Gevrey Hypoellipticity for Extended Grushin Class and FBI-Transformation

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§1. Introduction

In this monograph we shall mention only the main results obtained recently on Gevrey hypoellipticity for extended class of Grushin operators. Precise proof of them will be given in a forthcoming paper. We shall determine the non-isotropic Gevrey exponents for Grushin operators by using also the method of FBI-transformation given in [1] somewhat modifying it as well as by using the method of pseudodifferential operators. Thus, we get an amelioration of the results obtained in the previous papers [3] and [4].

§2. Gevrey functions and FBI-transformation

We denote $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $D_j = -i \partial_{x_j}, j = 1, \ldots, n$, as usual. We remember the definition of Gevrey functions.

Definition 2.1. Let $\Omega$ be an open set in $\mathbb{R}^n$ and $\phi \in C^\infty(\Omega)$. Then we say that $\phi \in G^{(s)}(\Omega), s = (s_1, \ldots, s_n), s_j > 0$, if for any compact subset $K$ of $\Omega$ there are positive constants $C_0$ and $C_1$ such that

$$\sup_{z \in K} |D^\alpha \phi(z)| \leq C_0 C_1^{\alpha} \alpha!^{(s, \alpha)}, \quad \alpha \in \mathbb{Z}_+^n,$$

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where \( \langle s, \alpha \rangle = s_1\alpha_1 + \cdots + s_n\alpha_n \).

**Proposition 2.1.** Let \( \varphi \in C_0^\infty(\Omega) \). If for any compact subset \( K \) of \( \Omega \) there are positive constants \( C_0 \) and \( C_1 \) such that

\[
\sup_{x \in K} |D_x^j \varphi(x)| \leq C_0 C_1^k k!^j, \quad j = 1, 2, \ldots, n, \; k \in \mathbb{Z}_+.
\]

Then we have \( \varphi \in G^{\{s_1, s_2, \ldots, s_n\}}(\Omega) \).

The proof can be obtained by using FBI-transformation whose definition will be given in (2.2).

**Proposition 2.2.** Let \( a \) be a positive parameter. For any \( \varepsilon, 0 < \varepsilon < 1 \), there exists a positive constant \( C_\varepsilon \) such that

\[
|\partial_x^k e^{-ax^2}| \leq C_\varepsilon^{k+1} a^{\frac{k}{2}} k!^{\frac{1}{2}} e^{-\varepsilon ax^2}, \quad -\infty < x < \infty, \quad k = 1, 2, \ldots.
\]

Let \( u(x) \in C_0^\infty(\mathbb{R}^n) \). Then we have the Fourier inversion formula

\[
u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} u(y) e^{i\langle x-y, \xi \rangle} dyd\xi.
\]

Now shift the contour of integration from \( \mathbb{R}^n \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{C}^n \) to the contour

\[
\Gamma(y, \xi) = (y, \xi + i\langle \xi \rangle(x-y)), \quad (y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.
\]

Then we have the formula

\[
u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} u(y) e^{i\langle x-y, \xi \rangle - \langle \xi \rangle(x-y)^2} \alpha(x-y, \xi) dyd\xi,
\]

where

\[
\langle \xi \rangle = (1 + \xi^2)^\frac{1}{2} = (1 + \xi_1^2 + \cdots + \xi_n^2)^\frac{1}{2}
\]

and

\[
\alpha(x-y, \xi) = \prod_{j=1}^n (1 + i(x_j - y_j)\xi_j(1 + \xi_j^2)^{-\frac{1}{2}}).
\]

From this formula we define the FBI-transformation of \( u(x) \) by

\[
\mathcal{F}u(x, \xi) = \int_{\mathbb{R}^n} u(y) e^{i\langle x-y, \xi \rangle - \langle \xi \rangle(x-y)^2} \alpha(x-y, \xi) dy.
\]
The result of M. Christ, [1 ] is modified slightly in the following theorem relating to the characterization of the class $G^{(s)}$.

**Theorem 2.3.** (cf. [1 ], Theorem 2.3.) Let $s = (s_1, s_2, \ldots, s_n), s_j \geq 1, j = 1, 2, \ldots, n$, and $u(x) \in C_0^\infty(\mathbb{R}^n)$. Then the following four assertions are mutually equivalent:

(a) $u(x) \in G^{(s)}$ in a neighborhood of $x_0 \in \mathbb{R}^n$.

(b) There exist $C, \delta \in \mathbb{R}_+$ and a neighborhood $V$ of $x_0$ such that

$$|\mathcal{F}u(x, \xi)| \leq Ce^{-\delta \sum_{j=1}^n |\xi_j|^\frac{1}{j}}, \quad (x, \xi) \in V \times \mathbb{R}^n.$$

(c) There exist an open neighborhood $U = U(x_0) \subset \mathbb{C}^n$ of $x_0$ and $C, \delta \in \mathbb{R}_+$ such that, for each $\lambda \in \mathbb{R}_+^n, |\lambda| \geq 1$, there exists a decomposition

$$u = g_\lambda + h_\lambda \quad \text{in} \quad U \cap \mathbb{R}^n$$

such that $g_\lambda$ is holomorphic in $U$,

$$|g_\lambda(z)| \leq Ce^{C|Im(z)|}, \quad z \in U$$

and

$$|h_\lambda(x)| \leq Ce^{-\delta \sum_{j=1}^n \lambda_j^\frac{1}{j}}, \quad x \in U \cap \mathbb{R}^n.$$

(d) There exist an open neighborhood $U = U(x_0) \subset \mathbb{C}^n$ of $x_0$ and $C, \delta \in \mathbb{R}_+$ such that for each $\lambda \in \mathbb{R}_+^n, |\lambda| \geq 1$, there exists a decomposition

$$u = g_\lambda + h_\lambda \quad \text{in} \quad U \cap \mathbb{R}^n$$

such that $g_\lambda$ is holomorphic in \{ $z \in U; |\text{Im}(z_j)| \leq \langle \lambda \rangle s |\lambda|^{-1}$ \} $\equiv U_\lambda$,

$$|g_\lambda(z)| \leq C, \quad z \in U_\lambda,$$

and

$$|h_\lambda(x)| \leq Ce^{-\delta \sum_{j=1}^n \lambda_j^\frac{1}{j}}, \quad x \in U \cap \mathbb{R}^n.$$

**Remark 2.1.** By using appropriate cut-off functions for $u$, the standard method of calculation goes well to prove (a) $\iff$ (b) with the aid of Proposition 2.2. Proof that (b) $\implies$ (c) $\implies$ (d) $\implies$ (b) can be obtained by the
same method as in [1]. However, it might be needed to add a sketch of the proof of (a) \(\iff\) (b). it will be sufficient to consider one dimensional case.

(i) The case where \(s = 1\). Let \(u \in C_0^\infty(\mathbb{R})\) and let \(u\) be real analytic in a neighborhood of \(x_0\), say in \(\omega_\delta = \{x; |x - x_0| < \delta\}\) for some \(\delta > 0\). Then for

\[
\mathcal{F}u(x, \xi) = \int u(y)e^{i(x-y,\xi) - \langle \xi, (x-y)^2 \rangle} \alpha(x - y, \xi)dy
\]

we make a deformation of the integral contour in \(\omega_\delta\) and we have

\[
|\mathcal{F}u(x, \xi)| \leq Ce^{-c(\xi)}, \quad (x, \xi) \in V \times \mathbb{R},
\]

where \(V\) is a small neighborhood of \(x_0\) and \(C\) and \(c\) are positive constants independent of \(\xi\).

(ii) The case where \(s > 1\). We may suppose that \(u \in C_0^\infty(\omega_\delta) \cap G^{(s)}\), so that we have \(u(y)\alpha(x - y, \xi) \in C_0^\infty(\omega_\delta) \cap G^{(s)}\). By Proposition 2.2, taking \(C_1, C, C'\) sufficiently large and \(c'\) sufficiently small we have

\[
|\xi^{-N} \int e^{i(x-y)\xi} D_y^N (u(y)\alpha e^{-\langle \xi, (x-y)^2 \rangle})dy| \\
\leq |\xi|^{-N} C_1^{N+1} \sum_{j=0}^{N} \left(\begin{array}{l}N \\ j \end{array}\right) j! \|\xi\|^{\frac{1}{2}} (N-j)! \\
\leq |\xi|^{-N} C_1^{N+1} N! \|\xi\|^{\frac{1}{2}} \\
\leq \frac{CN^s}{|\xi|} e^{\left(\frac{s}{2} - \frac{1}{2}\right) |\zeta|} \tau^{\llcorner} \\
\leq C'e^{-c'|\xi|!}.
\]

Now take \(N\) such that \(|N - (\epsilon|\xi|)^{\frac{1}{2}}| < 1\) with \(\epsilon\) sufficiently small. Then the above quantity is estimated by

\[
C(C\epsilon)^N e^{\left(\frac{s-1}{2}\right) |\xi|^{\frac{1}{2}-1}} \leq C'e^{-c'|\xi|^{\frac{1}{2}}}, \quad x \in \omega_{\frac{s}{2}}, \xi \in \mathbb{R}. \quad \square
\]

Remark 2.2. In Theorem 2.3, we can replace \(\mathcal{F}u(x, \xi)\) by \(\mathcal{F}_s u(x, \xi)\) as follows:

\[
(2.3) \quad \mathcal{F}_s u(x, \xi) = \int u(y)e^{i(x-y,\xi) - \langle \xi, (x-y)^2 \rangle} \alpha_*(x - y, \xi)dy
\]
\[ <\xi > = \sum_{j=1}^{n}(1 + \xi_j^2)^{1/2}, \]

\[ \alpha_s(x - y, \xi) = \prod_{j=1}^{n}(1 + \frac{i}{s_j}(x_j - y_j)\xi_j(1 + \xi_j^2)^{1/2} - 1). \]

The transform \( \mathcal{F}u(z, \xi) \) extends, for each \( \xi \), to an entire holomorphic function of \( z \in \mathbb{C}^n \). We can see that the same reasoning as in (c) and (d) gives

\[ |\mathcal{F}u(z, \xi)| \leq C e^{-\delta \sum_{j=1}^{n} |\xi|^2_j e^{|z||Im(z)|}} \]  

for \( z \) in a sufficiently small neighborhood \( V \subset \mathbb{C}^n \) of \( x_0 \).

§3. Main results

We shall give the definition of the extended class of Grushin operators. We write \( (x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_n) \in \mathbb{R}^{k+n} \). Let \( m \) be an even positive integer and let \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k), q = (q_1, q_2, \ldots, q_k) \) whose elements are rational numbers such that

\[ \sigma_1, \ldots, \sigma_p > 0, \sigma_{p+1} = \cdots = \sigma_k = 0, (0 \leq p \leq k) \]

\[ q_1 \geq q_2 \geq \cdots \geq q_p \geq 0, q_{p+1} \geq \cdots \geq 0, \quad q_1 > 0. \]

Furthermore, we assume

\[ mq_j \in \mathbb{Z}, j = 1, \ldots, k; \quad \frac{mq_j}{\sigma_j} \in \mathbb{Z}, j = 1, 2, \ldots, p. \]

We pose the following major hypothesis:

**Hypothesis (G)** We suppose \( 1 + q_p > \sigma_0 = \text{max}(\sigma_1, \ldots, \sigma_p) \).

**Remark 3.1.** Grushin's original major hypothesis given in [2] was \( 1 + q_k > \sigma_0 = \text{max}(\sigma_1, \ldots, \sigma_k) \). We shall see that we can weaken this condition as above. (See §5 and §6.) The assumption on \( q_1, \ldots, q_k \) given in [4] is also slightly weakened as above. When \( p = 0 \), we consider \( q_0 = 0, \sigma_0 = 0 \).
We divide $x$ into two parts such as $x = (x', x'')$ when $1 \leq p < k$, where $x' = (x_1, \ldots, x_p)$ and $x'' = (x_{p+1}, \ldots, x_k)$. We consider $x = x'$ when $p = k$ and $x = x''$ when $\sigma = (0, \ldots, 0)$. Now we shall consider a differential operator with polynomial coefficients under the hypothesis (M):

$$(3.1) \quad P(x', y, D_x, D_y) = \sum_{|\alpha + \beta| \leq m} a_{\alpha \beta \nu \gamma} x'^\nu y^\gamma D_x^\alpha D_y^\beta, \quad a_{\alpha \beta \nu \gamma} \in \mathbb{C},$$

where $a_{\alpha \beta \nu \gamma}$ can be non-zero only when $|\gamma| = \langle q, \alpha \rangle + |\alpha + \beta| - m - \langle \sigma, \nu \rangle$ is a non-negative integer and we write such as $|\alpha + \beta| = |\alpha| + |\beta|$. We may also consider $\nu = (\nu_1, \ldots, \nu_p, 0, \ldots, 0)$.

We can see the symbol $P(x', y, \xi, \eta)$ satisfies the following condition.

**Condition 1.** (quasi-homogeneity) We have

$$P(\lambda^{-\sigma}x', \lambda^{-1}y, \lambda^{1+q} \xi, \lambda \eta) = \lambda^m P(x', y, \xi, \eta), \quad \lambda > 0, x, \xi \in \mathbb{R}^k, y, \eta \in \mathbb{R}^n,$$

where $\lambda^{-\sigma}x' = (\lambda^{-\sigma_1}x_1, \ldots, \lambda^{-\sigma_p}x_p)$ and $\lambda^{1+q} \xi = (\lambda^{1+q_1} \xi_1, \ldots, \lambda^{1+q_k} \xi_k)$.

We add the two more conditions on $P$.

**Condition 2.** (ellipticity) The operator $P$ is elliptic for $|x'| + |y| = 1$.

**Condition 3.** (non-zero eigenvalue) For all $\omega$, $|\omega| = 1$, the equation

$$P(x', y, \omega, D_y) v(y) = 0 \quad \text{in} \quad \mathbb{R}^n_y$$

has no non-trivial solution in $S(\mathbb{R}^n_y)$.

We set the Gevrey indices as follows.

$$\theta_j = \max \left( \frac{1+q_j}{1+q_k}, \frac{1+q_p}{1+q_p - \sigma_0} \right) \quad \text{for} \quad j = 1, \ldots, p,$$

$$\theta_j = \frac{1+q_j}{1+q_k} \quad \text{for} \quad j = p+1, \ldots, k, \quad d = \max_{1 \leq j \leq k} \left\{ \frac{\theta_j + q_j}{1+q_j} \right\},$$

We also denote

$$d = \max_{1 \leq j \leq k} \left\{ \frac{\theta_j + q_j}{1+q_j} \right\} \cdot I_n = (d, \ldots, d).$$
Theorem 3.2. (cf. [4]) Let $\Omega$ be an open neighborhood of $(0,0)$, and consider the equation

\[(3.2) \quad P(x', yD_x, D_y)u(x,y) = f(x,y) \quad \text{in} \quad \Omega,\]

where $u(x,y) \in \mathcal{D}'(\Omega)$ and $f(x,y) \in G_{x,y}^{(\theta,d)}(\Omega)$. Then we have $G_{x,y}^{(\theta,d)}(\Omega)$.

Remark 3.3. In the above theorem we can see that

(i) $p = 0, \theta_1 = 1 \iff (\theta, d) = (1,\ldots,1),$

(ii) $p = 0, \theta_1 > 1 \Rightarrow 1 < d = \frac{\theta_1 + q_1}{1+q_1} < \theta_1.$

Examples (a) For the operator $P_1 = D_y^2 + y^{2k}D_x^2, (k = 1, 2,\ldots)$, $q_1 = k, \sigma_1 = 0$ and $\theta_1 = 1, d = 1.$

(b) For the operator $P_2 = D_y^2 + (x^{2l} + y^{2k})D_x^2, (k, l = 1, 2,\ldots)$, $q_1 = k, \sigma_1 = k/l$ and $\theta_1 = \frac{l(1+k)}{l(1+k)-k}, d = \frac{\theta_1 + k}{1+k}.$

(c) For the operator $P_3 = D_y^2 + (x^{2l} + y^{2k})(D_x^2 + D_z^2), (k, l = 1,$ have

$q_1 = q_2 = k, \sigma_1 = k/l, \sigma_2 = 0, x' = x, x'' = z; \theta_1 = \frac{l(1+k)}{l(1+k)-k}, \theta_2 = \frac{l(1+k)}{l(1+k)-k}.$

(d) For the operator $P_4 = D_y^2 + (x^{2l} + y^{2k})D_x^2 + D_z^2, (k, l = 1,$ have

$q_1 = k, q_2 = 0, \sigma_1 = k/l, \sigma_2 = 0; \theta_1 = 1+k, \theta_2 = 1, d = \frac{\theta_1 + k}{1+k}.$

We remark that this operator $P_4$ does not satisfy the original hypo of Grushin.
(e) An example with $1 < d_1 < d_2$ is given by $P_4 = D_y^2 + (x^4 + y^4)D_x^2 + (x^2 + y^2)D_z^2$, where we have

$$q_1 = 2, q_2 = 1, \sigma_1 = \sigma_2 = 1; \theta_1 = \theta_2 = 2, d_1 = \frac{4}{3} < d_2 = \frac{3}{2}, d = \frac{3}{2}.$$ 

**Remark 3.4.** We omit the proof of $C^\infty$-hypoellipticity of the operator $P$ given in Theorem 3.2 since it is much simpler than that of Gevrey hypoellipticity. Then by using a cut-off function for $u$, we may suppose that $u, f \in C_0^\infty(\Omega)$ and $f \in G^{(\theta,d)}_{x,y}$ in a neighborhood of $(0,0) \in \mathbb{R}_{x,y}^{k+n}$. By Theorem 2.3,(b), our main purpose becomes to prove that there exist a small neighborhood $V$ of $(0,0)$ and positive constants $C$ and $\delta$ such that

$$|F(u(\tilde{x}, \tilde{y}, \xi, \eta))| \leq e^{-\delta(\sum_{1} \epsilon_{j}^{\theta_{j}} + |\eta|^{\frac{1}{\theta}})}, \quad (\tilde{x}, \tilde{y}, \xi, \eta) \in V \times \mathbb{R}_{\xi,\eta}^{k+n},$$

where

$$F(\tilde{x}, \tilde{y}, \xi, \eta) = \int u(x, y)e^{i((\tilde{x} - x, \xi) + (\tilde{y} - y, \eta) - \langle \mu \rangle((\tilde{x} - x)^2 + (\tilde{y} - y)^2))}\alpha(\tilde{x} - x, \xi) \cdot \alpha(\tilde{y} - y, \eta)dxdy, \quad \mu = (\xi, \eta),$$

$$\alpha(\tilde{x} - x, \xi) = \prod_{j=1}^{k}(1 + i)(\tilde{x}_j - x_j)\xi_j(1 + \xi^2)^{-\frac{1}{2}},$$

$$\alpha(\tilde{y} - y, \eta) = \prod_{j=1}^{n}(1 + i)(\tilde{y}_j - y_j)\eta_j(1 + \eta^2)^{-\frac{1}{2}}.$$ 

We can prove the inequality (3.3) in three steps. We prove first the inequality (3.3) in the elliptic region:

$$R_E = \{(\xi, \eta); (\xi, \eta) \in \mathbb{R}_{\xi,\eta}^{k+n}, |\xi| \leq |\eta|\}.$$ 

Next, we prove the inequality (3.3) in the subelliptic region:

$$R_S = \{(\xi, \eta); (\xi, \eta) \in \mathbb{R}_{\xi,\eta}^{k+n}, \left(\frac{1}{c} \sum_{j=1}^{k} |\xi_j|^{\frac{1}{\theta_j}}\right)^d \leq |\eta| \leq |\xi|, \quad c > 0.$$ 

Finally, we obtain the inequality of the kind (3.3) in the $L^2$-sense in the degenerate region:

$$R_D = \{(\xi, \eta); (\xi, \eta) \in \mathbb{R}_{\xi,\eta}^{k+n}, c|\eta|^{\frac{1}{\theta}} \leq \sum_{j=1}^{k} |\xi_j|^{\frac{1}{\theta_j}}, \quad c > 0.$$
These steps will be completed by a precision of the method given in [1] and [4].

REFERENCES


