On the Wellposedness of the Cauchy Problem for Weakly Hyperbolic Equations of Higher Order (Microlocal Analysis and Related Topics)

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On the Wellposedness of the Cauchy Problem for Weakly Hyperbolic Equations of Higher Order

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§1. Introduction

We consider here the Cauchy problem on \([0, T] \times \mathbb{R}_x^n\)
\[
\begin{cases}
D_t^m u = \sum_{j+|\alpha|=m} c_{j,\alpha}(t) D_t^j D_x^\alpha u + \sum_{j+|\alpha| \leq d} c_{j,\alpha}(t) D_t^j D_x^\alpha u + f(t, x) \\
D_t^j u(0, x) = u_j(x) (j = 0, \ldots, m-1),
\end{cases}
\]
(1)
where \(D_t = -i\partial_t, D_x = -i(\partial_{x_1}, \ldots, \partial_{x_n}),\) and \(0 \leq d \leq m - 1.\) We shall write in short
\[p(t, \tau, \xi) = \tau^m - \sum_{j+|\alpha|=m} c_{j,\alpha}(t)\tau^j\xi^\alpha\]
for the principal part and
\[p_d(t, \tau, \xi) = \sum_{j+|\alpha| \leq d} c_{j,\alpha}(t)\tau^j\xi^\alpha\]
for the lower order terms. We shall assume that the principal part \(p\) is hyperbolic with respect to \(\tau,\) that is, for any \(t \in \mathbb{R}_t, \xi \in \mathbb{R}_\xi^n\) the roots in \(\tau\) of the algebraic equation \(p(t, \tau, \xi) = 0\) are all real. We name them \(\lambda_j(t, \xi),\) according to the rule
\[\lambda_1(t, \xi) \geq \lambda_2(t, \xi) \geq \cdots \geq \lambda_m(t, \xi),\]
thus \(p(t, \tau, \xi)\) can be written
\[p(t, \tau, \xi) = \prod_{k=1}^m (\tau - \lambda_k(t, \xi)).\]

We recall that the functions \(\lambda_j(t, \xi)\) are homogeneous of degree 1 in \(\xi.\)
There are many results on this problem. As to the $C^\infty$-wellposedness, we mention that T. Nishitani [N1] considered the case when the multiplicity of the characteristic roots is at most double. F. Colombini and N. Orrù [CO] assumed that the characteristic roots vanish of finite order at $t = 0$ and satisfy

\[ t^2 \sum_{k,j=1,k\neq j}^{m} \frac{|\lambda'_k(t,\xi)|^2 + |\lambda'_j(t,\xi)|^2}{|\lambda_k(t,\xi) - \lambda_j(t,\xi)|^2} < \infty \quad \text{near } t = 0. \]

Moreover, K. Kajitani, S. Wakabayashi and K. Yagdjian [KWY] dealt with the case of characteristic roots vanishing of infinite order. Concerning the Gevrey-wellposedness, F. Colombini and T. Kinoshita [CK] considered the Cauchy problem in the case when the characteristic roots are Hölder continuous in $t$. F. Colombini, H. Ishida [CI] and H. Ishida, K. Yagdjian [IY] assumed that the characteristic roots vanish of infinite order.

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Then they showed the wellposedness in the Gevrey classes of order $1 \leq s < \overline{s}$. We see that in most results concerning the higher order case $m > 2$ the roots are assumed to coincide only at isolated points, and then a precise behaviour is assumed at those points. In this paper we try to give a global assumption valid in more general cases, even when this happens at an arbitrary set of points (also infinite or dense). To this end we introduce the sets $\Omega^k_\sigma$, $\Omega_\sigma$ defined as follows: for any $0 < \sigma < 1$, $k = 1, \ldots, m - 1$,

\[ \Omega^k_\sigma(\xi) = \{t \in [0,T] : |\lambda_k(t,\xi) - \lambda_{k+1}(t,\xi)| \leq \sigma \} \]

and

\[ \Omega_\sigma(\xi) = \bigcup_{k=1}^{m-1} \Omega^k_\sigma(\xi). \]
These sets enclose, for each $\xi$, the points $t$ where the roots coincide; thus we can regard the measure $\mu(\Omega_{\sigma})$, which is a function of $\sigma$, $\xi$, as a measure of the defect of strict hyperbolicity of $p$. Here $\mu(A)$ is the Lebesgue measure in $\mathbb{R}_t$ of the set $A \subseteq [0, T]$. We denote by $AC([0, T])$ the space of absolutely continuous functions on $[0, T]$ and by $G^s(\mathbb{R}^n)$ the space of Gevrey functions $g(x)$ satisfying 

$$\sup_{x \in K} |D_x^a g(x)| \leq C_K p_K^a |\alpha|^s$$

for any compact set $K \subset \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$.

Our first result is the following:

**Theorem 1. (Gevrey-wellposedness).** Assume that the coefficients $c_{j, \alpha}(t)$ of $p$, $p_d$ belong to $C^0([0,T])$ and the characteristic roots of the principal part $\lambda_1, \ldots, \lambda_m$ belong to $AC([0,T])$ and that there exist constants $C > 0$, $a \geq 0$ and $b > 0$ such that for any $0 < \sigma < 1$, $|\xi| = 1$, $k = 1, \ldots, m - 1$

(2) \[ \mu(\Omega_{\sigma}(\xi)) \leq C\sigma^a, \]

(3) \[ \int_{[0,T] \setminus \Omega_{\sigma}^k(\xi)} \frac{|\lambda_k'(t,\xi)| + |\lambda_{k+1}'(t,\xi)|}{|\lambda_k(t,\xi) - \lambda_{k+1}(t,\xi)|} \, dt \leq C\sigma^{-b}. \]

Then, when the degree $d$ of the lower order terms satisfies

$$0 \leq d \leq \frac{m(a + b)}{a + b + 1},$$

the Cauchy problem (1) is wellposed in the Gevrey classes of order

(4) \[ 1 \leq s < 1 + \frac{a + 1}{b}, \]

i.e., for any data $u_j \in G^s(\mathbb{R}^n)$ and $f \in C^0([0, T]; G^s(\mathbb{R}^n))$ the Cauchy problem (1) has a unique solution $u \in C^m([0, T]; G^s(\mathbb{R}^n))$. Moreover, when the degree $d$ of the lower order terms satisfies

$$d > \frac{m(a + b)}{a + b + 1},$$

then the problem is wellposed for

(5) \[ 1 \leq s < \frac{m}{d + a(d - m)}. \]

**Remark 1.** In the cases mentioned above, when $\lambda_1(t, \xi), \ldots, \lambda_m(t, \xi)$ vanish of infinite order, assumption (2) can be dropped (one is forced to choose
Thus by Theorem 1 we see that the Cauchy Problem (1) is wellposed in the Gevrey classes of order
\[ 1 \leq s < \min \left\{ 1 + \frac{1}{b}, \frac{m}{d} \right\} . \]

**Remark 2.** M. D. Bronshtein [B], S. Wakabayashi [W] proved the Lipschitz (or Hölder) continuity in $t$ of the characteristic roots of hyperbolic polynomials with smooth coefficients (see also [M]). Thus if we assume that $c_{j,\alpha}$ are smooth for $j + |\alpha| = m$, we can drop the assumption that $\lambda_j$ belong to $AC([0,T])$.

**Remark 3.** It is well-known that the lower order terms do not influence the $C^\infty$-wellposedness for strictly hyperbolic equations (the multiplicity of the characteristic roots is equal to 1) and the lower order terms of order $d = m - 1$ give the Gevrey-wellposedness of order $1 \leq s < m/(m - 1)$ for weakly hyperbolic equations (the multiplicity of the characteristic roots is equal to $m$) (see [B], [C], [CDS], [CJS], [OT], etc.). As the parameter $a$ in (2) becomes greater, the type of $p$ approaches to strictly hyperbolic type. Especially, when $d = m - 1$, the second exponent in (4) is equal to $m/(m - 1 - a)$. Taking $0 \leq a < m - 1$, we can obtain an interpolation between $C^\infty$ and the Gevrey classes of order $m/(m - 1)$.

**Example A.** When the characteristic roots are
\[ \lambda_k(t, \xi) = kt^h \left\{ 1 + \sin^2 \left( \frac{1}{t^{h/\alpha - 1}} \right) \right\} \cdot \xi \]
for some $0 < \alpha \leq 1$, $\alpha < h < \alpha/(1 - \alpha)$ and $k = 1, \ldots, m$, we find that $\lambda_1, \ldots, \lambda_m$ belong to $AC([0,T])$ and also $C^\alpha([0,T])$ and vanish of finite order at $t = 0$ and satisfy (2) with $a = 1/h$ and (3) with $b = 1/\alpha - 1/h$, since
\[
\mu(\Omega_\sigma(\xi)) \leq C \int_0^{C^\sigma^1/h} dt \leq C^\sigma^{1/h}, \\
\int_{[0,T]\setminus \Omega^k_\sigma(\xi)} \frac{|\lambda_k(t, \xi)| + |\lambda_{k+1}(t, \xi)|}{|\lambda_k(t, \xi) - \lambda_{k+1}(t, \xi)|} dt \leq C \int_{C^\sigma^{1/h}}^{T} \left( \frac{1}{t^{h/\alpha - 1}} \right)' dt \leq C^\sigma^{1/h-1/\alpha}.
\]
Applying Theorem 1, we get the wellposedness in the Gevrey classes of order
\[ 1 \leq s < \frac{h}{h - \alpha}(1 + \alpha). \]
According to [CK] or [OT], if the characteristic roots belong to $C^\alpha([0, T])$, the Cauchy problem (1) is in the wellposed in the Gevrey classes of order

$$1 \leq s < 1 + \alpha.$$ 

For the second order polynomial $P(t, \tau, \xi) \equiv \tau^2 - A(t)\xi^2$ where $A(t) \geq 0$, if $A(t)$ belongs to $C^{2\alpha}([0, T])$, we also know the Gevrey order (7) (see [CJS], [D1] and [N2]). We remark that (6) approaches to (7) as $h$ tends to infinity and $s$ can be taken arbitrarily large as $h$ tends to $\alpha$ (the characteristic roots oscillate more slowly). This example implies that the oscillation and the degeneracy of the characteristic roots influence on the wellposedness independently of their regularity.

**Example B.** [CI] and [IY] gave an example of the following kind:

$$\lambda_k(t, \xi) = \begin{cases} k \exp\left(-\frac{1}{t^h}\right) \left\{1 + \sin^2\left(\exp\frac{\gamma}{t^h}\right)\right\} \cdot \xi \\ 0 \end{cases}$$

for some $\gamma > 0$, $h > 0$ and $k = 1, \ldots, m$. They proved the wellposedness in the Gevrey classes of order $1 \leq s < 1 + 1/\gamma$. Notice that $\lambda_1(t, \xi), \ldots, \lambda_m(t, \xi)$ belong to $AC([0, T])$ and vanish of infinite order at $t = 0$ (see Remark 1) and satisfy (3) with $b = \gamma$;

$$\int_{[0,T]\backslash \Omega^k_\sigma(\xi)} \frac{|\lambda_k'(t, \xi)| + |\lambda_{k+1}'(t, \xi)|}{|\lambda_k(t, \xi) - \lambda_{k+1}(t, \xi)|} dt \leq C \int_{1/(\log \sigma^{-1} + C)^{1/h}}^{T} (\exp \frac{\gamma}{t^h})' dt \leq C \sigma^{-\gamma}.$$ 

Thus we can apply Theorem 1 and we get the same Gevrey order $1 \leq s < 1 + 1/\gamma$.

Our theorems can be applied also when the vanishing order of characteristic roots is different from the order of contact between the roots. For instance, if the characteristic polynomial is

$$p(t, \tau, \xi) = \tau^2 - 2t^\alpha \tau \xi + (t^{2\alpha} - t^{2\beta})\xi^2$$ 

where $0 < \alpha < \beta$, we easily obtain $\lambda_1(t, \xi) = (t^\alpha + t^\beta)\xi$ and $\lambda_2(t, \xi) = (t^\alpha - t^\beta)\xi$ which implies that

$$|\lambda_k(t, \xi)| \leq 2t^\alpha|\xi| \text{ (} k = 1, 2\text{), } |\lambda_1(t, \xi) - \lambda_2(t, \xi)| \geq 2t^\beta|\xi| \text{ for } (t, \xi) \in [0, T] \times \mathbb{R}_\xi.$$ 

Since $\lambda_1(t, \xi)$ and $\lambda_2(t, \xi)$ satisfy (2) with $a = 1/\beta$ and (3) $b = 1 - \alpha/\beta$, applying Theorem 1 we have wellposedness in the Gevrey classes of order

$$1 \leq s < 1 + \frac{\beta + 1}{\beta - \alpha}.$$ 

In the favourable case of analytic characteristic roots, more generally from Theorem 1 we also obtain the following results:
COROLLARY 2. (Gevrey-wellposedness). Assume that the coefficients $c_{j,\alpha}(t)$ of $p$, $p_{d}$ belong to $C^{0}([0,T])$ and the characteristic roots of the principal part $\lambda_{1}(t,\xi),\cdots,\lambda_{m}(t,\xi)$ are analytic in $t$ and vanish at $t = 0$ and that there exist constants $C > 0$, $c > 0$ and $0 < \alpha < \beta$ such that for any $(t,\xi) \in [0,T] \times \mathbb{R}_{\xi}^{d}$

$$|\lambda_{k}(t,\xi)| \leq Ct^{\alpha}|\xi| \quad \text{for} \quad k = 1,\cdots,m,$$

$$|\lambda_{k+1}(t,\xi) - \lambda_{k}(t,\xi)| \geq ct^{\beta}|\xi| \quad \text{for} \quad k = 1,\cdots,m-1.$$

Then, when the degree $d$ of the lower order terms satisfies

$$0 \leq d \leq \frac{m(\beta - \alpha + 1)}{2\beta - \alpha + 1},$$

the Cauchy problem (1) is wellposed in the Gevrey classes of order

$$1 \leq s < 1 + \frac{\beta + 1}{\beta - \alpha}.$$

Moreover, when the degree $d$ of the lower order terms satisfies

$$d > \frac{m(\beta - \alpha + 1)}{2\beta - \alpha + 1},$$

then the wellposedness holds for

$$1 \leq s < \frac{\beta m}{\beta d + d - m}.$$

In Corollary 2 and Examples A and B, the characteristic roots coincide only at $t = 0$ or at a finite number of points. We give a final example to emphasize that our results allow the characteristic roots to coincide at an infinite number of points.

Example C (see also Example A). When the characteristic roots are

$$\lambda_{k}(t,\xi) = kt^{h}\sin^{h}\left(\frac{1}{t^{1-h}}\right) \cdot \xi$$

for some even number $h$ and $k = 1,\cdots,m$, we find that $\lambda_{1}(t,\xi),\cdots,\lambda_{m}(t,\xi)$ are absolutely continuous in $t$, more precisely Lipschitz continuous in $t$ and vanish at $t = (\pi j)^{1/(1-h)}$ ($j = 1,2,\cdots$), they satisfy (2) with $a < 1/h$ and (3) with $b > 1 - 1/h$. Applying Theorem 1, we get the wellposedness in the Gevrey classes of order $1 \leq s < 2h/(h-1)$ (see (7)).
§2. Sketch of the proof

When $s = 1$, the Cauchy problem (1) is wellposed in the class of real analytic functions. Therefore we can suppose that $s > 1$ for the proof. By Fourier transform with respect to $x$, the Cauchy problem (1) turns into

$$
\begin{align*}
\{ & p(t, D_t, \xi) \hat{u} = \hat{f}(t, \xi) + p_d(t, D_t, \xi) \hat{u} \\
& D_t^j \hat{u}(0, \xi) = \hat{u}_j(\xi) \quad (j = 0, \ldots, m - 1) \}
\end{align*}
$$

(8)

Let $0 < \sigma < 1$ and $\varphi(r)$ be a non-negative function such that $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(r) \equiv 0$ for $|r| \geq 2$ and $\varphi(r) \equiv 1$ for $|r| \leq 1$. We define

$$
\begin{align*}
\omega(t, \xi) &= \sigma|\xi| \sum_{l=1}^{m-1} \varphi(\sigma^{-1}\{ \lambda_l(t, \frac{\xi}{|\xi|}) - \lambda_{l+1}(t, \frac{\xi}{|\xi|}) \}), \\
\mu_k(t, \xi) &= \lambda_k(t, \xi) + ik\omega(t, \xi) \quad \text{for} \; k = 1, \ldots, m.
\end{align*}
$$

Moreover we denote by $q(t, \tau, \xi)$ the polynomial of degree $m$ in $\tau$

$$
q(t, \tau, \xi) = \prod_{k=1}^{m} (\tau - \mu_k(t, \xi)).
$$

Now we set the energy density

$$
E(t, \xi) = \frac{1}{2} \sum_{l=1}^{m} |q_l(t, D_t, \xi) \hat{u}|^2,
$$

where $q_l(t, \tau, \xi)$ is the polynomial of degree $m - 1$ in $\tau$ defined by

$$
q_l(t, \tau, \xi) = \frac{q(t, \tau, \xi)}{\tau - \mu_l(t, \xi)} = \prod_{k=1, k \neq l}^{m} (\tau - \mu_k(t, \xi)).
$$

We denote by $'$ the derivative in $t$. Differentiating $E(t, \xi)$ in $t$ and dividing by $2\sqrt{E(t, \xi)}$, by (8) we have

$$
\sqrt{E}' \leq C \left( \max_{1 \leq k \leq m-1} \frac{\lambda_k' + \lambda_{k+1}' + |\omega'|}{|\lambda_k - \lambda_{k+1}| + \omega} + \frac{|\xi|^d}{\prod_{k=1}^{m-1} |\lambda_k - \lambda_{k+1}| + \omega^{m-1}} \right) \sqrt{E} + |\hat{f}|.
$$

Thus, Gronwall's inequality yields the estimate

$$
\sqrt{E(t, \xi)} \leq \exp \left\{ C \int_0^T \left( \max_{1 \leq k \leq m-1} \frac{\lambda_k' + \lambda_{k+1}' + |\omega'|}{|\lambda_k - \lambda_{k+1}| + \omega} + \frac{|\xi|^d}{\prod_{k=1}^{m-1} |\lambda_k - \lambda_{k+1}| + \omega^{m-1}} \right) dt \right\} 
$$

$$
\times \left\{ \sqrt{E(0, \xi)} + \int_0^T |\hat{f}(t, \xi)| dt \right\}.
$$
We remark that there exists $C > 0$ such that for any $(t, \xi) \in [0, T) \times \mathbb{R}^n \setminus 0$

$$C^{-1}(\sigma|\xi|)^{m-1}|\xi|^{-j}|D_t^j \hat{u}| \leq \sqrt{E(t, \xi)} \leq C \sum_{j=0}^{m-1} |\xi|^{m-1-j}|D_t^j \hat{u}|.$$

**Lemma 1.** Let $b \geq 0$. Assume that $\lambda_1(t, \xi), \cdots, \lambda_m(t, \xi)$ belong to $AC([0, T])$ and satisfy (3). Then there exists $C > 0$ such that for any $0 < \sigma < 1$, $|\xi| = 1$ and $k = 1, \cdots, m$

$$\int_{\Omega^{k}_{\sigma}(\xi) \cup \Omega^{k-1}_{\sigma}(\xi)} |\lambda'_k(t, \xi)| dt \leq \begin{cases} C & \text{if } b \geq 1 \\ C\sigma^{1-b} & \text{if } 0 \leq b < 1 \end{cases} \leq C\sigma^{1-b},$$

where $\Omega^0_{\sigma}(\xi) = \Omega^m_{\sigma}(\xi) = \phi$ and $\Omega^k_{\sigma}(\xi)$ for $k = 1, \cdots, m - 1$ are defined in §.1.

**Lemma 2.** Let $0 \leq a < m - 1$. Assume that $\lambda_1, \cdots, \lambda_m$ satisfy (2). Then there exists $C > 0$ such that for any $0 < \sigma < 1$, $|\xi| = 1$

$$\int_{[0, T) \setminus \Omega_{\sigma}(\xi)} \frac{dt}{\prod_{k=1}^{m-1} |\lambda_k(t, \xi) - \lambda_{k+1}(t, \xi)|} \leq C\sigma^{a+1-m},$$

where $\Omega_{\sigma}(\xi)$ is defined in §.1.

Consequently, it follows that

$$\sum_{j=0}^{m-1} |\xi|^{-j}|D_t^j \hat{u}(t, \xi)| \leq C\sigma^{1-m}\exp\{C(\sigma^{-b} + \sigma^{a+1}|\xi| + \sigma^{a+1-m}|\xi|^{d+1-m})\}$$

$$\times \left\{ \sum_{j=0}^{m-1} |\xi|^{-j}|\hat{u}_j| + \int_{0}^{T} |\xi|^{1-m} |\hat{f}(t, \xi)| dt \right\}.$$ 

When

$$d \leq \frac{m(a+b)}{a+b+1},$$

the third term is smaller and this choice gives immediately

$$|\xi|^\gamma b + |\xi|^{1-\gamma(a+1)} + |\xi|^{\gamma(m-a-1)+d+1-m} \leq 3|\xi|^\frac{b}{a+b+1}.$$ 

Hence, there exists $\rho > 0$ such that for any $(t, \xi) \in [0, T) \times \mathbb{R}^n \setminus 0$

$$\sum_{j=0}^{m-1} |\xi|^{-j}|D_t^j \hat{u}(t, \xi)| \leq C\exp\left\{ \rho|\xi|^\frac{1}{a+b+1} \right\} \left\{ \sum_{j=0}^{m-1} |\xi|^{\frac{m-1}{a+b+1}-j}|\hat{u}_j(\xi)| + \int_{0}^{T} |\xi|^{\frac{(1-m)(a+1)}{a+b+1}} |\hat{f}(t, \xi)| dt \right\}$$

53
In virtue of Paley-Wiener theorem, \( \{D_t^j u(\cdot, t) ; t \in [0,T], j = 0, \ldots, m-1 \} \) is bounded in the Gevrey classes of order (5). Thus, taking into account that \( u \) is a solution of (1), we find \( u \in C^m([0,T]; G^s(\mathbb{R}^n)) \). This concludes the proof of Theorem 1 in the case when \( d \leq m(a+b)/(a+b+1) \).

On the other hand, when

\[
d > \frac{m(a+b)}{a+b+1},
\]

the dominant terms in

\[
|\xi|^\gamma b + |\xi|^{1-\gamma(a+1)} + |\xi|^\gamma(m-a-1)+d+1-m
\]

are the last two (the first one is smaller). In this case we choose

\[
\gamma = \frac{m-r}{m}
\]

and proceeding as above we conclude the proof of this case and we get (4).

REFERENCES


