$.

The term $\Re \left(B(x, D_x) v^{(\alpha\beta)}_k, v^{(\alpha\beta)}_k\right)$ can be estimated using Condition 2 and Garding’s inequality, or Lemma 1.1 and Gronwall’s Lemma.

References


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