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Solvability of non-linear totally characteristic partial differential equations in the complex domain - when resonances occur -

Hidetoshi TAHARA （田原 秀敏）
Department of Mathematics, Sophia University （上智大・理工）

Abstract

Let us consider the following non-linear singular partial differential equation $(t\partial_t)^m u = F(t,x,\{(t\partial_t)^j\partial_x^\alpha u\}_{j+\alpha\le m,j<m})$ in the complex domain. When the equation is of totally characteristic type, the author has proved with H. Chen in [2] the existence of the unique holomorphic solution provided that the equation satisfies the Poincaré condition and that no resonances occur. In this paper, he will solve the same equation in the case where some resonances occur.

§1. Introduction.

Notations: $(t,x)\in\mathbb{C}_t\times\mathbb{C}_x$, $\mathbb{N}=\{0,1,2,\ldots\}$, and $\mathbb{N}^*=\{1,2,\ldots\}$. Let $m\in\mathbb{N}^*$, set $N=\#\{(j,\alpha)\in\mathbb{N}\times\mathbb{N};\ j+\alpha\le m, j<m\}$ $(=m(m+3)/2)$, and write the complex variable $z=\{z_{j,\alpha}\}_{j+\alpha\le m,j<m}\in\mathbb{C}^N$.

In this paper we will consider the following non-linear partial differential equation:

$$\text{(E)}\qquad \left(t\frac{\partial}{\partial t}\right)^m u = F\left(t,x,\left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+\alpha\le m\\ j<m}}\right),$$

where $F(t,x,z)$ is a function in the variables (t,x,z) defined in a neighborhood Δ of the origin of $\mathbb{C}_t\times\mathbb{C}_x\times\mathbb{C}_z^N$, and $u=u(t,x)$ is the unknown function. Set $\Delta_0=\Delta\cap\{t=0,z=0\}$. We impose the following conditions on $F(t,x,z)$:

A$_1$) $F(t,x,z)$ is a holomorphic function on Δ;

A$_2$) $F(0,x,0)\equiv 0$ on Δ_0.

Set $I_m=\{(j,\alpha)\in\mathbb{N}\times\mathbb{N}\ ;\ j+\alpha\le m, j<m\}$ and $I_m(+)=\{(j,\alpha)\in I_m\ ;\ \alpha>0\}$. Then the situation is divided into the following three cases:

Case 1 : $\dfrac{\partial F}{\partial z_{j,\alpha}}(0,x,0)\equiv 0$ on Δ_0 for all $(j,\alpha)\in I_m(+)$;

Case 2 : $\dfrac{\partial F}{\partial z_{j,\alpha}}(0,0,0)\neq 0$ for some $(j,\alpha)\in I_m(+)$;

Case 3 : the other case.

In the case 1, equation (E) is called a *non-linear Fuchsian type* partial differential equation and it was studied quite well by Gérard-Tahara [3][4]. In the case 2, equation

(E) is called a ***spacially non-degenerate type*** partial differential equation and it gives us a kind of Grousat problem: Gérard-Tahara [5] discussed a particular class of the case 2 and proved the existence of holomorphic solutions and also singular solutions of (E). In the case 3, equation (E) is called a ***non-linear totally characteristic type*** partial differential equation. The main thema of this paper is to discuss the case 3 under the following condition:

$$\mathrm{A}_3)\quad \frac{\partial F}{\partial z_{j,\alpha}}(0,x,0)=O(x^{\alpha})\quad (\text{as } x\longrightarrow 0)\ \text{ for all } (j,\alpha)\in I_m(+).$$

§2. Review of the result of Chen-Tahara [2].

Under the condition A_3), Chen-Tahara [2] has proved the existence of the unique holomorphic solution provided that the equation satisfies both non-resonance condition and the Poincaré condition. We will recall this result now.

By the condition A_3) we have $(\partial F/\partial z_{j,\alpha})(0,x,0)=x^{\alpha}c_{j,\alpha}(x)$ for some holomorphic functions $c_{j,\alpha}(x)$. Set

$$L(\lambda,\rho)=\lambda^m-\sum_{\substack{j+\alpha\le m\\ j<m}} c_{j,\alpha}(0)\,\lambda^j\rho(\rho-1)\cdots(\rho-\alpha+1). \tag{2.1}$$

Then equation (E) is rewritten in the form

$$\begin{aligned} L\Big(t\frac{\partial}{\partial t},x\frac{\partial}{\partial x}\Big)u &= x\sum_{(j,\alpha)\in I_m} S(c_{j,\alpha})(x)\Big(t\frac{\partial}{\partial t}\Big)^j\Big(x\frac{\partial}{\partial x}\Big)\Big(x\frac{\partial}{\partial x}-1\Big)\cdots\Big(x\frac{\partial}{\partial x}-\alpha+1\Big)u \\ &\quad + a(x)\,t+R_2\Big(t,x,\Big\{\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^{\alpha}u\Big\}_{(j,\alpha)\in I_m}\Big), \end{aligned} \tag{2.2}$$

where $S(c_{j,\alpha})(x)=(c_{j,\alpha}(x)-c_{j,\alpha}(0))/x$, $a(x)$ is a holomorphic function on Δ_0, and $R_2(t,x,z)$ is a holomorphic function whose Taylor expansion in (t,z) consists of the terms with degree greater than or equal to 2 (with respect to (t,z)). Therefore, it is easy to see that if $L(k,l)\neq 0$ holds for any $(k,l)\in \mathbb{N}^*\times\mathbb{N}$ the equation (2.2) has a unique formal solution of the form

$$u(t,x)=\sum_{k\ge 1,l\ge 0} u_{k,l}\,t^k x^l. \tag{2.3}$$

Next, let us consider the convergence of this formal solution. Denote by $c_1,\dots,c_m$ the roots of the following equation in X:

$$X^m-\sum_{\substack{j+\alpha=m\\ j<m}} c_{j,\alpha}(0)\,X^j=0.$$

Then, if we factorize $L(\lambda, l)$ into the form

$$L(\lambda, l) = (\lambda - \xi_1(l)) \cdots (\lambda - \xi_m(l)) \quad \text{for } l \in \mathbb{N}, \tag{2.4}$$

by renumbering the subscript i of $\xi_i(l)$ suitably we have

$$\lim_{l \to \infty} \frac{\xi_i(l)}{l} = c_i \quad \text{for } i = 1, \dots, m.$$

Therefore, if $c_1, \dots, c_m \in \mathbb{C} \setminus [0, \infty)$ we can find a $\sigma > 0$ such that $|L(k,l)| \geq \sigma(k+l)^m$ holds for any $(k,l) \in \mathbb{N}^* \times \mathbb{N}$ with $k+l$ being sufficiently large. This condition leads us to the convergence of the formal solution.

Thus, set

(N)(non-resonance) $L(k,l) \neq 0$ holds for any $(k,l) \in \mathbb{N}^* \times \mathbb{N}$,
(P)(Poincaré condition) $c_i \in \mathbb{C} \setminus [0, \infty)$ for $i = 1, \dots, m$;

and we have:

Theorem 1 (Chen-Tahara [2]). *Assume A_1), A_2) and A_3). Then, if the conditions (P) and (N) are satisfied, equation (E) has a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x$ satisfying $u(0,x) \equiv 0$ near $x = 0$.*

The purpose of this paper is to solve the equation (E) in the case where the Poincaré condition (P) is satisfied but the non-resonance condition (N) is not satisfied.

§3. When resonances occur.

Let $L(\lambda, \rho)$ be the polynomial in (2.1), and let $\xi_i(l)$ $(i = 1, \dots, m)$ be as in (2.4). Set

$$\mathcal{M} = \{ (k,l) \in \mathbb{N}^* \times \mathbb{N};\ L(k,l) = 0 \},$$
$$\mathcal{M}_i = \{ (k,l) \in \mathbb{N}^* \times \mathbb{N};\ k - \xi_i(l) = 0 \} \quad (i = 1, \dots, m).$$

We have $\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_m$. Note that $\mathcal{M} = \varnothing$ is equivalent to the non-resonance condition (N). In the case $\mathcal{M} \neq \varnothing$ we note:

Lemma 1. *If the Poincaré condition (P) is satisfied, we have the following properties: (1) $\mathcal{M}$ is a finite set; (2) there is a $\sigma > 0$ such that $|k - \xi_i(l)| \geq \sigma\,(k+l)$ holds for any $(k,l) \in (\mathbb{N}^* \times \mathbb{N}) \setminus \mathcal{M}_i$ $(i = 1, \dots, m)$.*

For $(k,l) \in \mathcal{M}$ we set $\mu(k,l) = \#\{ i\,;\, \xi_i(l) = k \}$ and we say that $\mu(k,l)$ is *the multiplicity of resonance of $L(\lambda, \rho)$ at (k,l)*. We denote by μ the total number of the multiplicities of resonances of $L(\lambda, \rho)$, that is,

$$\mu = \sum_{(k,l) \in \mathcal{M}} \mu(k,l)\,. \tag{3.1}$$

The following is the main result of this paper.

Theorem 2 (when resonances occur). *Assume A_1), A_2), A_3) and (P). Then, equation (E) has a solution $u(t,x)$ of the form*

$$u(t,x) = W(t, t(\log t), t(\log t)^2, \dots, t(\log t)^\mu, x) \tag{3.2}$$

where μ is the number in (3.1) and $W(t_0, t_1, \dots, t_\mu, x)$ is a holomorphic function in a neighborhood of $(t_0, t_1, \dots, t_\mu, x) = (0, 0, \dots, 0, 0)$ satisfying $W(0, 0, \dots, 0, x) \equiv 0$ near $x = 0$.

Sketch of the proof. We set

$$t_0 = t, \;\; t_1 = t(\log t), \; \dots, \; t_\mu = t(\log t)^\mu$$

and set $u(t,x) = W(t_0, t_1, \dots, t_\mu, x)$. Then we have

$$t\frac{\partial u}{\partial t} = \tau W \quad \text{where } \tau = \sum_{i=0}^{\mu} t_i \frac{\partial}{\partial t_i} + \sum_{i=1}^{\mu} i\, t_{i-1} \frac{\partial}{\partial t_i}$$

and our equation (E) is written in the form

$$C\Big(x, \tau, x\frac{\partial}{\partial x}\Big) W = a(x)\, t + R_2\Big(t_0, x, \Big\{\tau^j \Big(\frac{\partial}{\partial x}\Big)^\alpha W\Big\}_{(j,\alpha)\in I_m}\Big), \tag{3.3}$$

where $W(t_0, t_1, \dots, t_\mu, x)$ is the new unknown function and

$$C(x, \lambda, \rho) = L(\lambda, \rho) - x \sum_{(j,\alpha)\in I_m} S(c_{j,\alpha})(x)\, \lambda^j \rho(\rho - 1) \cdots (\rho - \alpha + 1).$$

(Step 1) Construction of a formal solution of (3.3). Denote by $H_k[t_0, t_1, \dots, t_\mu]$ be the set of all the homogeneous polynomials of degree k in $(t_0, t_1, \dots, t_\mu)$. Let us look for a formal solution W of the form

$$W(t_0, t_1, \dots, t_\mu, x) = \sum_{k\geq 1, l\geq 0} w_{k,l}(t_0, t_1, \dots, t_\mu)\, x^l \tag{3.4}$$

with $w_{k,l} \in H_k[t_0, t_1, \dots, t_\mu]$ (for $k \geq 1$). Set

$$w_k = \sum_{l\geq 0} w_{k,l}(t_0, \dots, t_\mu)\, x^l \in H_k[t_0, \dots, t_\mu][[x]] \quad (k \geq 1); \tag{3.5}$$

we have $W = \sum_{k\geq 1} w_k$. By substituting this W into (3.3) and by comparing the homogeneous part of degree k with respect to $t_0, t_1, \dots, t_\mu$ in both sides of (3.3) we see that equation (3.3) is decomposed into the following recursive family:

$$C\Big(x, \tau, x\frac{\partial}{\partial x}\Big) w_k = f_k\Big(t_0, x, \Big\{D_{j,\alpha} w_p\, ;\, 1 \leq p \leq k-1, (j,\alpha) \in I_m\Big\}\Big), \tag*{(3.6)$_k$}$$
$$k = 1, 2, \dots,$$

where $f_1 = a(x)\, t_0$ and f_k (for $k \geq 2$) is a polynomial of $\{D_{j,\alpha} w_p\,;\, 1 \leq p \leq k-1, (j,\alpha) \in I_m\}$.

Moreover, by substituting (3.5) into $(3.6)_k$ and by comparing the homogeneous part of degree l with respect to x we see that equation $(3.6)_k$ is decomposed into the following recursive family:

$$(3.7)_{k,l} \qquad L(\tau, l) w_{k,l} = g_{k,l}, \quad l = 0, 1, 2, \ldots$$

with

$$(3.8) \qquad g_{k,l} = \sum_{(j,\alpha)\in I_m} \sum_{h=0}^{l-1} c_{j,\alpha,l-h}\, \tau^j\, [h]_\alpha\, w_{k,h} + \phi_{k,l},$$

where $c_{j,\alpha,l}$ are the coefficients of the Taylor expansion $c_{j,\alpha}(x) = \sum_{l\geq 0} c_{j,\alpha,l}\, x^l$, $[\lambda]_0 = 1$, $[\lambda]_\alpha = \lambda(\lambda-1)\cdots(\lambda-\alpha+1)$ for $\alpha \geq 1$, and $\phi_{k,l}$ are the coefficients of $f_k = \sum_{l\geq 0} \phi_{k,l}(t_0, t_1, \ldots, t_\mu)\, x^l \in H_k[t_0, t_1, \ldots, t_\mu][[x]]$ which are determined by $w_1, \ldots, w_{k-1}$ provided that $w_1, \ldots, w_{k-1}$ are of the form (3.5).

Thus, to get a formal solution W in the form (3.4) it is enough to solve $(3.7)_{k,l}$ inductively on (k,l) in the following way: 1) first we solve $(3.7)_{1,0}$, then we solve $(3.5)_{1,l}$ inductively on l and obtain w_1; 2) if $w_1, \ldots, w_{k-1}$ are already constructed, we solve $(3.7)_{k,0}$, then we solve $(3.7)_{k,l}$ inductively on l and obtain w_k; 3) repeating the same procedure, we can obtain a formal solution of (3.3).

Therefore, if the equation $(3.7)_{k,l}$ is always solvable in $H_k[t_0, t_1, \ldots, t_\mu]$ we can get a formal solution $W(t_0, t_1, \ldots, t_\mu, x)$ of the form (3.4). Though, the equation $(3.7)_{k,l}$ is not solvable in $H_k[t_0, t_1, \ldots, t_\mu]$ in a resonant case, and so in this case we must change our idea: we will consider the equation $(3.7)_{k,l}$ in a modulo class

$$(3.9)_{k,l} \qquad L(\tau, l) w_{k,l} \equiv g_{k,l} \pmod{\mathcal{R}_k},$$

where $\mathcal{R}_0 = \mathcal{R}_1 = \{0\}$ and for $k \geq 2$

$$\mathcal{R}_k = \bigcup_{i+j=p+q} H_{k-2}[t_0, t_1, \ldots, t_\mu] \times (t_i t_j - t_p t_q).$$

This causes no troubles in the solution of (3.2), because $f(t, t(\log t), \ldots, t(\log t)^\mu) \equiv 0$ holds for any $f(t_0, t_1, \ldots, t_\mu) \in \mathcal{R}_k$. Moreover we note that $(3.7)_{k,l}$ (or $(3.9)_{k,l}$) has the form $(\tau - \xi_1(l)) \cdots (\tau - \xi_m(l)) w_{k,l} = g_{k,l}$, and that if resonances occur we have $\xi_j(l) = k$ for some $j \in \{1, 2, \ldots, m\}$. The following lemma guarantees the solvability of $(3.7)_{k,l}$ (or $(3.9)_{k,l}$).

Lemma 2. *(1) If $\xi_j(l) \neq k$, for any $g \in H_k[t_0, t_1, \ldots, t_d]$ (with $0 \leq d \leq \mu$) the equation $(\tau - \xi_j(l))w = g$ has a unique solution $w \in H_k[t_0, t_1, \ldots, t_d]$. (2) If $\xi_j(l) = k$, for any $g \in H_k[t_0, t_1, \ldots, t_d]$ (with $0 \leq d \leq \mu - 1$) we can find a function $w \in H_k[t_0, t_1, \ldots, t_d, t_{d+1}]$ which satisfies $(\tau - \xi_j(l))w \equiv g$ (mod. $\mathcal{R}_k$).*

Thus, our way of solving the equation $(3.7)_{k,l}$ (or $(3.9)_{k,l}$) is as follows: if a resonance does not occur at (k,l) we use (1) of lemma 2 to solve $(3.7)_{k,l}$; while if some resonances occur at (k,l) we use (2) of Lemma 2 to solve $(3.9)_{k,l}$. Note that a new variable t_{d+1} is introduced whenever a resonance occurs. Since a resonance occurs μ-times, we must introduce a new variable also μ-times. Our starting point is $g_{1,0} = a(0)t_0 \in H_1[t_0]$. Hence, finally we obtain a formal solution $w_{k,l}$ at most in $H_k[t_0, t_1, \ldots, t_\mu]$.

Summing up, we have constructed a formal solution of the form (3.4) which satisfies

$$\tau^m W - F\left(t_0, x, \left\{\tau^j \left(\frac{\partial}{\partial x}\right)^\alpha W\right\}_{(j,\alpha)\in I_m}\right) \in \sum_{(k,l)\in\mathcal{M}} \mathcal{R}_k x^l.$$

(Step 2) Convergence of the formal solution (3.4). For $\vec{k} = (k_0, k_1, \ldots, k_\mu) \in \mathbb{N}^{\mu+1}$ we write $|\vec{k}| = k_0 + k_1 + \cdots + k_\mu$ and $\langle \vec{k} \rangle = k_1 + 2k_2 + \cdots + \mu k_\mu$. For $c > 0$ and $w = \sum_{|\vec{k}|=k} w_{\vec{k}} t_0^{k_0} t_1^{k_1} \cdots t_\mu^{k_\mu} \in H_k[t_0, t_1, \ldots, t_\mu]$ we define the norm $|w|_c$ by

$$|w|_c = \sum_{|\vec{k}|=k} \frac{|w_{\vec{k}}|}{c^{\langle \vec{k} \rangle}}.$$

It is easy to see

Lemma 3. *For any $w \in H_k[t_0, t_1, \ldots, t_\mu]$ we have*

$$|\tau w|_c \le (1 + c\mu)\, k\, |w|_c .$$

For $c > 0$, $\rho > 0$ and a function $f = \sum_{l\ge 0} f_{k,l}(t_0, t_1, \ldots, t_d)\, x^l \in H_k[t_0, t_1, \ldots, t_d][[x]]$ we define the norm $\|f\|_{c,\rho}$ (or the formal norm $\|f\|_{c,\rho}$) by

$$\|f\|_{c,\rho} = \sum_{l\ge 0} |f_{k,l}|_c\, \rho^l.$$

Similarly, for $\rho > 0$ and $f(x) = \sum_{l\ge 0} f_l\, x^l \in \mathbb{C}[[x]]$ (the ring of formal power series in x) we define the norm $\|f\|_\rho$ (or the formal norm $\|f\|_\rho$) by

$$\|f\|_\rho = \sum_{l\ge 0} |f_l|\, \rho^l.$$

In Gérard-Tahara [4], we have established a powerful method to prove the convergence of formal solutions of non-linear partial differential equations. We will be able to apply this method to this case and obtain the convergence of the formal solution W in (3.4), if we prove the following proposition:

Proposition 1. *Suppose the Poincaré condition (P). Then there are positive constants $c > 0$, $C > 0$ and $R > 0$ such that the following estimate holds for any $k \ge 1$:*

$$\|w_k\|_{c,\rho} \le \frac{C}{k^m}\, \|f_k\|_{c,\rho} \quad \textit{for any } 0 < \rho \le R. \tag{3.10}$$

Let us show this now. First we note the following basic lemma.

Lemma 4. *If the Poincaré condition (P) is satisfied, there are positive constants $c > 0$ and $A > 0$ which satisfy the following property. In Lemma 2 we can choose a solution $w_{k,l}$ (in both cases (1) and (2)) so that the following estimate holds for any $(k,l) \in \mathbb{N}^* \times \mathbb{N}$:*

$$|w_{k,l}|_c \le \frac{A}{(k+l)^m} |g_{k,l}|_c. \tag{3.11}$$

Proof of proposition 1. If we admit this lemma, Proposition 1 is proved in the following way. Let $c > 0$ and $A > 0$ be as in Lemma 4, and take $R > 0$ sufficiently small so that $0 < R \le 1$ and

$$A(1+c\mu)^{m-1} R \sum_{(j,\alpha)\in I_m} \|S(c_{j,\alpha})\|_R \le \frac{1}{2}. \tag{3.12}$$

By (3.8) and Lemma 3 we have

$$|g_{k,l}|_c \le \sum_{(j,\alpha)\in I_m} \sum_{h=0}^{l-1} |c_{j,\alpha,l-h}| \, (1+c\mu)^j k^j l^\alpha \, |w_{k,h}|_c + |\phi_{k,l}|_c$$

and therefore

$$\frac{A}{(k+l)^m} |g_{k,l}|_c \quad \le A(1+c\mu)^{m-1} \sum_{(j,\alpha)\in I_m} \sum_{h=0}^{l-1} |c_{j,\alpha,l-h}| \, |w_{k,h}|_c + \frac{A}{k^m} |\phi_{k,l}|_c.$$

Combining this with (3.11) and (3.12) we have

$$\begin{aligned}
\|w_k\|_{c,\rho} &= \sum_{l\ge 0} |w_{k,l}|_c \rho^l \le \sum_{l\ge 0} \frac{A}{(k+l)^m} |g_{k,l}|_c \rho^l \\
&\le A(1+c\mu)^{m-1} \rho \sum_{(j,\alpha)\in I_m} \|S(c_{j,\alpha})\|_\rho \, \|w_k\|_{c,\rho} + \frac{A}{k^m} \|f_k\|_{c,\rho} \\
&\le \frac{1}{2} \|w_k\|_{c,\rho} + \frac{A}{k^m} \|f_k\|_{c,\rho}.
\end{aligned}$$

Thus, by setting $C = 2A$ we obtain the estimate (3.10). □

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