数理解析研究所講究録 1261 巻 2002 年 123-132

NON-UNIQUE SOLVABILITY OF A CAUCHY PROBLEM FOR THE WAVE EQUATION IN QUASI-ANALYTIC ULTRADISTRIBUTION CATEGORY

TAKASHI TAKIGUCHI
(滝口 孝志)

Department of Mathematics, National Defense Academy
(防衛大学校数学教育室)

Introduction

In this article, we prove non-uniqueness in an overdetermined Cauchy problem

$$(1) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ \partial_x^\alpha u|_{x=x_0} = u_\alpha(t) \quad \text{for any } \alpha, \end{cases}$$

where Δ is the Laplacian on $\mathbb{R}^n$, $n \geq 2$.

This is an inverse problem to reconstruct the wave from observation at one space point. This problem was first introduced by L.Ehrenpreis [E], who proved uniqueness in this problem in distribution category, employing expansion by harmonic functions. As for uniqueness, F.John [J] also proved it globally with respect to general real analytic time-like curves. For distribution solutions, another uniqueness result was proved by M.Nacinovich [N] in a different way. In 1993, S.Tanabe-T.Takiguchi [TT] proved that

$$(2) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ \partial_x^\alpha u|_{x=x_0} = 0 \quad \text{for any } \alpha \end{cases}$$

would imply that $u = 0$ in a neighborhood of $x = x_0$ if u is a non-quasi-analytic ('NQA' for short) ultradistribution. In the same article, they introduced a counterexample by A.Kaneko which yields that uniqueness in this Cauchy problem does not hold for hyperfunctions.

For uniqueness in the Cauchy problem (1), the case where u is a quasi-analytic ('QA' for short) ultradistribution is left open, which we study in this article.

Ultradistributions

In this section, we review the definition of ultradistributions. Let $\Omega \subset \mathbb{R}^n$ be an open subset and $M_p, p = 0, 1, \cdots$, be a sequence of positive numbers.

Definition 1. $f \in \mathcal{E}(\Omega) = C^\infty(\Omega)$ is called an *ultradifferentiable function* of class $\{M_p\}$ (resp. (M_p)) if for any compact subset $K \subset \Omega$ there exist constants h and C (resp. for any K and for any $h > 0$ there exists some C) such that

$$\sup_{x \in K} |D^\alpha \varphi(x)| \leq C h^{|\alpha|} M_{|\alpha|} \quad \text{for all } \alpha$$

holds. Denote the set of the ultradifferentiable functions of class $\{M_p\}$ (resp. (M_p)) on Ω by $\mathcal{E}^{\{M_p\}}(\Omega)$ (resp. $\mathcal{E}^{(M_p)}(\Omega)$) and denote by $\mathcal{D}^*(\Omega)$ the set of all functions in $\mathcal{E}^*(\Omega)$ with support compact in Ω, where $* = \{M_p\}$ or (M_p).

For a compact subset $K \subset \Omega$ let

$$\mathcal{D}^*_K = \{\varphi \in \mathcal{D}^*(\mathbb{R}^n) \ ; \ \mathrm{supp} f \subset K\},$$

and we define

$$\mathcal{D}_K^{\{M_p\},h} = \{\varphi \in \mathcal{D}_K^{\{M_p\}} \ ; \ \exists C \text{ such that } \sup_{x \in K} |D^\alpha \varphi(x)| \leq C h^{|\alpha|} M_{|\alpha|}\}.$$

These spaces are endowed with natural structure of locally convex spaces.

For NQA class, we impose the following conditions on M_p.

$(M.0)$ (normalization)

$$M_0 = M_1 = 1.$$

$(M.1)$ (logarithmic convexity)

$$M_p^2 \leq M_{p-1} M_{p+1}, \quad p = 1, 2, \cdots.$$

$(M.2)$ (stability under ultradifferential operators)

$$\exists G, \ \exists H \text{ such that } M_p \leq G H^p \min_{0 \leq q \leq p} M_p M_{q-p}, \quad p = 0, 1, \cdots.$$

$(M.3)$ (strong non-quasi-analyticity)

$$\exists G \text{ such that } \sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leq Gp\frac{M_p}{M_{p+1}}, \quad p = 1, 2, \cdots .$$

$(M.2)$ and $(M.3)$ are often replaced by the following weaker conditions respectively;

$(M.2)'$ (stability under differential operators)

$$\exists G, \ \ \exists H \text{ such that } M_{p+1} \leq GH^p M_p, \quad p = 0, 1, \cdots .$$

$(M.3)'$ (non-quasi-analyticity)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$$

We note that if $\sigma > 1$ then the Gevrey sequence

$$M_p = (p!)^{\sigma}$$

satisfies all the above conditions. For more details about NQA ultradifferentiable functions and NQA ultradistributions confer [Ko1] and [Ko2].

In this article, we study QA ultradistributions. Let N_p, $p = 0, 1, \cdots$, be a sequence of positive numbers. We impose the following conditions ((QA) and (NA)) instead of $(M.3)$ or $(M.3)'$;

(QA) (quasi-analyticity)

$$N_p \geq p!, \ p = 0, 1, \cdots , \quad \sum_{p=1}^{\infty} \frac{N_{p-1}}{N_p} = \infty.$$

Let N_p be a sequence of positive numbers satisfying (QA). If

$$\liminf_{p \to \infty} \sqrt[p]{\frac{p!}{N_p}} > 0$$

then $\mathcal{E}^{\{N_p\}}$ is the class of analytic functions. We impose the condition that N_p does not define the analytic class;

(NA) (non-analyticity)

$$\lim_{p \to \infty} \sqrt[p]{\frac{p!}{N_p}} = 0.$$

If the sequence N_p satisfies $(M.1)$ and (QA), the sets $\mathcal{D}^{(N_p)}$ and $\mathcal{D}^{\{N_p\}}$ are $\{0\}$ (cf. [C]), however, we define the sheaves $\mathcal{D}^{*\prime}$ of QA ultradistributions of class $*$, where $* = \{N_p\}$ or (N_p).

For a sequence M_p of positive numbers, we define its *associated functions.* For $t > 0$, let

$$\widetilde{M}(t) := \sup_k \frac{t^k}{M_k},$$

$$M(t) := \sup_k \log \frac{t^k}{M_k},$$

$$M^*(t) := \sup_k \frac{t^k k!}{M_k}.$$

Definition 2. $f \in \mathcal{D}^{(M_p)\prime}$ (resp. $f \in \mathcal{D}^{\{M_p\}\prime}$) if f is expressed by the boundary value of the holomorphic functions,

$$f(x) = F_1(x + i\Gamma_1 0) + \cdots + F_m(x + i\Gamma_m 0),$$

where $i := \sqrt{-1}$, Γ_j, $j = 1, \cdots, m$ are open cones in $\mathbb{R}^n$, $F_j \in \mathcal{O}(\{z \in \mathbb{C}^n \ ; \ z \in \mathbb{R}^n + i\Gamma_j, \ |\mathrm{Im} z| < \exists\varepsilon\})$, $j = 1, \cdots, m$, for which, for any compact set $K \subset \mathbb{R}^n$ there exist constants L and C (resp. for any $L > 0$ there exists C) such that

$$\sup_{x \in K} |F_j(x + iy)| \leq C\widetilde{M}(L/|y|).$$

Note that, in NQA case, this definition is equivalent to the one by the duality (cf. [Ko1]).

For a function defined on $\mathbb{R}^n$, its Fourier-Laplace transform is

$$\widehat{f}(\zeta) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\zeta} dx, \qquad \zeta \in \mathbb{C}^n.$$

The Paley-Wiener theorem for NQA ultradistributions are proved by H.Komatsu (Theorem 1.1 in [Ko2]). We extend this theorem for QA ultradistributions which are not hyperfunctions. Note that the Paley-Wiener theorem for hyperfunctions are known (Theorem 8.1.1 in [Ka]).

Proposition 3. (the Paley-Wiener theorem for ultradistributions) *Assume that a sequence M_p of positive numbers satisfies* $(M.0)$, $(M.1)$, $(M.2)'$ *and* (NA). *The following conditions are equivalent.*

i) $\widehat{f}$ *is the Fourier-Laplace transform of* $f \in \mathcal{E}_K^{(M_p)'}$ (*resp.* $f \in \mathcal{E}_K^{\{M_p\}'}$), *where* $\mathcal{E}_K^{*}{}'$ *is the set of ultradistributions of the class* $*$ *whose supports are contained in* K.

ii) *There exist* $L > 0$ *and* $C > 0$ (*resp. for any* $L > 0$, *there exists* $C > 0$) *such that*

$$|\widehat{f}(\xi)| \le C\widetilde{M}(L|\xi|), \qquad \xi \in \mathbb{R}^n$$

and for any $\varepsilon > 0$ *there exists* C_ε *such that*

$$|\widehat{f}(\zeta)| \le C_\varepsilon \exp(H_K(\zeta) + \varepsilon|\zeta|), \qquad \zeta \in \mathbb{C}^n$$

where

$$H_K(\zeta) := \sup_{x \in K} \operatorname{Im} x \cdot \zeta$$

is the support function of K.

iii) *There exist* $L > 0$ *and* $C > 0$ (*resp. for any* $L > 0$, *there exists* $C > 0$) *such that*

$$|\widehat{f}(\zeta)| \le C\widetilde{M}(L|\zeta|)e^{H_K(\zeta)}, \qquad \zeta \in \mathbb{C}^n.$$

The proof of this Proposition is obtained by modifying the proof of Theorem 1.1 in [Ko2]. For this modification, we apply the Paley-Wiener theorem for hyperfunctions (Theorem 8.1.1 in [Ka]) and an estimate

$$M(L|\zeta|) = \sup_k \frac{L^k|\zeta|^k}{M_k} \le C \sup_k \frac{(\varepsilon|\zeta|)^k}{k!} \le Ce^{\varepsilon|\zeta|}.$$

Uniqueness of a function with analytic parameters

In this section, we review the results of the following problem.

Problem 4. *Let* f *be a function defined on* $\mathbb{R}^n$. *Assume that* f *contains* x'' *as analytic parameters at* $x = 0$, *where* $x = (x', x'') \in \mathbb{R}^n$ *and that the restrictions to* $x = 0$ *of* f *and all its derivatives in* x'' *vanish;*

$$\partial_{x''}^{\alpha} f|_{\{x=0\}} = 0 \quad \text{for all } \alpha.$$

Under these conditions, judge whether $f = 0$ *in some neighborhood of* $x = 0$.

The answer to this problem depends on the class where f belongs and is closely related to the uniqueness in (1), which we introduce in this section.

If f is a NQA ultradistribution, the answer to Problem 4 is positive (cf. [B1], [TT]). Applying this result, S.Tanabe-T.Takiguchi proved uniqueness in (1) in NQA ultradistribution category.

Theorem 5. (Theorem 6.2 in [TT]) *Assume that* u *is a NQA ultradistribution satisfying* (2). *Then* $u = 0$ *in some neighborhood of* $\{x = x_0\}$.

The proof of this theorem is too short and easy to omit, which we introduce.

Proof. Since all conormals to $\{x = 0\}$ are non-characteristic with respect to the wave operator, u contains x as analytic parameters at $x = 0$. Therefore the answer to Problem 4 proves the theorem. □

It is also known that uniqueness is proved for NQA ultradistributions even if the parameter x'' is weakened to QA one (cf. [B2]).

The answer to Problem 4 is negative when f is a hyperfunction. This case there is a famous counterexample by M.Sato (cf. Note 3.3 in [Ka]). J.Boman proved that the answer to Problem 4 is negative when f is a QA ultradistribution by modifying M.Sato's counterexample (cf. [B3]).

The idea of J.Boman's extension is the following. Assume that N_p satisfies $(M.0)$, $(M.1)$, $(M.2)'$, (QA) and (NA). Let

$$E := \{z \in \mathbb{C}\ ;\ |z| < 1,\ \mathrm{Im}z \neq 0\}.$$

Take such polynomials $p_k(z)$ which approximate $1/z$ uniformly in the wider sense in E that

$$|F(\tau, z)| \leq C_r M^*\Big(\frac{r}{|\mathrm{Im}z|}\Big),$$

for $\forall r > 0$, $\exists C_r$, where

$$F(\tau, z) := \sum_{k=0}^{\infty} \frac{p_k(z)}{k!}\tau^k \in \mathcal{O}((\mathbb{C} \setminus (-\infty, 0]) \times \mathbb{C})$$

F is a defining function of a QA ultradistribution f of class $\{N_p\}$,

$$f(\tau, x) = F(\tau, x + i0) - F(\tau, x - i0),$$

containing τ as a holomorphic parameter. It is not difficult to construct a counterexample in (N_p) class applying the inclusion relation between $\{N_p\}$ and (N_p) classes.

A.Kaneko proved that there exists a hyperfunction $u(t, x) \not\equiv 0$ in a neighborhood of $\{x = 0\}$ satisfying (2), applying Sato's counterexample (cf. [TT]). We modify A.Kaneko's idea and prove that uniqueness in (1) does not hold in the QA ultradistribution category neither, in the proof of which, we utilize J.Boman's counterexample.

Ultradistribution solutions to partially hyperbolic partial differential equations

In this section, we study solvability of partially hyperbolic partial differential equations in ultradistribution category. This solvability is one of the main tools to prove non-uniqueness in the Cauchy problem (1) in QA ultradistribution category.

We denote $x = (x_1, x') = (x_1, x'', x''') \in \mathbb{R}^n$, where $x'' = (x_2, \cdots, x_{k+1})$, $x''' = (x_{k+2}, \cdots, x_n)$. Let $P(D)$ be an m-th order linear partial differential operator with constant coefficients and $p_m(D)$ be its principal part. We assume that $\{x_1 = 0\}$ is non-characteristic with respect to P. We consider the complexification $z = x + iy$ of $x \in \mathbb{R}^n$ and apply similar notations for x'' and x'''. We put

$$\Omega_A := \{x'' \in \mathbb{R}^k \ ; \ |x''| < A\},$$
$$U_A := \{z''' \in \mathbb{C}^{n-k-1} \ ; \ |z'''| < A\},$$
$$T_A := \{x_1 \in \mathbb{R} \ ; \ |x_1| < A\}.$$

Let M_p, $p = 0, 1, \cdots$, be a sequence of positive numbers satisfying $(M.0)$, $(M.1)$ and $(M.2)'$. We denote by $\mathcal{D}^{*\prime}\mathcal{O}(\Omega_A \times U_A)$ the space of ultradistributions of the class $*$ defined on $\mathbb{R}^k \times \mathbb{C}^{n-k-1}$ containing $z''' \in U_A$ as holomorphic parameters. For the definition of hyperfunctions and holomorphic parameters, confer [Ka]. In the same way, we define $\mathcal{D}^{*\prime}\mathcal{O}(T_A \times \Omega_A \times U_A)$ on $\mathbb{R} \times \mathbb{R}^k \times \mathbb{C}^{n-k-1}$. We apply the same notions for $\mathcal{E}^{*\prime}\mathcal{O}$.

Our main purpose in this section is to prove the following theorem.

Theorem 6. *Let P be a partial differential operator defined above. Assume that the sequence M_p satisfies $(M.0)$, $(M.1)$, $(M.2)$ and (NA). Then the following conditions are equivalent.*

i) *For any $A > 0$, there exist such $0 < a$, $0 < B < A$ that the initial value problem*

$$\begin{cases} P(D)u(x) = 0, \\ \partial_{x_1}^j u|_{x_1=0} = u_j(x'', z'''), \quad j = 0, 1, \cdots, m-1, \end{cases}$$

where $u_j \in \mathcal{E}^{*\prime}\mathcal{O}(\Omega_A \times U_A)$, allows an ultradistribution solution $u(x_1, x'', z''')$ with support compact in x'' of class $* = (M_p)$ (resp. $\{M_p\}$) defined on $T_a \times \Omega_B \times U_B$ which contains $z''' \in U_B$ as holomorphic parameters.

ii) *There exist constants β, γ, C, l (resp. there exist β, γ and for any l there exists C) such that*

$$|\mathrm{Im}\zeta_1| \le \beta|\mathrm{Im}\zeta''| + \gamma|\zeta'''| + M(l|\zeta''|) + C,$$

for $P(\zeta) = 0$.

Remark 7. i) The counterparts of Theorem 7 for distributions and hyperfunctions are proved in [LK] in a stronger form, our proof is a modification of their theory. In E.G.Lee-A.Kaneko's theorems they do not assume that initial values and solutions are compactly supported in x''. For NQA ultradistributions, this extension is possible since $\mathcal{D}^{*\prime}\mathcal{O}$ is partially soft when $*$ defines NQA class. For QA case, we have to prove partial flabbiness of $\mathcal{D}^{*\prime}\mathcal{O}$ for this extension.

ii) In the proof of Theorem 6, we apply Proposition 3 to estimate the support with respect to x''.

iii) What we claim in Theorem 6 is that we have a solution with holomorphic parameter in ultradistribution category, especially in QA ones. Since the symbol of P is a polynomial it is not the case that the term $M(l|\zeta''|)$ is valid, however, our theorem holds for a general convolution operators. Therefore we state our theorem as Theorem 6.

The main theorem

In this section, we prove that uniqueness in (1) does not hold in QA ultradistribution category, to prove which, Theorem 6 and J.Boman's counterexample play important roles.

Theorem 8. *Assume that the sequence N_p satisfies $(M.0)$, $(M.1)$, $(M.2)$, (QA) and (NA). There exists such a QA ultradistribution $u(t,x)$ of class (N_p) or $\{N_p\}$ satisfying (2) that $u(t,x) \not\equiv 0$ in any neighborhood of $x = x_0$.*

Proof. For simplicity, let us assume that $x_0 = 0$. Consider the Cauchy problem

$$(3) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ u|_{x_1=0} = \varphi(x',t), \quad \dfrac{\partial u}{\partial x_1}|_{x_1=0} = \psi(x',t) = 0, \end{cases}$$

where $x = (x_1, x') \in \mathbb{R}^n$ and φ is J.Boman's counterexample with holomorphic parameter x'. By the construction, φ is compactly supported in t. By virtue of Theorem 6, the Cauchy problem (3) has a local QA ultradistribution solution $u(t,x)$ near $x_1 = 0$. By (1),

$$\partial_{x,t}^{\alpha} u = \sum_{\beta} c_\beta \partial_{x',t}^{\beta} \partial_{x_1} u + \sum_{\gamma} c_\gamma \partial_{x',t}^{\gamma} u,$$

where c_β, $c_\gamma = 1$ or -1. We have

$$\partial^\alpha_{x,t} u|_{x_1=0} = \sum_\gamma c_\gamma \partial^\gamma_{x',t}\varphi = \sum_{\gamma',\gamma''} c_{\gamma',\gamma''} \partial_t^{\gamma'} \partial_{x'}^{\gamma''} \varphi.$$

Restricting both sides to $\{x' = 0\}$ gives us

$$\partial^\alpha_{x,t} u|_{x=0} = \partial_t^{\gamma'} (\partial_{x'}^{\gamma''} \varphi|_{x'=0}) = 0,$$

because $\partial_{x'}^{\gamma''} \varphi|_{x'=0} = 0$. □

Theorem 8 completes the study of uniqueness in the Cauchy problem (1).

Remark 9. Even in NQA ultradistribution category, uniqueness does not hold if initial values are restricted to finite order. More strongly, we construct a counterexample in distribution category. Let $m \in \mathbb{N}$. We have a local distribution solution $u(t,x) \not\equiv 0$ to the Cauchy problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ \partial_x^\alpha u|_{x=x_0} = 0 \ \text{ for } |\alpha| \le m. \end{cases}$$

In fact, for simplicity, we assume that $x_0 = 0$. Consider the Cauchy problem

$$(4) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ u|_{x_1=0} = (x_2 \cdots x_n)^{m+1} g(t), \quad \dfrac{\partial u}{\partial x_1}|_{x_1=0} = \psi(x',t) = 0, \end{cases}$$

where $g(t)$ is a distribution of one variable. By Theorem 2 or 3 in [LK], (4) has a distribution solution $u(t,x)$ near $x_1 = 0$. It is easy to show that $\partial_x^\alpha u|_{x=0} = 0$ for $|\alpha| \le m$.

In smoother classes where the counterpart of Theorem 6 holds, the counterpart of Remark 9 is proved. For example, C^∞, ultradifferential and analytic classes are those ones. Note also that the argument in this article applies to a general linear partial differential equation with analytic coefficients and a real analytic submanifold whose conormals are non-characteristic.

References

[B1] Boman J., *A local vanishing theorem for distributions*, C. R. Acad. Sci. Paris **315** (1992), 1231-1234.

[B2] ———, *Microlocal quasianalyticity for distributions and ultradistributions*, Publ. RIMS, Kyoto Univ. **31** (1995), 1079-1095.

[B3] ———, *Uniqueness and non-uniqueness for microanalytic continuation of ultradistributions*, Contemporary Mathematics **259** (2000), 61-82.

[C] Carleman T., *Fonction quasi analytique*, Gauther-Villars et Cie, Éditeurs, Paris, 1926.

[E] Ehrenpreis L., *A new Cauchy problem for partial differential equations* (*lecture at Univ. of Tokyo*), Sugaku **26** (1974), 168 (notes by A.Kaneko), in Japanese.

[J] John F., *On linear partial differential equations with analytic coefficients*, Comm. Pure. Appl. Math. **2** (1949), 209-253.

[Ka] Kaneko A., *Introduction to Hyperfunctions*, Kluwer Academic Publishers,, Dordrecht, Boston, London, 1988.

[Ko1] Komatsu H., *Ultradistributions,I,Structure theorems and characterization*, J. Fac. Sci. Univ. Tokyo, Sect.1A **20** (1973), 25-105.

[Ko2] ———, *Ultradistributions,II,The kernel theorem and ultradistributions with support in a submanifold*, Ibid **24** (1977), 607-628.

[LK] Lee E.G. and Kaneko A., *Condition of partial hyperbolicity modulo a linear subvariety for operators with constant coefficients*, J. Fac. Sci. Univ. Tokyo,Sect.1A **36** (1989), 329-346.

[N] Nacinovich M., *Cauchy problem for overdetermined systems*, Talk at the meeting "Analytic Solutions of Partial Differential Equations ", Trento, 1992.

[TT] Tanabe S. and Takiguchi T., *A local vanishing theorem for ultradistributions with analytic parameters*, J. Fac. Sci. Univ. Tokyo Sec.1A **40** (1993), 607-621.

Department of Mathematics, National Defense Academy, 1-10-20, Hashirimizu, Yokosuka, Kanagawa, 239-8686 JAPAN

E-mail address: takashi@cc.nda.ac.jp