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On the Wellposedness of the Cauchy Problem for Weakly Hyperbolic Equations of Higher Order

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§1. Introduction

We consider here the Cauchy problem on $[0,T]\times\mathbf{R}_x^n$

$$(1)\qquad \begin{cases} D_t^m u = \displaystyle\sum_{j+|\alpha|=m} c_{j,\alpha}(t) D_t^j D_x^\alpha u + \sum_{j+|\alpha|\le d} c_{j,\alpha}(t) D_t^j D_x^\alpha u + f(t,x) \\ D_t^j u(0,x) = u_j(x) \quad (j=0,\cdots,m-1), \end{cases}$$

where $D_t=-i\partial_t$, $D_x=-i(\partial_{x_1},\cdots,\partial_{x_n})$, and $0\le d\le m-1$. We shall write in short

$$p(t,\tau,\xi)=\tau^m-\sum_{j+|\alpha|=m} c_{j,\alpha}(t)\tau^j\xi^\alpha$$

for the principal part and

$$p_d(t,\tau,\xi)=\sum_{j+|\alpha|\le d} c_{j,\alpha}(t)\tau^j\xi^\alpha$$

for the lower order terms. We shall assume that the principal part p is hyperbolic with respect to τ, that is, for any $t\in\mathbf{R}_t$, $\xi\in\mathbf{R}_\xi^n$ the roots in τ of the algebraic equation $p(t,\tau,\xi)=0$ are all real. We name them $\lambda_j(t,\xi)$, according to the rule

$$\lambda_1(t,\xi)\ge\lambda_2(t,\xi)\ge\cdots\ge\lambda_m(t,\xi),$$

thus $p(t,\tau,\xi)$ can be written

$$p(t,\tau,\xi)=\prod_{k=1}^m\bigl(\tau-\lambda_k(t,\xi)\bigr).$$

We recall that the functions $\lambda_j(t,\xi)$ are homogeneous of degree 1 in ξ.

There are many results on this problem. As to the C^∞-wellposedness, we mention that T. Nishitani [N1] considered the case when the multiplicity of the characteristic roots is at most double. F. Colombini and N. Orrú [CO] assumed that the characteristic roots vanish of finite order at $t = 0$ and satisfy

$$t^2 \sum_{k,j=1,k\neq j}^{m} \frac{|\lambda_k'(t,\xi)|^2 + |\lambda_j'(t,\xi)|^2}{|\lambda_k(t,\xi) - \lambda_j(t,\xi)|^2} < \infty \quad \text{near} \quad t = 0.$$

Moreover, K. Kajitani, S. Wakabayashi and K. Yagdjian [KWY] dealt with the case of characteristic roots vanishing of infinite order. Concerning the Gevrey-wellposedness, F. Colombini and T. Kinoshita [CK] considered the Cauchy problem in the case when the characteristic roots are Hölder continuous in t. F. Colombini, H. Ishida [CI] and H. Ishida, K. Yagdjian [IY] assumed that the characteristic roots vanish of infinite order at $t = 0$ and satisfy for some $\bar{s} > 1$

$$\frac{\Phi_1(t)^{2\bar{s}/(\bar{s}-1)}}{\phi_1(t)^2} \sum_{k,j=1,k\neq j}^{m} \frac{|\lambda_k'(t,\xi)|^2 + |\lambda_j'(t,\xi)|^2}{|\lambda_k(t,\xi) - \lambda_j(t,\xi)|^2} < \infty \quad \text{near} \quad t = 0,$$

where $\Phi_1(t) = \int_0^t \phi_1 dt$ and $\phi_1(t), \cdots, \phi_m(t)$ are real-valued functions such that
(i) $\phi_k(0) = \phi_k'(0) = 0$, $\phi_k'(t) > 0$ if $t \in (0,T]$ for any $k = 1, \cdots, m$.
(ii) $\phi_1(t) \geq \phi_2(t) \geq \cdots \geq \phi_m(t)$ for $t \in [0,T]$.
(iii) $|\lambda_k(t,\xi)| \leq C_k \phi_k(t)|\xi|$ $(^\exists C_k > 0)$ for $k = 1, \cdots, m$ and $(t,\xi) \in [0,T] \times \mathbf{R}^n_\xi \backslash 0$.
(iv) $|\lambda_k(t,\xi) - \lambda_j(t,\xi)| \geq c\phi_k(t)|\xi|$ $(^\exists c > 0)$ for $k < j$ and $(t,\xi) \in [0,T] \times \mathbf{R}^n_\xi \backslash 0$.
Then they showed the wellposedness in the Gevrey classes of order $1 \leq s < \bar{s}$.

We see that in most results concerning the higher order case $m > 2$ the roots are assumed to coincide only at isolated points, and then a precise behaviour is assumed at those points. In this paper we try to give a global assumption valid in more general cases, even when this happens at an arbitrary set of points (also infinite or dense). To this end we introduce the sets Ω_σ^k, Ω_σ defined as follows: for any $0 < \sigma < 1$, $k = 1, \ldots, m-1$,

$$\Omega_\sigma^k(\xi) = \{t \in [0,T] \ : \ |\lambda_k(t,\xi) - \lambda_{k+1}(t,\xi)| \leq \sigma\}$$

and

$$\Omega_\sigma(\xi) = \bigcup_{k=1}^{m-1} \Omega_\sigma^k(\xi).$$

These sets enclose, for each ξ, the points t where the roots coincide; thus we can regard the measure $\mu(\Omega_\sigma)$, which is a function of σ, ξ, as a measure of the defect of strict hyperbolicity of p. Here $\mu(A)$ is the Lebesgue measure in $\mathbf{R}_t$ of the set $A \subseteq [0,T]$. We denote by $AC([0,T])$ the space of absolutely continuous functions on $[0,T]$ and by $G^s(\mathbf{R}^n)$ the space of Gevrey functions $g(x)$ satisfying $\sup_{x\in K} |D_x^\alpha g(x)| \le C_K \rho_K^{|\alpha|} |\alpha|!^s$ for any compact set $K \subset \mathbf{R}^n$, $\alpha \in \mathbf{N}^n$.

Our first result is the following:

THEOREM 1. *(Gevrey-wellposedness). Assume that the coefficients $c_{j,\alpha}(t)$ of p, p_d belong to $C^0([0,T])$ and the characteristic roots of the principal part $\lambda_1, \cdots, \lambda_m$ belong to $AC([0,T])$ and that there exist constants $C > 0$, $a \ge 0$ and $b > 0$ such that for any $0 < \sigma < 1$, $|\xi| = 1$, $k = 1, \cdots, m-1$*

(2) $$\mu(\Omega_\sigma(\xi)) \le C\sigma^a,$$

(3) $$\int_{[0,T]\setminus\Omega_\sigma^k(\xi)} \frac{|\lambda_k'(t,\xi)| + |\lambda_{k+1}'(t,\xi)|}{|\lambda_k(t,\xi) - \lambda_{k+1}(t,\xi)|} dt \le C\sigma^{-b}.$$

Then, when the degree d of the lower order terms satisfies

$$0 \le d \le \frac{m(a+b)}{a+b+1},$$

the Cauchy problem (1) is wellposed in the Gevrey classes of order

(4) $$1 \le s < 1 + \frac{a+1}{b},$$

i.e., for any data $u_j \in G^s(\mathbf{R}^n)$ and $f \in C^0([0,T]; G^s(\mathbf{R}^n))$ the Cauchy problem (1) has a unique solution $u \in C^m([0,T]; G^s(\mathbf{R}^n))$. Moreover, when the degree d of the lower order terms satisfies

$$d > \frac{m(a+b)}{a+b+1},$$

then the problem is wellposed for

(5) $$1 \le s < \frac{m}{d + a(d-m)}.$$

Remark 1. In the cases mentioned above, when $\lambda_1(t,\xi), \cdots, \lambda_m(t,\xi)$ vanish of infinite order, assumption (2) can be dropped (one is forced to choose

$a = 0$). Thus by Theorem 1 we see that the Cauchy Problem (1) is wellposed in the Gevrey classes of order

$$1 \le s < \min\left\{1 + \frac{1}{b}, \frac{m}{d}\right\}.$$

Remark 2. M. D. Bronshtein [B], S. Wakabayashi [W] proved the Lipschitz (or Hölder) continuity in t of the characteristic roots of hyperbolic polynomials with smooth coefficients (see also [M]). Thus if we assume that $c_{j,\alpha}$ are smooth for $j + |\alpha| = m$, we can drop the assumption that λ_j belong to $AC([0,T])$.

Remark 3. It is well-known that the lower order terms do not influence the C^∞-wellposedness for strictly hyperbolic equations (the multiplicy of the characteristic roots is equal to 1) and the lower order terms of order $d = m - 1$ give the Gevrey- wellposedness of order $1 \le s < m/(m-1)$ for weakly hyperbolic equations (the multiplicy of the characteristic roots is equal to m) (see [B], [C], [CDS], [CJS], [OT], etc.). As the parameter a in (2) becomes greater, the type of p approaches to strictly hyperbolic type. Especially, when $d = m-1$, the second exponent in (4) is equal to $m/(m-1-a)$. Taking $0 \le a < m-1$, we can obtain an interpolation between C^∞ and the Gevrey classes of order $m/(m-1)$.

Example A. When the characteristic roots are

$$\lambda_k(t,\xi) = kt^h\Big\{1 + \sin^2\Big(\frac{1}{t^{h/\alpha-1}}\Big)\Big\} \cdot \xi$$

for some $0 < \alpha \le 1$, $\alpha < h < \alpha/(1-\alpha)$ and $k = 1, \cdots, m$, we find that $\lambda_1, \cdots, \lambda_m$ belong to $AC([0,T])$ and also $C^\alpha([0,T])$ and vanish of finite order at $t = 0$ and satisfy (2) with $a = 1/h$ and (3) with $b = 1/\alpha - 1/h$, since

$$\mu(\Omega_\sigma(\xi)) \le C \int_0^{C\sigma^{1/h}} dt \le C\sigma^{1/h},$$

$$\int_{[0,T]\setminus\Omega_\sigma^k(\xi)} \frac{|\lambda_k'(t,\xi)| + |\lambda_{k+1}'(t,\xi)|}{|\lambda_k(t,\xi) - \lambda_{k+1}(t,\xi)|} dt \le C \int_{C\sigma^{1/h}}^T \Big(\frac{1}{t^{h/\alpha-1}}\Big)' dt \le C\sigma^{1/h-1/\alpha}.$$

Applying Theorem 1, we get the wellposedness in the Gevrey classes of order

$$1 \le s < \frac{h}{h-\alpha}(1+\alpha). \tag{6}$$

According to [CK] or [OT], if the characteristic roots belong to $C^{\alpha}([0,T])$, the Cauchy problem (1) is in the wellposed in the Gevrey classes of order

$$1 \leq s < 1 + \alpha.$$

For the second order polynomial $P(t,\tau,\xi) \equiv \tau^2 - A(t)\xi^2$ where $A(t) \geq 0$, if $A(t)$ belongs to $C^{2\alpha}([0,T])$, we also know the Gevrey order (7) (see [CJS], [D1] and [N2]). We remark that (6) approaches to (7) as h tends to infinity and s can be taken arbitrarily large as h tends to α (the characteristic roots oscillate more slowly). This example implies that the oscillation and the degeneracy of the characteristic roots influence on the wellposedness independently of their regulality.

Example B. [CI] and [IY] gave an example of the following kind:

$$\lambda_k(t,\xi) = \begin{cases} k\exp\left(-\frac{1}{t^h}\right)\left\{1+\sin^2\left(\exp\frac{\gamma}{t^h}\right)\right\}\cdot\xi \\ 0 \end{cases}$$

for some $\gamma > 0$, $h > 0$ and $k = 1,\dots,m$. They proved the wellposedness in the Gevrey classes of order $1 \leq s < 1 + 1/\gamma$. Notice that $\lambda_1(t,\xi),\cdots,\lambda_m(t,\xi)$ belong to $AC([0,T])$ and vanish of infinite order at $t=0$ (see Remark 1) and satisfy (3) with $b=\gamma$;

$$\int_{[0,T]\setminus\Omega_\sigma^k(\xi)} \frac{|\lambda_k'(t,\xi)| + |\lambda_{k+1}'(t,\xi)|}{|\lambda_k(t,\xi)-\lambda_{k+1}(t,\xi)|}dt \leq C\int_{1/(\log\sigma^{-1}+C)^{1/h}}^{T}\left(\exp\frac{\gamma}{t^h}\right)'dt \leq C\sigma^{-\gamma}.$$

Thus we can apply Theorem 1 and we get the same Gevrey order $1 \leq s < 1+1/\gamma$.

Our theorems can be applied also when the vanishing order of characteristic roots is different from the order of contact between the roots. For instance, if the characteristic polynomial is

$$p(t,\tau,\xi) = \tau^2 - 2t^{\alpha}\tau\xi + (t^{2\alpha} - t^{2\beta})\xi^2 \quad \text{where} \quad 0 < \alpha < \beta,$$

we easily obtain $\lambda_1(t,\xi) = (t^{\alpha}+t^{\beta})\xi$ and $\lambda_2(t,\xi) = (t^{\alpha}-t^{\beta})\xi$ which implies that $|\lambda_k(t,\xi)| \leq 2t^{\alpha}|\xi|$ $(k=1,2)$, $|\lambda_1(t,\xi)-\lambda_2(t,\xi)| \geq 2t^{\beta}|\xi|$ for $(t,\xi) \in [0,T]\times\mathbf{R}_{\xi}$. Since $\lambda_1(t,\xi)$ and $\lambda_2(t,\xi)$ satisfy (2) with $a = 1/\beta$ and (3) $b = 1-\alpha/\beta$, applying Theorem 1 we have wellposedness in the Gevrey classes of order

$$1 \leq s < 1 + \frac{\beta+1}{\beta-\alpha}.$$

In the favourable case of analytic characteristic roots, more generally from Theorem 1 we also obtain the following results:

COROLLARY **2.** *(Gevrey-wellposedness). Assume that the coefficients* $c_{j,\alpha}(t)$ *of* p, p_d *belong to* $C^0([0,T])$ *and the characteristic roots of the principal part* $\lambda_1(t,\xi),\cdots,\lambda_m(t,\xi)$ *are analytic in* t *and vanish at* $t=0$ *and that there exist constants* $C>0$, $c>0$ *and* $0<\alpha<\beta$ *such that for any* $(t,\xi)\in[0,T]\times\mathbf{R}^n_\xi$

$$|\lambda_k(t,\xi)|\le Ct^\alpha|\xi| \quad \textit{for} \quad k=1,\cdots,m,$$
$$|\lambda_{k+1}(t,\xi)-\lambda_k(t,\xi)|\ge ct^\beta|\xi| \quad \textit{for} \quad k=1,\cdots,m-1.$$

Then, when the degree d *of the lower order terms satisfies*

$$0\le d\le\frac{m(\beta-\alpha+1)}{2\beta-\alpha+1},$$

the Cauchy problem (1) is wellposed in the Gevrey classes of order

$$1\le s<1+\frac{\beta+1}{\beta-\alpha}.$$

Moreover, when the degree d *of the lower order terms satisfies*

$$d>\frac{m(\beta-\alpha+1)}{2\beta-\alpha+1},$$

then the wellposedness holds for

$$1\le s<\frac{\beta m}{\beta d+d-m}.$$

In Corollary 2 and Examples A and B, the characteristic roots coincide only at $t=0$ or at a finite number of points. We give a final example to emphasize that our results allow the characteristic roots to coincide at an infinite number of points.

Example C (see also Example A). When the characteristic roots are

$$\lambda_k(t,\xi)=kt^h\sin^h\left(\frac{1}{t^{h-1}}\right)\cdot\xi$$

for some even number h and $k=1,\cdots,m$, we find that $\lambda_1(t,\xi),\cdots,\lambda_m(t,\xi)$ are absolutely continuous in t, more precisely Lipschitz continuous in t and vanish at $t=(\pi j)^{1/(1-h)}$ $(j=1,2,\cdots)$, they satisfy (2) with $a<1/h$ and (3) with $b>1-1/h$. Applying Theorem 1, we get the wellposedness in the Gevrey classes of order $1\le s<2h/(h-1)$ (see (7)).

§2. Sketch of the proof

When $s = 1$, the Cauchy problem (1) is wellposed in the class of real analytic functions. Therefore we can suppose that $s > 1$ for the proof. By Fourier transform with respect to x, the Cauchy problem (1) turns into

$$\text{(8)}\qquad \begin{cases} p(t, D_t, \xi)\hat{u} = \hat{f}(t, \xi) + p_d(t, D_t, \xi)\hat{u} \\ D_t^j \hat{u}(0, \xi) = \hat{u}_j(\xi) \quad (j = 0, \cdots, m-1). \end{cases}$$

Let $0 < \sigma < 1$ and $\varphi(r)$ be a non-negative function such that $\varphi \in C_0^\infty(\mathbf{R})$, $\varphi(r) \equiv 0$ for $|r| \geq 2$ and $\varphi(r) \equiv 1$ for $|r| \leq 1$. We define

$$\omega(t, \xi) = \sigma|\xi| \sum_{l=1}^{m-1} \varphi\Big(\sigma^{-1}\Big\{\lambda_l\Big(t, \frac{\xi}{|\xi|}\Big) - \lambda_{l+1}\Big(t, \frac{\xi}{|\xi|}\Big)\Big\}\Big),$$
$$\mu_k(t, \xi) = \lambda_k(t, \xi) + ik\omega(t, \xi) \quad \text{for} \quad k = 1, \cdots, m.$$

Moreover we denote by $q(t, \tau, \xi)$ the polynomial of degree m in τ

$$q(t, \tau, \xi) = \prod_{k=1}^{m} (\tau - \mu_k(t, \xi)).$$

Now we set the energy density

$$E(t, \xi) = \frac{1}{2} \sum_{l=1}^{m} |q_l(t, D_t, \xi)\hat{u}|^2,$$

where $q_l(t, \tau, \xi)$ is the polynomial of degree $m - 1$ in τ defined by

$$q_l(t, \tau, \xi) = \frac{q(t, \tau, \xi)}{\tau - \mu_l(t, \xi)} \Big(= \prod_{k=1, k\neq l}^{m} (\tau - \mu_k(t, \xi))\Big).$$

We denote by $'$ the derivative in t. Differentiating $E(t, \xi)$ in t and dividing by $2\sqrt{E(t, \xi)}$, by (8) we have

$$\sqrt{E}\,' \leq C\left(\max_{1\leq k\leq m-1} \frac{|\lambda_k'| + |\lambda_{k+1}'| + |\omega'|}{|\lambda_k - \lambda_{k+1}| + \omega} + \omega + \frac{|\xi|^d}{\prod_{k=1}^{m-1} |\lambda_k - \lambda_{k+1}| + \omega^{m-1}}\right) \sqrt{E} + |\hat{f}|.$$

Thus, Gronwall's inequality yields the estimate

$$\sqrt{E(t, \xi)} \leq \exp\Big\{C \int_0^T \Big(\max_{1\leq k\leq m-1} \frac{|\lambda_k'| + |\lambda_{k+1}'| + |\omega'|}{|\lambda_k - \lambda_{k+1}| + \omega} + \omega + \frac{|\xi|^d}{\prod_{k=1}^{m-1} |\lambda_k - \lambda_{k+1}| + \omega^{m-1}}\Big) dt\Big\}$$
$$\times \Big\{\sqrt{E(0, \xi)} + \int_0^T |\hat{f}(t, \xi)| dt\Big\}.$$

We remark that there exists $C > 0$ such that for any $(t, \xi) \in [0, T] \times \mathbf{R}^n_\xi \backslash 0$

$$C^{-1}(\sigma|\xi|)^{m-1}|\xi|^{-j}|D_t^j\hat{u}| \leq \sqrt{E(t,\xi)} \leq C \sum_{j=0}^{m-1} |\xi|^{m-1-j}|D_t^j\hat{u}|.$$

Lemma 1. *Let $b \geq 0$. Assume that $\lambda_1(t,\xi), \cdots, \lambda_m(t,\xi)$ belong to $AC([0,T])$ and satisfy (3). Then there exists $C > 0$ such that for any $0 < \sigma < 1$, $|\xi| = 1$ and $k = 1, \cdots, m$*

$$\int_{\Omega_\sigma^k(\xi)\cup\Omega_\sigma^{k-1}(\xi)} |\lambda_k'(t,\xi)|dt \leq \begin{cases} C & \text{if } b \geq 1 \\ C\sigma^{1-b} & \text{if } 0 \leq b < 1 \end{cases} \leq C\sigma^{1-b},$$

where $\Omega_\sigma^0(\xi) = \Omega_\sigma^m(\xi) = \phi$ and $\Omega_\sigma^k(\xi)$ for $k = 1, \cdots, m-1$ are defined in §.1.

Lemma 2. *Let $0 \leq a < m-1$. Assume that $\lambda_1, \cdots, \lambda_m$ satisfy (2). Then there exists $C > 0$ such that for any $0 < \sigma < 1$, $|\xi| = 1$*

$$\int_{[0,T]\backslash\Omega_\sigma(\xi)} \frac{dt}{\prod_{k=1}^{m-1} |\lambda_k(t,\xi) - \lambda_{k+1}(t,\xi)|} \leq C\sigma^{a+1-m}, \tag{21}$$

where $\Omega_\sigma(\xi)$ is defined in §.1.

Consequently, it follows that

$$\sum_{j=0}^{m-1} |\xi|^{-j}|D_t^j\hat{u}(t,\xi)| \leq C\sigma^{1-m}\exp\{C(\sigma^{-b} + \sigma^{a+1}|\xi| + \sigma^{a+1-m}|\xi|^{d+1-m})\}$$
$$\times \Big\{\sum_{j=0}^{m-1} |\xi|^{-j}|\hat{u}_j| + \int_0^T |\xi|^{1-m}|\hat{f}(t,\xi)|dt\Big\}.$$

When

$$d \leq \frac{m(a+b)}{a+b+1},$$

the third term is smaller and this choice gives immediately

$$|\xi|^{\gamma b} + |\xi|^{1-\gamma(a+1)} + |\xi|^{\gamma(m-a-1)+d+1-m} \leq 3|\xi|^{\frac{b}{a+b+1}}.$$

Hence, there exists $\rho > 0$ such that for any $(t,\xi) \in [0,T] \times \mathbf{R}^n_\xi \backslash 0$

$$\sum_{j=0}^{m-1} |\xi|^{-j}|D_t^j\hat{u}(t,\xi)| \leq C\exp\{\rho|\xi|^{\frac{b}{a+b+1}}\}\Big\{\sum_{j=0}^{m-1} |\xi|^{\frac{m-1}{a+b+1}-j}|\hat{u}_j(\xi)| + \int_0^T |\xi|^{\frac{(1-m)(a+1)}{a+b+1}}|\hat{f}(t,\xi)|dt\Big\}.$$

In virtue of Paley-Wiener theorem, $\{D_t^j u(\cdot,t)\ ;\ t \in [0,T],\ j = 0,\cdots,m-1\}$ is bounded in the Gevrey classes of order (5). Thus, taking into account that u is a solution of (1), we find $u \in C^m([0,T];G^s(\mathbf{R}^n))$. This concludes the proof of Theorem 1 in the case when $d \le m(a+b)/(a+b+1)$.

On the other hand, when

$$d > \frac{m(a+b)}{a+b+1},$$

the dominant terms in

$$|\xi|^{\gamma b} + |\xi|^{1-\gamma(a+1)} + |\xi|^{\gamma(m-a-1)+d+1-m}$$

are the last two (the first one is smaller). In this case we choose

$$\gamma = \frac{m-r}{m}$$

and proceeding as above we conclude the proof of this case and we get (4).

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