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On Kernel Function for Borel Sum of Divergent Solution to a Certain Non-Kowalevski Type Equation

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1 Statement of Results

We study the Borel summability for the formal solution of the following Cauchy problem for a partial differential equation of non-Kowalevski type

$$(1.1)\quad \begin{cases} P(\partial_t,\partial_x)u(t,x) \equiv \left(\partial_t^{p\nu}-\sum_{j=1}^{\nu}a_j\partial_t^{p(\nu-j)}\partial_x^{jq}\right)u(t,x) \\ \qquad = \prod_{j=1}^{\mu}\left(\partial_t^p-\alpha_j\partial_x^q\right)^{\ell_j}u(t,x)=0, \\ \partial_t^k u(0,x)=0\ (k=0,1,\ldots,p\nu-2),\quad \partial_t^{p\nu-1}u(0,x)=\varphi(x), \end{cases}$$

where $t,x\in\mathbb{C}$, $p,q,\nu,\mu,\ell_j\in\mathbb{N}$ $(p<q)$ $(\sum_{j=1}^{\mu}\ell_j=\nu)$, $a_j,\alpha_j(\neq 0)\in\mathbb{C}$ and the Cauchy data $\varphi(x)\in\mathcal{O}$, which denotes the set of holomorphic functions in a neighbourhood of the origin.

The purpose in this paper is to investigate the condition for the k-summability with respect to t-variable of a formal power series solution of the Cauchy problem (1.1) which is divergent in general. The main result of this paper is to give the integral representation of the Borel sum of the formal power series solution under some condition (Theorem 1.2).

The Cauchy problem (1.1) has a unique formal solution $\hat{u}(t,x)$ which is given by

$$(1.2)\quad \hat{u}(t,x)=\sum_{n\geq p\nu-1}u_n(x)\frac{t^n}{n!}=\sum_{n\geq 0}u_{pn+p\nu-1}(x)\frac{t^{pn+p\nu-1}}{(pn+p\nu-1)!}.$$

If we put

$$(1.3)\quad u_{pn+p\nu-1}(x)=A(n)\varphi^{(qn)}(x),\quad n\geq 0,$$

then $A(n)$ satisfy the following difference equation

$$(1.4)\quad A(n+\nu)-\sum_{j=1}^{\nu}a_jA(n+\nu-j)=0,\quad n=-\nu+1,-\nu+2,\ldots,$$

with the Cauchy data

$$(1.5)\quad A(-\nu+1)=A(-\nu+2)=\cdots=A(-1)=0,\quad A(0)=1.$$

Since the fundamental system of solutions of this difference equation are given by $\{n^{k-1}\alpha_j^n ; k = 1, 2, \dots, \ell_j,\ j = 1, 2, \dots, \mu\}$, $A(n)$ are given by

$$A(n) = \sum_{j=1}^{\mu} \alpha_j^n \sum_{k=1}^{\ell_j} c_{jk} n^{k-1}, \tag{1.6}$$

with the coefficients c_{jk} $(j = 1, 2, \dots, \mu,\ k = 1, 2, \dots, \ell_j)$ satisfying the following linear system of equations

$$\mathcal{A}\vec{c} = \vec{e}, \tag{1.7}$$

where $\mathcal{A}$ denotes a $\nu \times \nu$ matrix which is given by

$$\mathcal{A} = \begin{pmatrix} \overbrace{(1 \quad 0 \quad 0 \quad \cdots \quad 0)}^{\ell_1} & \cdots & \overbrace{(1 \quad 0 \quad 0 \quad \cdots \quad 0)}^{\ell_\mu} \\ \alpha_1^{-1}(1,\ -1,\ (-1)^2,\ \cdots,\ (-1)^{\ell_1-1}) & \cdots & \alpha_\mu^{-1}(1,\ -1,\ (-1)^2,\ \cdots,\ (-1)^{\ell_1-1}) \\ \vdots & \vdots & \vdots \\ \alpha_1^{-j}(1,\ (-j),\ (-j)^2,\ \cdots,\ (-j)^{\ell_1-1}) & \cdots & \alpha_\mu^{-j}(1,\ (-j),\ (-j)^2,\ \cdots,\ (-j)^{\ell_\mu-1}) \\ \vdots & \vdots & \vdots \\ \alpha_1^{-\nu+1}(1, (1-\nu), \cdots, (1-\nu)^{\ell_1-1}) & \cdots & \alpha_\mu^{-\nu+1}(1, (1-\nu), \cdots, (1-\nu)^{\ell_1-1}) \end{pmatrix},$$

$\vec{c} = {}^t(c_{11}, c_{12}, \cdots, c_{1\ell_1}, c_{21}, c_{22}, \cdots, c_{2\ell_2}, \cdots, c_{\mu 1}, c_{\mu 2}, \cdots, c_{\mu\ell_\mu})$ and $\vec{e} = {}^t(1, 0, 0, \cdots, 0)$ denote the ν-column vectors, respectively.

We study the Borel summability of the formal solution (1.2) and its Borel sum.

Before stating our results, we shall prepare some definitions and notations.

1. Sector. For $d \in \mathbb{R}$, $\beta > 0$ and $\rho\,(0 < \rho \leq \infty)$, we define a sector $S = S(d, \beta, \rho)$ by

$$S(d, \beta, \rho) := \left\{ t \in \mathbb{C};\ |d - \arg t| < \frac{\beta}{2},\ 0 < |t| < \rho \right\}, \tag{1.8}$$

where d, β and ρ are called the direction, the opening angle and the radius of S, respectively.

2. Gevrey formal power series. We denote by $\mathcal{O}[[t]]$ the ring of formal power series in t-variable with coefficients in $\mathcal{O}$. For $k > 0$, we define that $\hat{f}(t, x) = \sum_{n=0}^{\infty} f_n(x) t^n \in \mathcal{O}[[t]]_{1/k} (\subset \mathcal{O}[[t]])$, which is the ring of formal power series of Gevrey order $1/k$ in t-variable, if there exists a positive constant r such that the coefficients $f_n(x) \in \mathcal{O}(B_r)$, which denotes the set of holomorphic functions on a comman closed disk $B_r := \{x \in \mathbb{C}; |x| \leq r\}$, and there exist some positive constants C and K such that for any n, we have

$$\max_{|x| \leq r} |f_n(x)| \leq C K^n \Gamma\left(1 + \frac{n}{k}\right), \tag{1.9}$$

where Γ denotes the Gamma function.

By using this terminology, we can easily prove that our formal solution $\hat{u}(t,x)$ of the Cauchy problem (1.1) belongs to $\mathcal{O}[[t]]_{(q-p)/p}$, that is, $k = p/(q-p)$.

3. Gevrey asymptotic expansion. Let $k > 0$, $\hat{f}(t,x) = \sum_{n=0}^{\infty} f_n(x)t^n \in \mathcal{O}[[t]]_{1/k}$ and $f(t,x)$ be an analytic function on $S(d,\beta,\rho) \times B_r$. Then we define that

$$f(t,x) \cong_k \hat{f}(t,x) \quad \text{in } S = S(d,\beta,\rho), \tag{1.10}$$

if for any closed subsector S' of S, there exist some positive constants C and K such that for any N, we have

$$\max_{|x|\le r}\left| f(t,x) - \sum_{n=0}^{N-1} f_n(x)t^n \right| \le CK^N|t|^N\Gamma\left(1+\frac{N}{k}\right), \quad t \in S'. \tag{1.11}$$

4. Borel summability. For $k > 0$, $d \in \mathbb{R}$ and $\hat{f}(t,x) \in \mathcal{O}[[t]]_{1/k}$, we define that $\hat{f}(t,x)$ is k-summable or Borel summable in d direction if there exist a sector $S = S(d,\beta,\rho)$ with $\beta > \pi/k$ and an analytic function $f(t,x)$ on $S \times B_r$ such that $f(t,x) \cong_k \hat{f}(t,x)$ in S.

We remark that the function $f(t,x)$ above for a k-summable $\hat{f}(t,x)$ is unique if it exists. Therefore such a function $f(t,x)$ is called the Borel sum of $\hat{f}(t,x)$ in d direction and it is written by $f^d(t,x)$.

5. Let $\{\alpha_j\}_{j=1}^{\mu} \subset \mathbb{C}\setminus\{0\}$. For a direction $d \in \mathbb{R}$ and an opening angle ε, we define the multi-sectors Ω_{α_j} and Ω_x by

$$\Omega_{\alpha_j} = \Omega_{\alpha_j}(p,q;d,\varepsilon) := \bigcup_{m=0}^{q-1} S\left(\frac{pd + \arg\alpha_j + 2\pi m}{q}, \varepsilon, \infty\right), \quad j = 1,2,\dots,\mu,$$

$$\Omega_x = \Omega_x(p,q;d,\varepsilon) := \bigcup_{j=1}^{\mu} \Omega_{\alpha_j}(p,q;d,\varepsilon). \tag{1.12}$$

We remark that if $\arg\alpha_j = \arg\alpha_i$ for any j and i, then we have $\Omega_x = \Omega_{\alpha_j}$.

Now, our first result for the Borel summability is stated as follows.

Theorem 1.1 (Borel summability) *Let $\hat{u}(t,x)$ be the formal solution of the Cauchy problem* (1.1)*, which is given by* (1.2)*. Then the following three propositions are equivalent:*

(i) *$\hat{u}(t,x)$ is $p/(q-p)$-summable in d direction.*

($\tilde{\text{i}}$) *$\hat{u}(t,x)$ is $p/(q-p)$-summable in d' direction with $d' \equiv d \pmod{2\pi/p}$.*

(ii) *The Cauchy data $\varphi(x)$ can be continued analytically in Ω_x and has a growth condition of exponential order at most $q/(q-p)$ there, which means that there exist some positive constants C and δ such that we have*

$$|\varphi(x)| \leq C \exp\left(\delta |x|^{q/(q-p)}\right), \quad x \in \Omega_x. \tag{1.13}$$

This result is a generalization of results in [LMS] and [Miy]. Here in [LMS] they proved that the condition (ii) is necessary and sufficient for the Borel summability in case heat equation and in [Miy] he proved that the condition (ii) is necessary and sufficient in the case $\nu = 1$ for our differential operator $P(\partial_t, \partial_x)$.

Our main purpose is to give an integral representation of the Borel sum $u^d(t,x)$ of the formal solution $\hat{u}(t,x)$ under the condition (ii) of Theorem 1.1 for the Cauchy data $\varphi(x)$. Therefore we suppose the condition (ii) of Theorem 1.1.

Before stating our main results, we need some preparations for the special functions.

1. *The Generalized Hypergeometric Series* (cf. [Luk, p. 41])

For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p) \in \mathbb{C}^p$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_q) \in \mathbb{C}^q$, we define

$${}_pF_q(\boldsymbol{\alpha}; \boldsymbol{\gamma}; z) = {}_pF_q\left(\begin{matrix} \boldsymbol{\alpha} \\ \boldsymbol{\gamma} \end{matrix}; z\right) := \sum_{n=0}^{\infty} \frac{(\boldsymbol{\alpha})_n}{(\boldsymbol{\gamma})_n} \frac{z^n}{n!}, \tag{1.14}$$

where

$$(\boldsymbol{\alpha})_n = \prod_{\ell=1}^{p} (\alpha_\ell)_n, \quad (\boldsymbol{\gamma})_n = \prod_{j=1}^{q} (\gamma_j)_n, \quad (c)_n = \frac{\Gamma(c+n)}{\Gamma(c)} \quad (c \in \mathbb{C}).$$

2. *Meijer G-Function* (cf. [MS, p.2])

For $\boldsymbol{\alpha} \in \mathbb{C}^p$ and $\boldsymbol{\gamma} \in \mathbb{C}^q$ with $\alpha_\ell - \gamma_j \notin \mathbb{N}$ $(\ell = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$, we define

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} \boldsymbol{\alpha} \\ \boldsymbol{\gamma} \end{matrix}\right) = \frac{1}{2\pi i} \int_I \frac{\prod_{j=1}^{m} \Gamma(\gamma_j + \tau) \prod_{\ell=1}^{n} \Gamma(1 - \alpha_\ell - \tau)}{\prod_{j=m+1}^{q} \Gamma(1 - \gamma_j - \tau) \prod_{\ell=n+1}^{p} \Gamma(\alpha_\ell + \tau)} z^{-\tau} d\tau, \tag{1.15}$$

where the path of integration I runs from $\kappa - i\infty$ to $\kappa + i\infty$ for any fixed $\kappa \in \mathbb{R}$ in such a manner that all poles of $\Gamma(\gamma_j + \tau)$, $\{-\gamma_j - k\,;\, k \geq 0, j = 1, 2, \ldots, m\}$, lie to the left of the path and all poles of $\Gamma(1 - \alpha_\ell - \tau)$, $\{1 - \alpha_\ell + k\,;\, k \geq 0, \ell = 1, 2, \ldots, n\}$, lie to the right of the path.

We use the following abbreviations:

$$\boldsymbol{p} = (1, 2, \ldots, p) \in \mathbb{N}^p, \qquad \boldsymbol{q} = (1, 2, \ldots, q) \in \mathbb{N}^q$$

$$\boldsymbol{p} + c = (1 + c, 2 + c, \ldots, p + c) \in \mathbb{C}^p \quad (c \in \mathbb{C}), \qquad \frac{\boldsymbol{p}}{p} = \left(\frac{1}{p}, \frac{2}{p}, \ldots, \frac{p}{p}\right)$$

$$\bar{\boldsymbol{p}} = \boldsymbol{p} - 1 = (0, 1, \ldots, p - 1), \qquad \widehat{\boldsymbol{q}_\ell} = (1, 2, \ldots, \ell - 1, \ell + 1, \ldots, q) \in \mathbb{N}^{q-1}$$

$$\boldsymbol{\Gamma}\left(\frac{\boldsymbol{p}}{p}\right) = \prod_{j=1}^{p} \Gamma\left(\frac{j}{p}\right)$$

In the following, the integration $\int_0^{\infty(\theta)}$ denotes the integration from 0 to ∞ along the half line of the argument θ.

We recall that the formal solution $\hat{u}(t,x)$ is given by

$$
\begin{aligned}
(1.16)\qquad \hat{u}(t,x) &= \sum_{j=1}^{\mu}\sum_{k=1}^{\ell_j} c_{jk} \sum_{n\geq 0} n^{k-1}\varphi^{(qn)}(x)\frac{\alpha_j^n\, t^{pn+p\nu-1}}{(pn+p\nu-1)!} \\
&= t^{p\nu-1}\sum_{j=1}^{\mu}\sum_{k=1}^{\ell_j} c_{jk}p^{1-k}(t\partial_t)^{k-1}\sum_{n\geq 0}\varphi^{(qn)}(x)\frac{\alpha_j^n\, t^{pn}}{(pn+p\nu-1)!}.
\end{aligned}
$$

Now, our main result for the Borel sum is stated as follows.

Theorem 1.2 (Borel sum) *The Borel sum $u^d(t,x)$ is obtained by an analytic continuation in t-plane of the following function by rotating the argument of the integral path, which can be permitted by the assumption* (ii)

$$
(1.17)\quad u^d(t,x) = \left(\frac{t}{p}\right)^{p\nu-1}\sum_{j=1}^{\mu}\sum_{k=1}^{\ell_j} c_{jk}p^{1-k}(t\partial_t)^{k-1}\int_0^{\infty((pd+\arg\alpha_j)/q)} \Phi(x,\zeta)\mathcal{K}_{\alpha_j}(t,\zeta)d\zeta,
$$

where $(t,x)\in S(d,\alpha,\rho)\times B_r$ with $\alpha<\pi(q-p)/p$ and some sufficiently small $r>0$,

$$
(1.18)\qquad \Phi(x,\zeta) = \sum_{m=0}^{q-1}\varphi(x+\zeta\omega_q^m), \quad \omega_q = e^{2\pi i/q},
$$

and the functions $\mathcal{K}_{\alpha_j}(t,\zeta)$ $(j=1,2,\dots,\mu)$ are given by

$$
(1.19)\qquad \mathcal{K}_{\alpha_j}(t,\zeta) = \frac{C_{p,q}}{\zeta}\, G_{p,q}^{q,0}\left(Z_{\alpha_j}\,\middle|\, \begin{matrix}\nu+\bar{\boldsymbol{p}}/p\\ \boldsymbol{q}/q\end{matrix}\right),
$$

with

$$
(1.20)\qquad Z_{\alpha_j}\left(=\frac{p^p}{q^q}\frac{1}{\alpha_j}\frac{\zeta^q}{t^p}\right)\in S(0,(q-p)\pi,\infty), \quad C_{p,q} = \frac{\Gamma(\boldsymbol{p}/p)}{\Gamma(\boldsymbol{q}/q)}.
$$

In the paper [Ich], we gave the explicit formula for the Borel sum, in case of $\nu=1$ for our differential operator $P(\partial_t,\partial_x)$, in a different form by using ${}_pF_{q-1}$ but is the same one. This theorem is a generalization of the previous result.

The G-function in the expression (1.19) has the following integral representation with an integral path $I=\{\tau\in\mathbb{C}\,|\,\mathrm{Re}\,\tau=\kappa>-1/q\}$

$$
(1.21)\quad G_{p,q}^{q,0}\left(Z_{\alpha_j}\,\middle|\, \begin{matrix}\nu+\bar{\boldsymbol{p}}/p\\ \boldsymbol{q}/q\end{matrix}\right) = \frac{1}{2\pi i}\int_I \frac{\Gamma(\boldsymbol{q}/q+\tau)}{\Gamma(\nu+\bar{\boldsymbol{p}}/p+\tau)}Z_{\alpha_j}^{-\tau}d\tau, \quad Z_{\alpha_j} = \frac{p^p}{q^q}\frac{1}{\alpha_j}\frac{\zeta^q}{t^p}.
$$

By calculating the residues of the left side of the path I, we obtain the following propo-

Proposition 1.3 (A relationship between G-function and ${}_pF_{q-1}$)

$$G^{q,0}_{p,q}\left(Z_{\alpha_j}\,\middle|\,\begin{matrix}\nu+\bar{\boldsymbol{p}}/p\\ \boldsymbol{q}/q\end{matrix}\right)=F_{\alpha_j}(t,\zeta)-P_{\alpha_j}(t,\zeta), \tag{1.22}$$

where

$$F_{\alpha_j}(t,\zeta)=\sum_{\ell=1}^{q-1}C_{p,q,\nu}(\ell)Z_{\alpha_j}^{\ell/q}\,{}_pF_{q-1}\left(\begin{matrix}1+\ell/q-\nu-\bar{\boldsymbol{p}}/p\\ 1+\ell/q-\widehat{\boldsymbol{q}_\ell}/q\end{matrix};(-1)^{p-q}Z_{\alpha_j}\right), \tag{1.23}$$

$$P_{\alpha_j}(t,\zeta)=C_{p,q,\nu}\sum_{m=1}^{\nu-1}\frac{(1-\nu-\bar{\boldsymbol{p}}/p)_m}{(1-\widehat{\boldsymbol{q}_q}/q)_m\Gamma(m)}\left((-1)^{p-q}Z_{\alpha_j}\right)^m,\quad (Polynomial), \tag{1.24}$$

$$C_{p,q,\nu}(\ell)=\frac{\boldsymbol{\Gamma}(\widehat{\boldsymbol{q}_\ell}/q-\ell/q)}{\boldsymbol{\Gamma}(\nu+\bar{\boldsymbol{p}}/p-\ell/q)},\quad C_{p,q,\nu}=\frac{\boldsymbol{\Gamma}(\widehat{\boldsymbol{q}_q}/q)}{\boldsymbol{\Gamma}(\nu+\bar{\boldsymbol{p}}/p)}. \tag{1.25}$$

In the case where the arguments of α_j are all the same, we can prove that the polynomial parts P_{α_j} will disappear from the function $\mathcal{K}_{\alpha_j}$ in Theorem 1.2.

Corollary 1.4 *If* $\arg\alpha_j=\theta$ *for all* j, *then the Borel sum* $u^d(t,x)$ *is given by*

$$u^d(t,x)=\left(\frac{t}{p}\right)^{p\nu-1}\sum_{j=1}^{\mu}\sum_{k=1}^{\ell_j}c_{jk}p^{1-k}(t\partial_t)^{k-1}\int_0^{\infty((pd+\theta)/q)}\Phi(x,\zeta)\times\frac{C_{p,q}}{\zeta}F_{\alpha_j}(t,\zeta)d\zeta. \tag{1.26}$$

Especially, if $\mu=1$, *that is,* $\alpha_j=\alpha$ *for all* j, *then the Borel sum* $u^d(t,x)$ *is given by*

$$u^d(t,x)=\left(\frac{t}{p}\right)^{p\nu-1}\int_0^{\infty((pd+\arg\alpha)/q)}\Phi(x,\zeta)K_\alpha(t,\zeta)d\zeta, \tag{1.27}$$

where $\Phi(x,\zeta)$ *is given by* (1.18) *and the function* $K_\alpha(t,\zeta)$ *is given by*

$$\begin{aligned}K_\alpha(t,\zeta)&=\frac{D_{p,q,\nu}}{\zeta}G^{q-1,0}_{p-1,q-1}\left(Z_\alpha\,\middle|\,\begin{matrix}\nu+\widehat{\boldsymbol{p}_p}/p\\ \widehat{\boldsymbol{q}_q}/q\end{matrix}\right),\quad Z_\alpha=\frac{p^p}{q^q}\frac{1}{\alpha}\frac{\zeta^q}{t^p}.\\ &=\frac{D_{p,q,\nu}}{\zeta}\sum_{\ell=1}^{q-1}D_{p,q,\nu}(\ell)Z_\alpha^{\ell/q}\,{}_{p-1}F_{q-2}\left(\begin{matrix}1+\ell/q-\nu-\widehat{\boldsymbol{p}_p}/p\\ 1+\ell/q-(\widehat{\boldsymbol{q}_q})_\ell/q\end{matrix};(-1)^{p-q}Z_\alpha\right),\end{aligned} \tag{1.28}$$

with

$$D_{p,q,\nu}=\frac{\boldsymbol{\Gamma}(\boldsymbol{p}/p)}{\Gamma(\nu)\boldsymbol{\Gamma}(\boldsymbol{q}/q)},\quad D_{p,q,\nu}(\ell)=\frac{\boldsymbol{\Gamma}((\widehat{\boldsymbol{q}_q})_\ell/q-\ell/q)}{\boldsymbol{\Gamma}(\nu+\widehat{\boldsymbol{p}_p}/p-\ell/q)}. \tag{1.29}$$

If $p=1$ *or* $q=2$, *then* $\widehat{\boldsymbol{p}_p}$ *or* $(\widehat{\boldsymbol{q}_q})_\ell$ *are empty.*

In order to illustrate the difference between our results, we shall give the following example.

Example. Let us consider the following Cauchy problem

$$(1.30) \qquad \begin{cases} \prod_{j=1}^{2}(\partial_t - \alpha_j\partial_x^2)u(t,x) = 0, \\ u(0,x) = 0, \quad \partial_t u(0,x) = \varphi(x), \end{cases}$$

where $\alpha_j \in \mathbb{C}\backslash\{0\}$ and assume that $\varphi(x)$ satisfies the conditions for the Borel summability.

When $\arg\alpha_1 \neq \arg\alpha_2$, the Borel sum $u^d(t,x)$ is given by

$$(1.31) \qquad u^d(t,x) = \sum_{j=1}^{2} c_j \int_0^{\infty((d+\arg\alpha_j)/2)} \Phi(x,\zeta)\left\{\sqrt{\frac{t}{\pi\alpha_j}}\,{}_1F_1\left(\begin{matrix}-1/2\\1/2\end{matrix};-\frac{\zeta^2}{4\alpha_j t}\right) - \frac{\zeta}{2\alpha_j}\right\}d\zeta$$

where $c_1 = \alpha_1/(\alpha_1-\alpha_2),\ c_2 = \alpha_2/(\alpha_2-\alpha_1)$.

When $\arg\alpha_1 = \arg\alpha_2 = \theta \ (\alpha_1 \neq \alpha_2)$, we have

$$(1.32) \qquad u^d(t,x) = \int_0^{\infty((d+\theta)/2)} \Phi(x,\zeta)\left\{\sum_{j=1}^{2} c_j\sqrt{\frac{t}{\pi\alpha_j}}\,{}_1F_1\left(\begin{matrix}-1/2\\1/2\end{matrix};-\frac{\zeta^2}{4\alpha_j t}\right)\right\}d\zeta.$$

This expression is obtained from $c_1/\alpha_1 + c_2/\alpha_2 = 0$ in the equality (1.31).

When $\alpha_1 = \alpha_2 = \alpha$, we have

$$(1.33) \qquad u^d(t,x) = \int_0^{\infty((d+\arg\alpha)/2)} \Phi(x,\zeta)\sqrt{\frac{t}{4\alpha\pi}}\exp\left(-\frac{\zeta^2}{4\alpha t}\right)d\zeta.$$

This expression is also derived from (1.32) by taking $\alpha_2 \to \alpha_1 (= \alpha)$.

2 Proof of Theorem 1.2

In order to prove Theorem 1.2, we use the following important lemma for the Borel summability (cf. [Bal], [LMS], [Miy]).

Lemma 2.1 *Let $k > 0$, $d \in \mathbb{R}$ and $\hat{u}(t,x) \in \mathcal{O}[[t]]_{1/k}$. Then the following three statements are equivalent:*

(i) *$\hat{u}(t,x)$ is k-summable in d direction.*

(ii) *Let $v_1(s,x)$ be the formal k-Borel transform of $\hat{u}(t,x)$*

$$(2.1) \qquad v_1(s,x) = (\hat{\mathcal{B}}_k\hat{u})(s,x) := \sum_{n=0}^{\infty} u_n(x)\frac{s^n}{\Gamma(1+n/k)},$$

which is holomorphic in a neighbourhood of the origin. Then $v_1(s,x)$ can be continued analytically in a sector $S(d,\varepsilon,\infty)$ in s-plane for some positive constant ε and satisfies a growth condition of exponential order at most k there, that is, there exsit some positive constants C and γ such that

$$\max_{|x|\le r}|v_1(s,x)| \le C\exp\{\gamma|s|^k\}, \quad s\in S(d,\varepsilon,\infty). \tag{2.2}$$

(iii) *Let $j\ge 2$ and $k_1>0,\dots,k_j>0$ satisfy $1/k = 1/k_1+\cdots+1/k_j$. Let $v_2(s,x)$ be the following iterated formal Borel transforms of $\hat{u}(t,x)$*

$$v_2(s,x) = (\hat{\mathcal{B}}_{k_j}\circ\cdots\circ\hat{\mathcal{B}}_{k_1}\hat{u})(s,x). \tag{2.3}$$

Then $v_2(s,x)$ holds the same properties as $v_1(s,x)$ above.

In case (ii), the Borel sum $u^d(t,x)$ is obtained after an analytic continuation from the following k-Laplace integral by shifting the argument in the path of integration of the half line of argument d.

$$u^d(t,x) = (\mathcal{L}_k v_1)(t,x) := \frac{1}{t^k}\int_0^{\infty(d)} \exp\left[-\left(\frac{s}{t}\right)^k\right] v_1(s,x)d(s^k), \tag{2.4}$$

where $(t,x)\in S(d,\beta,\rho)\times B_r$ with $\beta<\pi/k$ and $\rho>0$.

In case (iii), the Borel sum $u^d(t,x)$ is obtained after an analytic continuation from the following iterated Laplace integrals

$$u^d(t,x) = (\mathcal{L}_{k_1}\circ\cdots\circ\mathcal{L}_{k_j}v_2)(t,x). \tag{2.5}$$

Proof of Theorem 1.2. Let $v(s,x)$ be the $(q-p)$ times iterated formal p-Borel transform of $\hat{u}(t,x)$ which is given by (1.2)-(1.3) with (1.6)

$$\begin{aligned}(2.6)\quad v(s,x) &= (\hat{\mathcal{B}}_p^{q-p}\hat{u})(s,x)\\ &= \sum_{n\ge 0}\frac{A(n)\varphi^{(qn)}(x)}{(pn+p\nu-1)!}\frac{s^{pn+p\nu-1}}{\Gamma(1+(pn+p\nu-1)/p)^{q-p}}\\ &= s^{p\nu-1}\sum_{j=1}^{\mu}\sum_{k=1}^{\ell_j}c_{jk}\sum_{n\ge 0}n^{k-1}\frac{\varphi^{(qn)}(x)\,\alpha_j^n\,s^{pn}}{(pn+p\nu-1)!\Gamma(1+(pn+p\nu-1)/p)^{q-p}}.\end{aligned}$$

By the Cauchy integral formula for the sufficiently small $|s|$ and $|x|$, we have

$$\begin{aligned}(2.7)\qquad v(s,x) &= \frac{s^{p\nu-1}}{2\pi i}\sum_{j=1}^{\mu}\sum_{k=1}^{\ell_j}c_{jk}p^{1-k}(s\partial_s)^{k-1}\oint_{|\zeta|=r}\frac{\varphi(x+\zeta)}{\zeta}h_{\alpha_j}(s,\zeta)d\zeta\\ &\underset{\text{put}}{\equiv} \frac{s^{p\nu-1}}{2\pi i}\sum_{j=1}^{\mu}\sum_{k=1}^{\ell_j}c_{jk}p^{1-k}(s\partial_s)^{k-1}I_{\alpha_j}(s,x)d\zeta,\end{aligned}$$

where for $j = 1, 2, \ldots, \mu$

$$(2.8)\qquad h_{\alpha_j}(s,\zeta) = \sum_{n\ge 0} \frac{(qn)!}{(pn+p\nu-1)!\Gamma(1+(pn+p\nu-1)/p)^{q-p}} \left(\frac{\alpha_j s^p}{\zeta^q}\right)^n = \frac{1}{C_0} {}_{q+1}F_q\left(\begin{matrix} \boldsymbol{q}/q, 1 \\ \nu+\bar{\boldsymbol{p}}/p, \nu+\frac{p-1}{p}, \ldots, \nu+\frac{p-1}{p} \end{matrix} ; \frac{q^q}{p^p}\cdot\frac{\alpha_j s^p}{\zeta^q}\right),$$

with $C_0 = \Gamma(p\nu)\Gamma(\nu+(p-1)/p)^{q-p}$, and $h_{\alpha_j}(s,x)$ has the following Barnes type integral representation (cf. [IKSY], [Luk])

$$(2.9)\qquad h_{\alpha_j}(s,\zeta) = \frac{C_1}{2\pi i}\int_I \frac{\Gamma(\boldsymbol{q}/q+\tau)\Gamma(1+\tau)\Gamma(-\tau)}{\Gamma(\nu+\bar{\boldsymbol{p}}/p+\tau)\Gamma(\nu+(p-1)/p+\tau)^{q-p}}\left(-\alpha_j\frac{q^q s^p}{p^p\zeta^q}\right)^\tau d\tau,$$

where the path of integration I runs from $\kappa - i\infty$ to $\kappa + i\infty$ with $-1/q < \kappa < 0$ and

$$(2.10)\qquad C_1 = \frac{\Gamma(\boldsymbol{p}/p)}{\Gamma(\boldsymbol{q}/q)p^{p\nu-1}} = \frac{C_{p,q}}{p^{p\nu-1}}.$$

In the expression (2.7), by the assumption that $\varphi(x)$ is analytic in Ω_x, we can deform the each path of integration of $I_{\alpha_j}(s,x)$ as follows (cf. [Ich]).

$$(2.11)\qquad I_{\alpha_j}(s,x) = \int_0^{\infty((pd+\arg\alpha_j)/q)} \frac{\Phi(x,\zeta)}{\zeta} H_{\alpha_j}(s,\zeta)d\zeta,$$

where $\Phi(x,\zeta)$ is given by (1.18) and

$$(2.12)\qquad H_{\alpha_j}(s,\zeta) = h_{\alpha_j}(s,\zeta) - h_{\alpha_j}(s,\omega_q^{-1}\zeta) = C_1\int_I \frac{\Gamma(\boldsymbol{q}/q+\tau)}{\Gamma(\nu+\bar{\boldsymbol{p}}/p+\tau)\Gamma(\nu+(p-1)/p+\tau)^{q-p}}\left(\alpha_j\frac{q^q s^p}{p^p\zeta^q}\right)^\tau d\tau.$$

Since the Borel sum $u^d(t,x)$ is given by the analytic continuation of the following $(q-p)$ times iterated p-Laplace integral

$$(2.13)\qquad u^d(t,x) = (\mathcal{L}_p^{q-p}v)(t,x),$$

we have

$$(2.14)\qquad u^d(t,x) = \left[\mathcal{L}_p^{q-p}\left(\frac{s^{p\nu-1}}{2\pi i}\sum_{j=1}^{\mu}\sum_{k=1}^{\ell_j} c_{jk}p^{1-k}(s\partial_s)^{k-1}\int_0^{\infty((pd+\arg\alpha_j)/q)} \frac{\Phi(x,\zeta)}{\zeta}H_{\alpha_j}(s,\zeta)d\zeta\right)\right](t,x).$$

By exchanging the order of integrations, we have

$$(2.15)\qquad u^d(t,x) = \sum_{j=1}^{\mu}\sum_{k=1}^{\ell_j} c_{jk}\int_0^{\infty((pd+\arg\alpha_j)/q)} \Phi(x,\zeta) \times\frac{1}{2\pi i\zeta}\left(\mathcal{L}_p^{q-p}(s^{p\nu-1}p^{1-k}(s\partial_s)^{k-1}H_{\alpha_j}(s,\zeta))\right)(t,\zeta)d\zeta.$$

By substituting the integral representation (2.12) of H_{α_j}, we calculate the iterated Laplace integral carefully as follows.

$$\begin{aligned}
(2.16)\qquad & \frac{1}{2\pi i\zeta}\left(\mathcal{L}_p^{q-p}(s^{p\nu-1}p^{1-k}(s\partial_s)^{k-1}H_{\alpha_j}(s,\zeta))\right)(t,\zeta)\\
&= C_1\frac{t^{p\nu-1}}{2\pi i\zeta}\int_I\frac{\boldsymbol{\Gamma}(\boldsymbol{q}/q+\tau)\tau^{k-1}}{\boldsymbol{\Gamma}(\nu+\bar{\boldsymbol{p}}/p+\tau)}\left(\alpha_j\frac{q^qt^p}{p^p\zeta^q}\right)^{\tau}d\tau\\
&= \frac{t^{p\nu-1}}{p^{p\nu-1}}\frac{C_{p,q}}{\zeta}p^{1-k}(t\partial_t)^{k-1}G_{p,q}^{q,0}\left(Z_{\alpha_j}\,\middle|\,\begin{matrix}\nu+\bar{\boldsymbol{p}}/p\\ \boldsymbol{q}/q\end{matrix}\right),\quad Z_{\alpha_j}=\frac{p^p}{q^q}\frac{1}{\alpha_j}\frac{\zeta^q}{t^p}.
\end{aligned}$$

In the above equality are obtained by the following reasons. The first equality is obtained by exchanging the differentiation and integration, and calculating $(q-p)$ Gamma integrals after exchanging the order of Laplace integrals and Barnes type integral. The second equality is obtained by exchanging the integration and differentiation, and employing the G-function representation.

Thus we get the desired formula (1.17). □

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