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Structure of the solutions to Fuchsian systems

Takeshi MANDAI (万代武史)
Osaka Electro-Communication University (大阪電気通信大学)

Hidetoshi TAHARA (田原秀敏)
Sophia University (上智大学)

Abstract: To a certain Volevič system of homogeneous singular partial differential equations in a complex domain, called a Fuchsian system, holomorphic solutions which have singularities only on the initial surface are considered. All the solutions are constructed and parametrized in a good way, without any assumptions on the characteristic exponents.

1 Introduction

We consider a system of linear partial differential operators

\[ P = tD_t I_m - A(t, x; D_x), \quad (t, x) \in C \times C^n, \quad (1.1) \]

where \( I_m \) is the \( m \times m \) unit matrix, and

\[ A = (A_{i,j}(t, x; D_x))_{1 \leq i, j \leq m}, \quad A_{i,j} = \sum_{\alpha \text{ finite}} a_{i,j;\alpha}(t, x)D_x^\alpha, \]

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Key Words: Fuchsian system, Volevič system, regular singularity, characteristic exponent, characteristic index.
$a_{i,j;\alpha}$ are holomorphic in a neighborhood of the origin $(0,0) \in C^{n+1}$. We use $D_{t} = \frac{\partial}{\partial t}$, $D_{z} = (D_{1}, \ldots, D_{n})$, $D_{j} = \frac{\partial}{\partial x_{j}}$, without dividing by $\sqrt{-1}$.

$P$ is called a Fuchsian system if $P$ satisfies the following two conditions.

(A-1) There exists $n_{j} \in \mathbb{N} := \{0, 1, 2, \ldots \}$ such that
\[
\operatorname{ord}_{D_{x}} A_{i,j}(t, x; D_{x}) \leq n_{i} - n_{j} + 1.
\]

(A-2) $A(0, x; D_{x}) := A_{0}(x)$ is independent of $D_{x}$.

The condition (A-1) is equivalent to each of the following condition ([4], [5]).

(A-1)' $\max_{1 \leq p \leq m} \left( \frac{1}{p} \max_{1 \leq i_{1} < \cdots < i_{p} \leq m} \sum_{k=1}^{p} \operatorname{ord}_{D_{x}} A_{i_{k}, i_{r(k)}} \right) =: \rho(A) \leq 1$

($\rho(A)$ is called the matrix order of $A$.)

When this condition is satisfied, the system $D_{t} I_{m} - A(t, x; D_{x})$ is called a kowalevskian system in Volević's sense ([4]).

The polynomial
\[
C(x; \lambda) := \det(\lambda I_{m} - A_{0}(x))
\]
of $\lambda$ is called the indicial polynomial of $P$, and a root $\lambda$ of $C(x; \lambda) = 0$ is called a characteristic exponent or a characteristic index of $P$ at $x$.

The second author ([6], [7]) has shown the following fundamental theorems corresponding to the Cauchy-Kowalevsky theorem and the Holmgren theorem. Let $\mathcal{O}_{(0,0)}$ denote the germ space of holomorphic functions at $(0,0) \in C \times C^{n}$.

**Theorem 1.1** ([6, Theorem 1.2.10]). If $C(0;j) \equiv 0 \, (j \in \mathbb{N})$, then for every $\vec{f} \in (\mathcal{O}_{(0,0)})^{m}$, there exists a unique $\vec{u} \in (\mathcal{O}_{(0,0)})^{m}$ such that $P \vec{u} = \vec{f}(t,x)$.

**Theorem 1.2** ([7, Theorem 2]). Let $\Omega$ be an open neighborhood of $0 \in R^{n}$ and $T > 0$. Let $L \in R$ satisfy that if $C(x; \lambda) = 0 \, (x \in \Omega)$, then $\Re \lambda < L$. If $\vec{u}(t) = \vec{U}(t,x) \in C^{1}((0,T], \mathcal{D}'(\Omega))^{m}$ satisfies $P \vec{u} = \vec{0}$ in $(0,T) \times \Omega$, and if $t^{-L} \vec{u} \in C^{0}([0,T], \mathcal{D}'(\Omega))^{m}$, then $\vec{u} = \vec{0}$ near $(0,0)$ in $(0,T) \times \Omega$. Here, $\mathcal{D}'(\Omega)$ denotes the space of Schwartz distributions on $\Omega$. 
Now, we introduce the following notation.

\[ \mathcal{O}(\Omega) := \{ \text{holomorphic functions on } \Omega \} , \]
\[ B_R := \{ x \in \mathbb{C}^n : |x| < R \} , \quad \Delta_T := \{ t \in \mathbb{C} : |t| < T \} \quad (T > 0) , \]
\[ \mathcal{O}_0 := \bigcup_{R>0} \mathcal{O}(B_R) , \quad \mathcal{O}_{(0,0)} := \bigcup_{R>0,T>0} \mathcal{O}(\Delta_T \times B_R) , \]
\[ S_{\infty,T} := \mathcal{R}(\Delta_T \setminus \{0\}) \quad \text{(the universal covering of } \Delta_T \setminus \{0\}) , \]
\[ S_{\theta,T} := \{ t \in S_{\infty,T} : |\arg t| \leq \theta \} , \quad \tilde{\mathcal{O}} := \bigcup_{T>0,R>0} \mathcal{O}(S_{\infty,T} \times B_R) . \]

Now, we consider solutions of \( P \tilde{u} = \vec{0} \) which are singular only at \( t = 0 \), that is, \( \tilde{u} \in (\tilde{\mathcal{O}})^m \). Under the assumption that the characteristic exponents \( \lambda_j(x) \) \( (j = 1, 2, \ldots, m) \) of \( P \) do not differ by integers, that is, \( \lambda_i(0) - \lambda_j(0) \not\in \mathbb{Z} \quad (i \neq j) \), the structure of the kernel \( \text{Ker}_{(\tilde{\mathcal{O}})^m} P \) of the map \( P : (\tilde{\mathcal{O}})^m \to (\tilde{\mathcal{O}})^m \) has been studied by the second author([6]).

Our purpose of this talk is to construct a solution map, that is, a linear isomorphism
\[
(\mathcal{O}_0)^m \xrightarrow{\sim} \text{Ker}_{(\tilde{\mathcal{O}})^m} P := \{ \tilde{u} \in (\tilde{\mathcal{O}})^m : P \tilde{u} = \vec{0} \} ,
\]
rather explicitly, with no assumptions on the characteristic exponents (Theorem 2.2).

In the case of single Fuchsian partial differential equations, the first author([2]) have constructed a good solution map. These single equations can be reduced to our Fuchsian systems as follows.

Remark 1.3. Let \( P' \) be a single Fuchsian partial differential operator with weight 0 ([1], [6], [2], etc.); that is, \( P' = (tD_t)^m + \sum_{j=1}^m P_j(t,x;D_x)(tD_t)^{m-j} \), \( \text{ord}_{D_x} P_j \leq j \), and \( P_j(0,x;D_x) = a_j(x) \) is a function of \( x \). Then, by \( u_j = (tD_t)^{j-1}u \) \( (1 \leq j \leq m) \),
the equation $Pu = f$ is reduced to

$$tD_t I_m - \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-P'_m & -P'_{m-1} & -P'_{m-2} & \cdots & -P'_1 \end{pmatrix} \rightarrow \vec{u} = \begin{pmatrix} 0 \\
0 \\
\vdots \\
0 \\
f \end{pmatrix}.$$ 

Since this system satisfies (A-1) with $n_j = j$ and (A-2), it is a Fuchsian system. Further, this system has the same indicial polynomial $C(x; \lambda)$ as $P'$, where the indicial polynomial of $P'$ is defined by

$$C[P'](x; \lambda) := \lambda^m + \sum_{j=1}^{m} a_j(x) \lambda^{m-j} = [t^{-\lambda}P'(t^\lambda)]|_{t=0}.$$ 

2 Construction of the solution map

Let $\mu_l (l = 1, \ldots, d)$ be all the distinct roots of $C(0; \lambda) = 0$, and let $r_l$ be the multiplicity of $\mu_l$. There exists $Q(x) \in GL_m(\mathcal{O}_0)$ such that

- $Q(x)^{-1}A_0(x)Q(x) = A_1(x) \otimes \cdots \otimes A_d(x) := \begin{pmatrix} A_1(x) & O & \cdots & O \\
O & A_2(x) & O & \vdots \\
\vdots & O & \ddots & \vdots \\
O & \cdots & O & A_d(x) \end{pmatrix},$

- $A_l \in M_{r_l}(\mathcal{O}_0)$ ($l = 1, \ldots, d$),

- $\det(\lambda I_{r_l} - A_l(0)) = (\lambda - \mu_l)^{r_l}$ ($l = 1, \ldots, d$).

Corresponding to the blocks of $Q(x)^{-1}A_0(x)Q(x)$, we denote the $l$-th block of $\vec{u}$ by $\vec{u}^{(l)} \in C^{r_l}$, that is, $\vec{u} = \begin{pmatrix} \vec{u}^{(1)} \\
\vdots \\
\vec{u}^{(d)} \end{pmatrix}$. Conversely, for an $r_l$-vector $\vec{v} \in C^{r_l}$, we
denote by $\vec{\gamma}^{[(i)]} \in \mathbb{C}^m$ the $m$-vector

$$
\vec{\gamma}^{[(i)]} = \begin{pmatrix}
0 \\
\vdots \\
\vec{\gamma} \\
\vdots \\
0
\end{pmatrix}
$$

(l th block)

with the entries $\vec{\gamma}$ in the $l$-th block and the entries 0 in the other blocks.

Set

$$
\Lambda_P := \{ \mu_l - j \in \mathbb{C} : 1 \leq l \leq d, j \in \mathbb{N} \} .
$$

(2.1)

Take $\epsilon \geq 0$ as $\text{Re} \mu_l - \epsilon \notin \mathbb{Z}$ for all $l$. For each $l$, take $L_l \in \mathbb{Z}$ as $L_l + \epsilon < \text{Re} \mu_l < L_l + \epsilon + 1$.

**Lemma 2.1.** (1) For each $l$, there exists a domain $D_l$ in $\mathbb{C}$ enclosed by a simple closed curve $\Gamma_l$ such that

(a) $\mu_l \in D_l \ (1 \leq l \leq d)$,

(b) $\overline{D_l} \cap \overline{D_{l'}} = \emptyset \ (l \neq l')$, where $\overline{D}$ denotes the closure of $D$.

(c) $\overline{D_l} \cap \Lambda_P = \{ \mu_l \}$ for every $l$.

(d) $\overline{D_l} \subset \{ \lambda \in \mathbb{C} : L_l + \epsilon < \text{Re} \lambda < L_l + \epsilon + 1 \}$ for every $l$.

(2) There exists $R_0 > 0$ such that

(e) $C(x; \lambda + j) \neq 0$ for every $x \in B_{R_0}$, every $\lambda \in \bigcup_{l=1}^{d} \Gamma_l$, and every $j \in \mathbb{N}$.

The main result is

**Theorem 2.2.** For every $l$ and every $\varphi_l \in (\mathcal{O}_0)^n$, there exists a unique $\vec{V} = \vec{V}[l, \varphi_l](t, x; \lambda) \in \mathcal{O}((0, 0)) \times (\bigcup_{l=1}^{d} \Gamma_l)^m$ such that

$$
P(t^l \vec{V}) = t^l Q(x) \varphi_{l}^{[(i)]}(x)
$$

(2.2)
in a neighborhood of \( \{(0,0)\} \times (\bigcup_{l=1}^{d} \Gamma_{l}) \).

Set \( \vec{u}_{l}[\varphi_{l}](t, x) := \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_{l}} t^{\lambda} V[l, \varphi_{l}](t, x; \lambda) d\lambda \). Then, the map

\[
(\mathcal{O}_{0})^{m} \ni \begin{pmatrix} \varphi_{1} \\ \vdots \\ \varphi_{d} \end{pmatrix} \mapsto \sum_{l=1}^{d} \vec{u}_{l}[\varphi_{l}] \in \text{Ker}_{} P \quad (2.3)
\]

is a linear isomorphism.

3 Expansion of the solutions

Expand the operator \( A \) and the vector \( \vec{V} \) as follows.

\[
A(t, x; D_{x}) = A_{0}(x) + \sum_{l=1}^{\infty} t^{l} B_{l}(x; D_{x}) ,
\]

\[
\vec{V}[l, \varphi_{l}](t, x; \lambda) = \sum_{j=0}^{\infty} t^{j} \vec{V}_{j}(x; \lambda) .
\]

Then, the equation (2.2) for \( \vec{V} \) is equivalent to

\[
(\lambda I_{m} - A_{0}(x)) \vec{V}_{0}(x; \lambda) = Q(x) \varphi_{l}(x) ,
\]

\[
(\lambda + j) I_{m} - A_{0}(x)) \vec{V}_{j}(x; \lambda) = \sum_{l=1}^{j} B_{l}(x; D_{x}) \vec{V}_{j-1}(x; \lambda) \quad (j \geq 1) .
\]

From these equations, we can determine \( \vec{V}_{j} \) by Lemma 2.1 (e), and we get an expansion of \( \vec{u}_{l}[\varphi_{l}] \) as follows.

\[
\vec{u}_{l}[\varphi_{l}](t, x) = \sum_{j=0}^{\infty} t^{j} \vec{u}_{l,j}(t, x) ,
\]

\[
\vec{u}_{l,j}(t, x) := \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_{l}} t^{\lambda} \vec{V}_{j}(x; \lambda) d\lambda .
\]

Especially, the leading term of \( \vec{u}_{l}[\varphi_{l}] \) is

\[
\vec{u}_{l,0}(t, x) = t^{A_{0}(x)} Q(x) \varphi_{l}(x) = Q(x) \{ t^{A_{0}(x)} \varphi_{l}(x) \}^{\#} (3.3)
\]
4 Sketch of the proof of the existence of $\vec{V}$

We change the letter $\lambda$ to $\zeta$. Then, the system $P(t^\zeta \vec{V}) = t^\zeta Q(x)\vec{\varphi}_l^\#(t)(x)$ is equivalent to another system

$$\vec{P}\vec{V} := ((tD_t + \zeta)I_m - A(t,x;D_x)) \vec{V} = Q(x)\vec{\varphi}_l^\#(t)(x).$$

This is also a Fuchsian system in $(t, x, \zeta)$. Note that we consider $(x, \zeta)$ as the space variables. Further, the indicial polynomial of $\vec{P}$ is

$$C[\vec{P}](x, \zeta; \lambda) = C[P](x; \lambda + \zeta).$$

Since $C[\vec{P}](0, \zeta; j) \neq 0$ $(\zeta \in \Gamma_l, \ j \in \mathbb{N})$ by Lemma 2.1 (e), we can use Theorem 1.1 to this new system. Thus, there exists a unique $\vec{V} = \vec{V}[t, \vec{\varphi}](t, x; \zeta) \in \mathcal{O}((0,0) \times \Gamma_l)^m$ such that $\vec{P}\vec{V} = Q(x)\vec{\varphi}_l^\#(t)(x)$.

5 Function spaces to estimate the order

**Definition 5.1.** ([2, Definition 5.1]) For $a \in \mathbb{R}$, we set

$$W^{(a)} := \bigcup_{R > 0, \ T > 0} \left\{ \phi \in \mathcal{O}(S_{\infty,T} \times B_R) : \sup_{|x| < R} |\phi(t, x)| \to 0 \ (\text{as } t \to 0 \text{ in } S_{\theta,T}) \text{ for every } \theta > 0 \right\}$$

**Lemma 5.2.** ([2, Lemma 5.2])

1. $a' < a \implies W^{(a)} \subset W^{(a')}.
2. $t \times W^{(a)} \subset W^{(a+1)}, \quad \partial_t W^{(a)} \subset W^{(a-1)}.$
3. If $B(t, x; D_x)$ is a partial differential operator in $x$ with $\mathcal{O}(0,0)$ coefficients, then $B(t, x; D_x)(W^{(a)}) \subset W^{(a)}$.

6 Keys to the proof of the theorem

The first key is the temperedness of the solutions in $(\vec{\mathcal{O}})^m$. 


Proposition 6.1. There exists $a \in \mathbb{R}$ such that if $\vec{u} \in (\mathcal{O})^m$ and $P \vec{u} = \vec{0}$, then $\vec{u} \in (W'(a))^m$.

The second key is an estimate of the remainder terms of our solutions $\vec{u} \vec{[\varphi]}(t, x)$.

Lemma 6.2. For $\vec{\varphi} \in (\mathcal{O}_0)^{r}$, we have

$$\vec{u} \vec{[\varphi]}(t, x) = Q(x)\{ t^{A_l(x)} \varphi_l(x) \}^{\#(l)} + t \cdot \vec{r} \vec{[\varphi]}(t, x),$$

and $\vec{r} \vec{[\varphi]} \in (W^{(L_l+\epsilon)})^m$. Note that $t^{A_l(x)} \varphi_l(x) \in (W^{(L_l+\epsilon)})^r$ and $\vec{u} \vec{[\varphi]} \in (W^{(L_l+\epsilon)})^m$.

The third key is the two facts on the Euler system $(tD_t - A_0(x)) \vec{u} = f(t, x)$ with holomorphic parameters $x$.

Lemma 6.3. If $\vec{u} \in (\mathcal{O})^m$ and $(tD_t I_m - A_0(x)) \vec{u} = \vec{0}$, then there exists $\vec{\varphi} \in (\mathcal{O}_0)^{r}$ ($1 \leq l \leq d$) such that

$$\vec{u} = \sum_{l=1}^{d} Q(x) \{ t^{A_l(x)} \varphi_l(x) \}^{\#(l)} = Q(x) \begin{pmatrix} t^{A_1(x)} \varphi_1(x) \\ \vdots \\ t^{A_d(x)} \varphi_d(x) \end{pmatrix}.$$ 

Further, if $L \in \mathbb{Z}$ and $\vec{u} \in W^{(L+\epsilon)}(\theta, R)^m$, then $\varphi_l = 0$ for all $l$ such that $L_l < L$.

Proposition 6.4. For any $L \in \mathbb{Z}$ and any $\vec{g} \in W^{(L+\epsilon)}$, there exists $\vec{u} \in W^{(L+\epsilon)}$ such that $(tD_t I_m - A_0(x)) \vec{u} = \vec{g}(t, x)$.

If a root $\lambda(x)$ of $C(x; \lambda) = 0$ touches the line $\text{Re} \lambda = L + \epsilon$ in $\lambda$-plane, then this proposition does not hold, as the simplest example $tD_t v = \frac{1}{\log t}$ shows ($m = 1$, $L + \epsilon = 0$, no parameter $x$). This proposition is the reason why we took $\epsilon$.

7 Proof of the injectivity of the solution map

Assume that $\vec{\varphi} \in (\mathcal{O}_0)^{r}$ ($1 \leq l \leq d$), $\sum_{l=1}^{d} \vec{u} \vec{[\varphi]} = \vec{0}$, and that there exists $l$ such that $\varphi_l \neq \vec{0}$. Take $l_0$ as $L_{l_0} = \min \{ L_l : \varphi_l \neq \vec{0} \}$.
For each $l$ with $\varphi_l \neq 0$, consider $(\varphi_l)^{k(l_0)}$: the $l_0$-th block of $\varphi_l := Q^{-1} \bar{u}_l^r [\varphi_l]$. Then, we have by Lemma 6.2

$$(\bar{u}_l^{r})^{k(l_0)} = t^{A_{l_0}(x)} \varphi_{l_0}^r(x) + (W^{(L_0+1+\epsilon)})^{r_{l_0}}.$$ 

On the other hand, if $l \neq l_0$, then $L_l \geq L_{l_0}$ and hence

$$(\bar{u}_l^{r})^{k(l_0)} \in (W^{(L+1+\epsilon)})^{r_{l_0}} \subset (W^{(L_0+1+\epsilon)})^{r_{l_0}}.$$ 

Thus,

$$\bar{v} = \sum_{i=1}^{d}(Q^{-1} \bar{u}_l^r [\varphi_l])^{k(l_0)} = t^{A_{l_0}(x)} \varphi_{l_0}^r(x) + (W^{(L_0+1+\epsilon)})^{r_{l_0}}.$$ 

Namely, $t^{A_{l_0}(x)} \varphi_{l_0}^r(x) \in (W^{(L_0+1+\epsilon)})^{r_{l_0}}$. It is easy to show that this implies $\varphi_{l_0}^r = \bar{v}$, which contradicts the definition of $l_0$.

8 Proof of the surjectivity of the solution map

Let $\bar{u} \in (\bar{O})^m$ and $P \bar{u} = \bar{v}$. Decompose $A(t, x; D_x) = A_0(x) + tB(t, x; D_x)$.

(I) By Proposition 6.1, there exists $L \in \mathbb{Z}$ such that $\bar{u} \in (W^{(L)})^m$.

By Lemma 5.2, we have $tB(\bar{v}) \in (W^{(L+1)})^m$.

(II) By Proposition 6.4, there exists $\bar{v} \in (W^{(L+1)})^m$ such that $(tD_t I_m - A_0(x)) \bar{v} = tB(\bar{v}) = (tD_t I_m - A_0(x)) \bar{u}$.

(III) Since $(tD_t I_m - A_0(x))(\bar{u} - \bar{v}) = \bar{v}$ and $\bar{u} - \bar{v} \in (W^{(L)})^m$, there exists $\varphi_l[1] \in (O_0)^r$ such that $\varphi_l[1] = \bar{v}$ if $L_l < L$, and that

$$\bar{u} - \bar{v} = \sum_{l=1}^{d} Q(x)\{t^{A_l(x)} \varphi_l[1](x)\}^{\#(l)}.$$ 

by Lemma 6.3.

(IV) Set

$$\bar{u}[1] := \bar{u} - \sum_{l=1}^{d} \bar{u}_l [\varphi_l[1]] \in (W^{(L+1)})^m.$$ 

Then, we have $P(\bar{u}[1]) = \bar{v}$, $u[1] \in (W^{(L+1)})^m$. 

Now, we can return to the step (I) taking $\vec{u}[1]$ instead of $\vec{u}$, with order $L + 1 + \epsilon$ instead of $L + \epsilon$.

Repeating such arguments, we have $\vec{\varphi[j]} \in (O_0)^{r_i}$ ($j = 2, 3, \ldots$) such that $\vec{\varphi[j]} = \vec{0}$ if $L_i < L + j - 1$, and that

$$\vec{u}[j] := \vec{u}[j - 1] - \sum_{l=1}^{d} \vec{u}[\vec{\varphi}[j]] \left( = \vec{u} - \sum_{k=1}^{j} \sum_{l=1}^{d} \vec{u}[\vec{\varphi}[k]] \right) \in (W^{(L+j+\epsilon)})^m ,$$

and $P(\vec{u}[j]) = \vec{0}$.

By Theorem 1.2, we have $\vec{u}[M] = \vec{0}$ for sufficiently large $M$. Thus, we get $\vec{\varphi} := \sum_{k=1}^{M} \vec{\varphi}[k] \in (O_0)^{r_i}$.

References


