<table>
<thead>
<tr>
<th>Title</th>
<th>On the Wellposedness of the Cauchy Problem for Weakly Hyperbolic Equations of Higher Order (Microlocal Analysis and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>D'Ancona, Piero; Kinoshita, Tamotu</td>
</tr>
<tr>
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<td></td>
</tr>
</tbody>
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On the Wellposedness of the Cauchy Problem for Weakly Hyperbolic Equations of Higher Order

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§1. Introduction

We consider here the Cauchy problem on \([0, T] \times \mathbb{R}^n_x\)

\[
\begin{aligned}
D_t^m u &= \sum_{j+|\alpha|=m} c_{j,\alpha}(t) D_t^j D_x^\alpha u + \sum_{j+|\alpha| \leq d} c_{j,\alpha}(t) D_t^j D_x^\alpha u + f(t, x) \\
D_t^j u(0, x) &= u_j(x) \quad (j = 0, \cdots, m-1),
\end{aligned}
\]

where \(D_t = -i \partial_t\) and \(D_x = -(\partial_{x_1}, \cdots, \partial_{x_n})\), and \(0 \leq d \leq m - 1\). We shall write in short

\[
p(t, \tau, \xi) = \tau^m - \sum_{j+|\alpha|=m} c_{j,\alpha}(t) \tau^j \xi^\alpha
\]

for the principal part and

\[
p_d(t, \tau, \xi) = \sum_{j+|\alpha| \leq d} c_{j,\alpha}(t) \tau^j \xi^\alpha
\]

for the lower order terms. We shall assume that the principal part \(p\) is hyperbolic with respect to \(\tau\), that is, for any \(t \in \mathbb{R}_t, \xi \in \mathbb{R}^n_\xi\) the roots in \(\tau\) of the algebraic equation \(p(t, \tau, \xi) = 0\) are all real. We name them \(\lambda_j(t, \xi)\), according to the rule

\[
\lambda_1(t, \xi) \geq \lambda_2(t, \xi) \geq \cdots \geq \lambda_m(t, \xi),
\]

thus \(p(t, \tau, \xi)\) can be written

\[
p(t, \tau, \xi) = \prod_{k=1}^m (\tau - \lambda_k(t, \xi)).
\]

We recall that the functions \(\lambda_j(t, \xi)\) are homogeneous of degree 1 in \(\xi\).
There are many results on this problem. As to the $C^\infty$-wellposedness, we mention that T. Nishitani [N1] considered the case when the multiplicity of the characteristic roots is at most double. F. Colombini and N. Orrù [CO] assumed that the characteristic roots vanish of finite order at $t = 0$ and satisfy

$$t^2 \sum_{k,j=1,k\neq j}^m \frac{|\lambda'_k(t,\xi)|^2 + |\lambda'_j(t,\xi)|^2}{|\lambda_k(t,\xi) - \lambda_j(t,\xi)|^2} < \infty \text{ near } t = 0.$$  

Moreover, K. Kajitani, S. Wakabayashi and K. Yagdjian [KWY] dealt with the case of characteristic roots vanishing of infinite order. Concerning the Gevrey-wellposedness, F. Colombini and T. Kinoshita [CK] considered the Cauchy problem in the case when the characteristic roots are Hölder continuous in $t$. F. Colombini, H. Ishida [CI] and H. Ishida, K. Yagdjian [IY] assumed that the characteristic roots vanish of infinite order at $t = 0$ and satisfy for some $\bar{s} > 1$

$$\frac{\Phi_1(t)^{2\bar{s}/(\bar{s}-1)}}{\phi_1(t)^2} \sum_{k,j=1,k\neq j}^m \frac{|\lambda'_k(t,\xi)|^2 + |\lambda'_j(t,\xi)|^2}{|\lambda_k(t,\xi) - \lambda_j(t,\xi)|^2} < \infty \text{ near } t = 0,$$

where $\Phi_1(t) = \int_0^t \phi_1 \, dt$ and $\phi_1(t), \ldots, \phi_m(t)$ are real-valued functions such that

(i) $\phi_k(0) = \phi'_k(0) = 0$, $\phi'_k(t) > 0$ if $t \in (0, T]$ for any $k = 1, \ldots, m$.

(ii) $\phi_1(t) \geq \phi_2(t) \geq \cdots \geq \phi_m(t)$ for $t \in [0, T]$.

(iii) $|\lambda_k(t,\xi)| \leq C_k \phi_k(t)|\xi|$ ($^3C_k > 0$) for $k = 1, \ldots, m$ and $(t, \xi) \in [0, T] \times \mathbb{R}_{\xi}^n \setminus 0$.

(iv) $|\lambda_k(t,\xi) - \lambda_j(t,\xi)| \geq c \phi_k(t)|\xi|$ ($^3c > 0$) for $k < j$ and $(t, \xi) \in [0, T] \times \mathbb{R}_{\xi}^n \setminus 0$.

Then they showed the wellposedness in the Gevrey classes of order $1 \leq s < \bar{s}$.

We see that in most results concerning the higher order case $m > 2$ the roots are assumed to coincide only at isolated points, and then a precise behaviour is assumed at those points. In this paper we try to give a global assumption valid in more general cases, even when this happens at an arbitrary set of points (also infinite or dense). To this end we introduce the sets $\Omega^k_\sigma$, $\Omega_\sigma$ defined as follows: for any $0 < \sigma < 1$, $k = 1, \ldots, m - 1$,

$$\Omega^k_\sigma(\xi) = \{t \in [0, T] : |\lambda_k(t,\xi) - \lambda_{k+1}(t,\xi)| \leq \sigma\}$$

and

$$\Omega_\sigma(\xi) = \bigcup_{k=1}^{m-1} \Omega^k_\sigma(\xi).$$
These sets enclose, for each \( \xi \), the points \( t \) where the roots coincide; thus we can regard the measure \( \mu(\Omega_\sigma) \), which is a function of \( \sigma, \xi \), as a measure of the defect of strict hyperbolicity of \( p \). Here \( \mu(A) \) is the Lebesgue measure in \( \mathbb{R}_t \) of the set \( A \subseteq [0, T] \). We denote by \( AC([0, T]) \) the space of absolutely continuous functions on \([0, T]\) and by \( G^s(\mathbb{R}^n) \) the space of Gevrey functions \( g(x) \) satisfying 
\[
\sup_{x \in K} |D^\alpha_x g(x)| \leq C_K |\alpha|!^s \quad \text{for any compact set } K \subset \mathbb{R}^n, \quad \alpha \in \mathbb{N}^n.
\]

Our first result is the following:

**Theorem 1.** (Gevrey-wellposedness). Assume that the coefficients \( c_{j,\alpha}(t) \) of \( p \), \( p_d \) belong to \( C^0([0,T]) \) and the characteristic roots of the principal part \( \lambda_1, \ldots, \lambda_m \) belong to \( AC([0,T]) \) and that there exist constants \( C > 0, a \geq 0 \) and \( b > 0 \) such that for any \( 0 < \sigma < 1, |\xi| = 1, k = 1, \ldots, m - 1 \)

\[
(2) \quad \mu(\Omega_\sigma(\xi)) \leq C\sigma^a,
\]

\[
(3) \quad \int_{[0,T] \setminus \Omega_{\sigma}^k(\xi)} \frac{|\lambda'_k(t,\xi)| + |\lambda'_{k+1}(t,\xi)|}{|\lambda_k(t,\xi) - \lambda_{k+1}(t,\xi)|} \, dt \leq C\sigma^{-b}.
\]

Then, when the degree \( d \) of the lower order terms satisfies

\[
0 \leq d \leq \frac{m(a + b)}{a + b + 1},
\]

the Cauchy problem (1) is wellposed in the Gevrey classes of order

\[
(4) \quad 1 \leq s < 1 + \frac{a + 1}{b},
\]

i.e., for any data \( u_j \in G^s(\mathbb{R}^n) \) and \( f \in C^0([0, T]; G^s(\mathbb{R}^n)) \) the Cauchy problem (1) has a unique solution \( u \in C^m([0, T]; G^s(\mathbb{R}^n)) \). Moreover, when the degree \( d \) of the lower order terms satisfies

\[
d > \frac{m(a + b)}{a + b + 1},
\]

then the problem is wellposed for

\[
(5) \quad 1 \leq s < \frac{m}{d + a(d - m)}.
\]

**Remark 1.** In the cases mentioned above, when \( \lambda_1(t, \xi), \ldots, \lambda_m(t, \xi) \) vanish of infinite order, assumption (2) can be dropped (one is forced to choose
\(a = 0\). Thus by Theorem 1 we see that the Cauchy Problem (1) is wellposed in the Gevrey classes of order

\[
1 \leq s < \min \left\{ 1 + \frac{1}{b}, \frac{m}{d} \right\}.
\]

**Remark 2.** M. D. Bronshtein [B], S. Wakabayashi [W] proved the Lipschitz (or Hölder) continuity in \(t\) of the characteristic roots of hyperbolic polynomials with smooth coefficients (see also [M]). Thus if we assume that \(c_{j,\alpha}\) are smooth for \(j + |\alpha| = m\), we can drop the assumption that \(\lambda_{j}\) belong to \(AC([0, T])\).

**Remark 3.** It is well-known that the lower order terms do not influence the \(C^{\infty}\)-wellposedness for strictly hyperbolic equations (the multiplicity of the characteristic roots is equal to 1) and the lower order terms of order \(d = m - 1\) give the Gevrey-wellposedness of order \(1 \leq s < m/(m - 1)\) for weakly hyperbolic equations (the multiplicity of the characteristic roots is equal to \(m\)) (see [B], [C], [CDS], [CJS], [OT], etc.). As the parameter \(a\) in (2) becomes greater, the type of \(p\) approaches to strictly hyperbolic type. Especially, when \(d = m - 1\), the second exponent in (4) is equal to \(m/(m - 1 - a)\). Taking \(0 \leq a < m - 1\), we can obtain an interpolation between \(C^{\infty}\) and the Gevrey classes of order \(m/(m - 1)\).

**Example A.** When the characteristic roots are

\[
\lambda_{k}(t, \xi) = k t^{h} \left\{ 1 + \sin^{2} \left( \frac{1}{t^{h/\alpha - 1}} \right) \right\} \cdot \xi
\]

for some \(0 < \alpha \leq 1, \alpha < h < \alpha/(1 - \alpha)\) and \(k = 1, \ldots, m\), we find that \(\lambda_{1}, \ldots, \lambda_{m}\) belong to \(AC([0, T])\) and also \(C^{\alpha}([0, T])\) and vanish of finite order at \(t = 0\) and satisfy (2) with \(a = 1/h\) and (3) with \(b = 1/\alpha - 1/h\), since

\[
\mu(\Omega_{\sigma}(\xi)) \leq C \int_{0}^{C \sigma^{1/h}} dt \leq C \sigma^{1/h},
\]

\[
\int_{[0, T] \setminus \Omega_{\sigma}^{\delta}(\xi)} \frac{|\lambda'_{k}(t, \xi)| + |\lambda'_{k+1}(t, \xi)|}{|\lambda_{k}(t, \xi) - \lambda_{k+1}(t, \xi)|} dt \leq C \int_{C \sigma^{1/h}}^{T} \left( \frac{1}{t^{h/\alpha - 1}} \right)' dt \leq C \sigma^{1/h - 1/\alpha}.
\]

Applying Theorem 1, we get the wellposedness in the Gevrey classes of order

\[
1 \leq s < \frac{h}{h - \alpha} (1 + \alpha).
\]
According to [CK] or [OT], if the characteristic roots belong to $C^\alpha([0, T])$, the Cauchy problem (1) is in the wellposed in the Gevrey classes of order

$$1 \leq s < 1 + \alpha.$$  

For the second order polynomial $P(t, \tau, \xi) \equiv \tau^2 - A(t)\xi^2$ where $A(t) \geq 0$, if $A(t)$ belongs to $C^{2\alpha}([0, T])$, we also know the Gevrey order (7) (see [CJS], [D1] and [N2]). We remark that (6) approaches to (7) as $h$ tends to infinity and $s$ can be taken arbitrarily large as $h$ tends to $\alpha$ (the characteristic roots oscillate more slowly). This example implies that the oscillation and the degeneracy of the characteristic roots influence on the wellposedness independently of their regularity.

**Example B.** [CI] and [IY] gave an example of the following kind:

$$\lambda_k(t, \xi) = \begin{cases} k \exp\left(\frac{1}{t^h}\right)\left(1 + \sin^2\left(\exp\frac{\gamma}{t^h}\right)\right) \cdot \xi \\ 0 \end{cases}$$

for some $\gamma > 0$, $h > 0$ and $k = 1, \ldots, m$. They proved the wellposedness in the Gevrey classes of order $1 \leq s < 1 + 1/\gamma$. Notice that $\lambda_1(t, \xi), \ldots, \lambda_m(t, \xi)$ belong to $AC([0, T])$ and vanish of infinite order at $t = 0$ (see Remark 1) and satisfy (3) with $b = \gamma$;

$$\int_{[0,T]\backslash \Omega^k_{\sigma}(\xi)} \frac{|\lambda_k'(t, \xi)| + |\lambda_{k+1}'(t, \xi)|}{|\lambda_k(t, \xi) - \lambda_{k+1}(t, \xi)|} dt \leq C \int_{1/(\log \sigma^{-1} + C)^{1/h}}^{T} \left(\exp\frac{\gamma}{t^h}\right)^' dt \leq C \sigma^{-\gamma}.$$  

Thus we can apply Theorem 1 and we get the same Gevrey order $1 \leq s < 1+1/\gamma$.

Our theorems can be applied also when the vanishing order of characteristic roots is different from the order of contact between the roots. For instance, if the characteristic polynomial is

$$p(t, \tau, \xi) = \tau^2 - 2t^\alpha \tau \xi + (t^{2\alpha} - t^{2\beta})\xi^2$$  

where $0 < \alpha < \beta$,

we easily obtain $\lambda_1(t, \xi) = (t^\alpha + t^\beta)\xi$ and $\lambda_2(t, \xi) = (t^\alpha - t^\beta)\xi$ which implies that $|\lambda_k(t, \xi)| \leq 2t^\alpha|\xi|$ ($k = 1, 2$), $|\lambda_1(t, \xi) - \lambda_2(t, \xi)| \geq 2t^\beta|\xi|$ for $(t, \xi) \in [0, T] \times \mathbb{R}_\xi$. Since $\lambda_1(t, \xi)$ and $\lambda_2(t, \xi)$ satisfy (2) with $a = 1/\beta$ and (3) $b = 1 - \alpha/\beta$, applying Theorem 1 we have wellposedness in the Gevrey classes of order

$$1 \leq s < 1 + \frac{\beta+1}{\beta - \alpha}.$$  

In the favourable case of analytic characteristic roots, more generally from Theorem 1 we also obtain the following results:
**Corollary 2.** (Gevrey-wellposedness). Assume that the coefficients $c_j, \alpha(t)$ of $p, p_d$ belong to $C^0([0,T])$ and the characteristic roots of the principal part $\lambda_1(t, \xi), \cdots, \lambda_m(t, \xi)$ are analytic in $t$ and vanish at $t = 0$ and that there exist constants $C > 0, c > 0$ and $0 < \alpha < \beta$ such that for any $(t, \xi) \in [0, T] \times \mathbb{R}^n$

$$|\lambda_k(t, \xi)| \leq Ct^\alpha|\xi| \quad \text{for} \quad k = 1, \cdots, m,$$

$$|\lambda_{k+1}(t, \xi) - \lambda_k(t, \xi)| \geq ct^{\beta}|\xi| \quad \text{for} \quad k = 1, \cdots, m - 1.$$ 

Then, when the degree $d$ of the lower order terms satisfies

$$0 \leq d \leq \frac{m(\beta - \alpha + 1)}{2\beta - \alpha + 1},$$

the Cauchy problem (1) is wellposed in the Gevrey classes of order

$$1 \leq s < 1 + \frac{\beta + 1}{\beta - \alpha}.$$

Moreover, when the degree $d$ of the lower order terms satisfies

$$d > \frac{m(\beta - \alpha + 1)}{2\beta - \alpha + 1},$$

then the wellposedness holds for

$$1 \leq s < \frac{\beta m}{\beta d + d - m}.$$

In Corollary 2 and Examples A and B, the characteristic roots coincide only at $t = 0$ or at a finite number of points. We give a final example to emphasize that our results allow the characteristic roots to coincide at an infinite number of points.

**Example C** (see also Example A). When the characteristic roots are

$$\lambda_k(t, \xi) = kt^h \sin^h\left(\frac{1}{t^h - 1}\right) \cdot \xi$$

for some even number $h$ and $k = 1, \cdots, m$, we find that $\lambda_1(t, \xi), \cdots, \lambda_m(t, \xi)$ are absolutely continuous in $t$, more precisely Lipschitz continuous in $t$ and vanish at $t = (\pi j)^{1/(1-h)}$ ($j = 1, 2, \cdots$), they satisfy (2) with $a < 1/h$ and (3) with $b > 1 - 1/h$. Applying Theorem 1, we get the wellposedness in the Gevrey classes of order $1 \leq s < 2h/(h - 1)$ (see (7)).
§2. Sketch of the proof

When $s = 1$, the Cauchy problem (1) is wellposed in the class of real analytic functions. Therefore we can suppose that $s > 1$ for the proof. By Fourier transform with respect to $x$, the Cauchy problem (1) turns into

\begin{equation}
\begin{cases}
p(t, D_t, \xi) \hat{u} = \hat{f}(t, \xi) + p_d(t, D_t, \xi) \hat{u} \\
D_t^j \hat{u}(0, \xi) = \hat{u}_j(\xi) \quad (j = 0, \ldots, m - 1).
\end{cases}
\end{equation}

Let $0 < \sigma < 1$ and $\varphi(r)$ be a non-negative function such that $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(r) \equiv 0$ for $|r| \geq 2$ and $\varphi(r) \equiv 1$ for $|r| \leq 1$. We define

$$
\omega(t, \xi) = \sigma|\xi| \sum_{l=1}^{m-1} \varphi(\sigma^{-1}\{\lambda_l(t, \frac{\xi}{|\xi|}) - \lambda_{l+1}(t, \frac{\xi}{|\xi|})\}) ,
$$

$$
\mu_k(t, \xi) = \lambda_k(t, \xi) + ik\omega(t, \xi) \quad \text{for} \quad k = 1, \ldots, m.
$$

Moreover we denote by $q(t, \tau, \xi)$ the polynomial of degree $m$ in $\tau$

$$
q(t, \tau, \xi) = \prod_{k=1}^{m} (\tau - \mu_k(t, \xi)).
$$

Now we set the energy density

$$
E(t, \xi) = \frac{1}{2} \sum_{l=1}^{m} |q_l(t, D_t, \xi) \hat{u}|^2 ,
$$

where $q_l(t, \tau, \xi)$ is the polynomial of degree $m - 1$ in $\tau$ defined by

$$
q_l(t, \tau, \xi) = \frac{q(t, \tau, \xi)}{\tau - \mu_l(t, \xi)} (= \prod_{k=1, k \neq l}^{m} (\tau - \mu_k(t, \xi))).
$$

We denote by $'$ the derivative in $t$. Differentiating $E(t, \xi)$ in $t$ and dividing by $2\sqrt{E(t, \xi)}$, by (8) we have

$$
\sqrt{E}' \leq C \left( \max_{1 \leq k \leq m-1} \frac{|\lambda'_k| + |\lambda'_{k+1}| + |\omega'|}{|\lambda_k - \lambda_{k+1}| + \omega} + \frac{|\xi|^d}{\prod_{k=1}^{m-1} |\lambda_k - \lambda_{k+1}| + \omega^{m-1}} \right) \sqrt{E} + |\hat{f}| .
$$

Thus, Gronwall's inequality yields the estimate

$$
\sqrt{E(t, \xi)} \leq \exp \left\{ C \int_0^T \left( \max_{1 \leq k \leq m-1} \frac{|\lambda'_k| + |\lambda'_{k+1}| + |\omega'|}{|\lambda_k - \lambda_{k+1}| + \omega} + \frac{|\xi|^d}{\prod_{k=1}^{m-1} |\lambda_k - \lambda_{k+1}| + \omega^{m-1}} \right) dt \right\} \
\times \left\{ \sqrt{E(0, \xi)} + \int_0^T |\hat{f}(t, \xi)| dt \right\}.
$$
We remark that there exists $C > 0$ such that for any $(t, \xi) \in [0, T] \times \mathbb{R}_\xi^n \setminus 0$

$$C^{-1}(\sigma|\xi|)^{m-1}|\xi|^{-j}|D_t^j \hat{u}| \leq \sqrt{E(t, \xi)} \leq C \sum_{j=0}^{m-1} |\xi|^{m-1-j}|D_t^j \hat{u}|.$$

**Lemma 1.** Let $b \geq 0$. Assume that $\lambda_1(t, \xi), \cdots, \lambda_m(t, \xi)$ belong to $AC([0, T])$ and satisfy (3). Then there exists $C > 0$ such that for any $0 < \sigma < 1, |\xi| = 1$ and $k = 1, \cdots, m$

$$\int_{\Omega_\sigma^k(\xi) \cup \Omega_\sigma^{k-1}(\xi)} |\lambda_k'(t, \xi)| dt \leq \begin{cases} C & \text{if } b \geq 1 \\ C\sigma^{1-b} & \text{if } 0 \leq b < 1 \end{cases} \leq C\sigma^{1-b},$$

where $\Omega_\sigma^0(\xi) = \Omega_\sigma^m(\xi) = \phi$ and $\Omega_\sigma^k(\xi)$ for $k = 1, \cdots, m-1$ are defined in §.1.

**Lemma 2.** Let $0 \leq a < m-1$. Assume that $\lambda_1, \cdots, \lambda_m$ satisfy (2). Then there exists $C > 0$ such that for any $0 < \sigma < 1, |\xi| = 1$

(21) $$\int_{[0, T] \setminus \Omega_\sigma(\xi)} \frac{dt}{\prod_{k=1}^{m-1} |\lambda_k(t, \xi) - \lambda_{k+1}(t, \xi)|} \leq C\sigma^{a+1-m},$$

where $\Omega_\sigma(\xi)$ is defined in §.1.

Consequently, it follows that

$$\sum_{j=0}^{m-1} |\xi|^{-j}|D_t^j \hat{u}(t, \xi)| \leq C\sigma^{1-m} \exp\left\{C(\sigma^{-b} + \sigma^{a+1}|\xi| + \sigma^{a+1-m}|\xi|^{d+1-m})\right\}$$

$$\times \left\{\sum_{j=0}^{m-1} |\xi|^{-j}|\hat{u}_j| + \int_0^T |\xi|^{1-m}|\hat{f}(t, \xi)| dt\right\}.$$

When

$$d \leq \frac{m(a+b)}{a+b+1},$$

the third term is smaller and this choice gives immediately

$$|\xi|^\gamma b + |\xi|^{1-\gamma(a+1)} + |\xi|^{\gamma(m-a-1)+d+1-m} \leq 3|\xi|^{\frac{b}{a+b+1}}.$$

Hence, there exists $\rho > 0$ such that for any $(t, \xi) \in [0, T] \times \mathbb{R}_\xi^n \setminus 0$

$$\sum_{j=0}^{m-1} |\xi|^{-j}|D_t^j \hat{u}(t, \xi)| \leq C \exp\left\{\rho|\xi|^{\frac{b}{a+b+1}}\right\} \left\{\sum_{j=0}^{m-1} |\xi|^{\frac{m-1}{a+b+1}-j}|\hat{u}_j(\xi)| + \int_0^T |\xi|^{\frac{(1-m)(a+1)}{a+b+1}}|\hat{f}(t, \xi)| dt\right\}.$$
In virtue of Paley-Wiener theorem, \( \{D_t^j u(\cdot, t) ; t \in [0, T], j = 0, \ldots, m - 1 \} \) is bounded in the Gevrey classes of order (5). Thus, taking into account that \( u \) is a solution of (1), we find \( u \in C^m([0, T]; G^s(\mathbb{R}^n)) \). This concludes the proof of Theorem 1 in the case when \( d \leq m(a + b)/(a + b + 1) \).

On the other hand, when

\[
d > \frac{m(a + b)}{a + b + 1},
\]

the dominant terms in

\[
|\xi|^{\gamma b} + |\xi|^{1-\gamma(a+1)} + |\xi|^{\gamma(m-a-1)+d+1-m}
\]

are the last two (the first one is smaller). In this case we choose

\[
\gamma = \frac{m-r}{m}
\]

and proceeding as above we conclude the proof of this case and we get (4).

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