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Gevrey Regularity of Solutions of Semilinear Hypoelliptic Equations on the Plane

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§1. Introduction.

In this note we discuss the Gevrey regularity (in particular, the analyticity) of solutions of semilinear elliptic degenerate equations of Grushin's type on $\mathbb{R}^2$. Most of the results will appear in [1]. Some results are new and they are presented here for the first time. We confine ourself with consideration of a model equation. Precisely, we will consider the following equation

\begin{equation}
G_{k,\lambda}f + \Psi(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}) = 0 \text{ in a domain } \Omega \subset \mathbb{R}^2,
\end{equation}

where

\[
G_{k,\lambda} = \frac{\partial^2}{\partial x^2} + x^{2k} \frac{\partial^2}{\partial y^2} + i\lambda x^{k-1} \frac{\partial}{\partial y}
\]

with $(x, y) \in \Omega \subset \mathbb{R}^2, \lambda \in \mathbb{C}, i = \sqrt{-1}$ and $k \in \mathbb{Z}_+$. $\Omega$ is a bounded domain in $\mathbb{R}^2$.

Let us define the following quantities

\[
R = (x^{k+1} + u^{k+1})^2 + (k + 1)^2(y - v)^2, \quad p = \frac{4x^{k+1}u^{k+1}}{R},
\]

\[
A_+ = x^{k+1} + u^{k+1} + (k + 1)(y - v), \quad A_- = x^{k+1} + u^{k+1} - (k + 1)(y - v),
\]

\[
M = A_+^{-\frac{k+\lambda}{2k+2}}A_-^{-\frac{k-\lambda}{2k+2}},
\]

here we take $z_1^{z_2} = e^{z_2 \ln z_1}$ for $z_1, z_2 \in \mathbb{C}$ and if $z_1 = re^{i\varphi}, -\pi < \varphi \leq \pi$ then $\ln z_1 = \ln r + i\varphi$. First, we will find the uniform fundamental solution of $G_{k,\lambda}$, that is

\[
G_{k,\lambda}F_{k,\lambda}(x, y, u, v) = \delta(x - u, y - v),
\]

in the following form

\[
F_{k,\lambda}(x, y, u, v) = F(p)M.
\]

After some computations we arrive at

\[
G_{k,\lambda}F_{k,\lambda} = 16(k + 1)^2u^{2k+2}x^{2k} \left[(u^{k+1} - x^{k+1})^2 + (k + 1)^2(y - v)^2\right] \times
\]

\[
\times MR^{-3}F''(p) + +4(k + 1)x^{k-1}u^{k+1}[k(x^{2k+2} + u^{2k+2} + (k + 1)^2(y - v)^2) - (6k + 4)x^{k+1}u^{k+1}]MR^{-2}F(p) + + (\lambda^2 - k^2)x^{k-1}u^{k+1}MR^{-1}F(p).
\]

Therefore, if $F(p)$ satisfies the following hypergeometric equation

\[(2) \quad p(1-p)F''(p) + [c - (1 + a + b)p]F'(p) - abF(p) = 0,\]

with $a = \frac{k+\lambda}{2k+2}, b = \frac{k-\lambda}{2k+2}, c = \frac{k}{k+1}$, then formally we will have

$$G_{k,\lambda}F_{k,\lambda} = 0.$$ 

The general solutions of (2) are

$$F(p) = C_1 F\left(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\right) + C_2 p^{\Gamma^{1}} + \mathrm{T} F\left(\frac{k+2+\lambda}{2k+2}, \frac{k+2-\lambda}{2k+2}, \frac{k+2}{k+1}, p\right),$$

where $F(a, b, c, p)$ is the Gauss hypergeometric function and $C_1, C_2$ are some complex constants [2].

\[\text{§2. Case } k \text{ is odd.}\]

Since $k$ is odd, we note that $0 \leq p \leq 1$. Moreover, $p = 1$ if and only if $x = \pm u \neq 0, y = v$. If $u = 0, v = 0$ then $p = 0$; therefore, from the result of [3]

$$G_{k,\lambda}F\left(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\right) M = -\frac{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k}{k+1}\right)}{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)} \delta(x, y)$$

we should choose

$$C_1 = -\frac{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)}{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k}{k+1}\right)}.$$

If $u \neq 0$ then the singularities of $F_{k,\lambda}(x, y, u, v)$ will be located at the one of $F(p)$. On the other hand, $F(p)$, with $0 \leq p \leq 1$, has singularity only when $p = 1$. As $p \to 1$ we have the following asymptotic expansions (see [2])

$$F\left(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\right) = -\frac{\Gamma\left(\frac{k}{k+1}\right)}{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)} \log(1-p) + O(1),$$

$$F\left(\frac{k+2+\lambda}{2k+2}, \frac{k+2-\lambda}{2k+2}, \frac{k+2}{k+1}, p\right) = -\frac{\Gamma\left(\frac{k+2}{k+1}\right)}{\Gamma\left(\frac{k+2+\lambda}{2k+2}\right)\Gamma\left(\frac{k+2-\lambda}{2k+2}\right)} \log(1-p) + O(1).$$

We expect that $F_{k,\lambda}(x, y, u, v)$ has singularity only when $x = u, y = v$. Since $p^{\frac{3}{2k+1}} = (4R^{-1})^{\frac{3}{2k+1}}xu \to -1$ when $(x, y) \to (-u, v)$, we should choose

$$C_2 = -\frac{\Gamma\left(\frac{k+2+\lambda}{2k+2}\right) \Gamma\left(\frac{k+2-\lambda}{2k+2}\right)}{2^{2+\frac{1}{k+1}} \pi \Gamma\left(\frac{k+2}{k+1}\right)}$$

such that $F(p)$ has no singularity at $x = -u, y = v$. Note that the following conditions

(3) \[ \lambda \neq \pm[2N(k + 1) + k], \lambda \neq \pm[2N(k + 1) + k + 2], \]

where $N$ is a non-negative integer, guarantee that $C_1, C_2 < \infty$ and hence $F(p)$ has logarithm growth (if $u \neq 0$) at $(x, y) = (u, v)$.

**Definition.** The parameter $\lambda$ is called admissible if $\lambda$ satisfies the condition (3).

Therefore, if $\lambda$ is admissible then we expect that the function $F(p)M$, or

$$F_{k,\lambda}(x, y, u, v) = -\frac{\Gamma\left(\frac{k+2+\lambda}{2k+2}\right) \Gamma\left(\frac{k+2-\lambda}{2k+2}\right) F\left(\frac{k+2+\lambda}{2k+2}, \frac{k+2-\lambda}{2k+2}, \frac{k+1}{k+1}, p\right)}{2^{2+\frac{1}{k+1}} \pi \Gamma\left(\frac{k+2}{k+1}\right) A_+^{\frac{k+2+\lambda}{2k+2}} A_-^{\frac{k+2-\lambda}{2k+2}}} - xu\Gamma\left(\frac{k+2+\lambda}{2k+2}\right) \Gamma\left(\frac{k+2-\lambda}{2k+2}\right) F\left(\frac{k+2+\lambda}{2k+2}, \frac{k+2-\lambda}{2k+2}, \frac{k+2}{k+1}, p\right)$$

will be our desired uniform fundamental solution. Indeed, we have

**Theorem 1.** Assume that $\lambda$ is admissible. Then

$$G_{k,\lambda}F_{k,\lambda}(x, y, u, v) = \delta(x - u, y - v).$$

**Remark 1.** A similar expression for $F_{k,0}$ is also given in [4].

Let us denote $X'_1 = \frac{\partial}{\partial u} - iu^k \frac{\partial}{\partial v}, X'_2 = \frac{\partial}{\partial u} + iu^k \frac{\partial}{\partial v}$, and $G'_{k,\lambda} = X'_2 X'_1 + i(\lambda + k)u^{k-1} \frac{\partial}{\partial v}$. Noting that $F_{k,\lambda}(x, y, u, v) = F_{k,-\lambda}(u, v, x, y)$, from Theorem 1 we can easily deduce

Proposition 1 (Representation formula). Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with piece-wise smooth boundary, $f \in C^2(\Omega)$ and $\lambda$ is admissible then we have

\[ f(x, y) = \int_{\Omega} F_{k,\lambda}(x, y, u, v) G'_{k,\lambda} f(u, v) du dv - \int_{\partial \Omega} F_{k,\lambda}(x, y, u, v) B_1'(f(u, v), k, -\lambda) ds + \int_{\partial \Omega} f(u, v) B_2'(F_{k,\lambda}(x, y, u, v), k) ds, \]

where

\[ B_1'(f(u, v), k, -\lambda) = (\nu_1 - iu^k \nu_2) X'_2 f(u, v) - i(-\lambda + k)u^{k-1}\nu_2 f(u, v), \]
\[ B_2'(F_{k,\lambda}(x, y, u, v), k) = (\nu_1 + iu^k \nu_2) X'_1 F_{k,\lambda}(x, y, u, v), \]

and $\nu = (\nu_1, \nu_2)$ is the unit outward normal vector on $\partial \Omega$.

Now, we re-state a well-known theorem on hypoellipticity of $G_{k,\lambda}$ as follows

Theorem 2. $G_{k,\lambda}$ is hypoelliptic if and only if the hypergeometric equation (2) has no bounded solution on the interval $[0, 1]$.

Proof. Here, with the help of $F_{k,\lambda}$, we give a proof, which is alternative to a well-known classical proof based on the theory of pseudo-differential operators. Suppose that $f \in C^2(\Omega)$ and $G_{k,\lambda} f(x, y) = h(x, y)$ where $h \in C^\infty(\Omega)$. Then we can express $f$ through $h$ as in (4), with $G'_{k,\lambda} f(u, v)$ replaced by $h(u, v)$. It is clear that the boundary integrals give $C^\infty(\Omega)$ functions. For the volume integral, we see that $\frac{\partial F_{k,\lambda}}{\partial y} = -\frac{\partial F_{k,\lambda}}{\partial v}$. Therefore, by integration by parts, we can differentiate the integral in $x$ one time and in $y$ as many times as we want to. And the resulting functions are continuous. We will complete the proof if we are able to show that if $f \in C^{n-1}(\Omega)$ then $f \in C^n(\Omega)$ for every positive integer $n$. This is the case because we already have $\frac{\partial^n f}{\partial y^n}, \frac{\partial^n f}{\partial y^{n-1}\partial x}$ and $\frac{\partial^{\alpha+\beta} f}{\partial y^\alpha \partial x^\beta}$, $\alpha + \beta \leq n - 1$ belong to $C(\Omega)$ from the above argument and assumption. We have to show that $\frac{\partial^n f}{\partial y^{n-j}\partial x^j}, \frac{\partial^n f}{\partial x^j}, 1 \leq j \leq n - 1$ are continuous. We shall prove that $\frac{\partial^n f}{\partial y^{n-j}\partial x^j}$ is in $C(\Omega)$, $\frac{\partial^n f}{\partial x^j}$ is in $C(\Omega)$. Indeed, we have

\[ \frac{\partial^2 f}{\partial x^2} = h - x^{2k} \frac{\partial^2 f}{\partial y^2} - i\lambda x^{k-1} \frac{\partial f}{\partial y}. \]
Therefore, differentiating $\frac{\partial^{n-2}}{\partial y^{n-j-1}\partial x^{j}}$ both sides of (5) gives

$$\frac{\partial^{n}f}{\partial y^{n-j-1}\partial x^{j+1}} = \frac{\partial^{n-2}h}{\partial y^{n-j-1}\partial x^{j-1}} - \sum_{i=0}^{j} \binom{j}{i} 2k(2k-1)\cdots(2k-i+1)x^{2k-i}\frac{\partial^{n-i-1}f}{\partial y^{n-j+1}\partial x^{j-i-1}} - i\lambda \sum_{i=0}^{j} \binom{j}{i} (k-1)(k-2)\cdots(k-i)x^{k-i}\frac{\partial^{n-i-1}f}{\partial y^{n-j}\partial x^{j-i-1}} \in C(\Omega).$$

Actually, a more detailed examination of the proof of Theorem 2 would show that the integral operators

$$K : h \rightarrow K(h)(x,y) = \int_{\Omega} F_{k,\lambda}(x,y,u,v)h(u,v)dudv,$$

$$tK : h \rightarrow tK(h)(x,y) = \int_{\Omega} F_{k,\lambda}(u,v,x,y)h(u,v)dudv$$

map $C^{\infty}_{0}(\Omega)$ into $C^{\infty}(\Omega)$. In other words, $K$ and $tK$ are separately regular. Since $F_{k,\lambda}$ is a $C^{\infty}$ function in the complement of the diagonal of $\Omega \times \Omega$, we conclude that $K$ and $tK$ are very regular.

Next, we introduce some notations

$$\Xi_{t} = \{(\alpha,\beta,\gamma) \in \mathbb{Z}_{+}^{3} : \alpha + \beta \leq t, kt \geq \gamma \geq \alpha + (1+k)\beta - t\}.$$ 

For a function $f(x,y)$ on $\mathbb{R}^{2}$, we write $\partial_{1}^{\alpha}f$, $\partial_{2}^{\beta}f$, $\partial_{1,2}^{\alpha,\beta}f$, $\gamma\partial_{1,2,\beta}f$ for $\frac{\partial^{\alpha}f(x,y)}{\partial x^{\alpha}}$, $\frac{\partial^{\beta}f(x,y)}{\partial y^{\beta}}$, $\frac{\partial^{\alpha+\beta}f(x,y)}{\partial x^{\alpha}\partial y^{\beta}}$, $x^{\gamma}\frac{\partial^{\alpha+\beta}f(x,y)}{\partial x^{\alpha}\partial y^{\beta}}$, respectively. For $m \in \mathbb{Z}_{+}$, let us denote by $\mathbb{H}^{m}_{loc}(\Omega)$ the space of all function $f \in L_{loc}^{2}(\Omega)$ such that for any compact $K$ of $\Omega$ we have $\sum_{(\alpha,\beta,\gamma) \in \Xi_{m}} \|\gamma \partial_{\alpha,\beta}f\|_{L_{x}(K)} < \infty$. Now we are in a position to formulate the main theorem of this section.

**Theorem 3.** Assume that $m \geq 2k^{2}+6k+5$. Let $f$ be a $\mathbb{H}^{m}_{loc}(\Omega)$ solution of the equation (1) and $\Psi \in G^{s}$. Then $f \in G^{s}$. In particular, if $\Psi$ is analytic in its arguments then so is $f$.

**Proof.** The proof of Theorem 3 consists of Theorem 4 and Theorem 5 below. The proof follows the scheme : $f \in \mathbb{H}^{m}_{loc} \implies f \in C^{\infty}(\Omega) \implies f \in A(\Omega)$. □

**Theorem 4.** Let $\Psi$ be a $C^{\infty}$ function of its arguments and $m \geq 2k^{2}+6k+5$. Assume that $f \in \mathbb{H}^{m}_{loc}(\Omega)$ is a solution of the equation (1) then $f \in C^{\infty}(\Omega)$.

**Proof.** Theorem 4 can be proved with the help of Proposition 2 . □
Proposition 2. Let \( m \geq 2k^2 + 6k + 5 \). Assume that \( f \in \mathbb{H}_{loc}^{m}(\Omega) \). Then 
\[ \Psi(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}) \in \mathbb{H}_{loc}^{m-1}(\Omega). \]

Next, put \( r_0 = 2k + 2 \). For \( r \in \mathbb{Z}_+ \) let \( \Gamma_r \) denote the set of pairs of multi-indices \((\alpha, \beta)\) such that \( \Gamma_r = \Gamma^1_r \cup \Gamma^2_r \) where
\[ \Gamma^1_r = \{ (\alpha, \beta) : \alpha \leq r_0, 2\alpha + \beta \leq r \}, \]
\[ \Gamma^2_r = \{ (\alpha, \beta) : \alpha \geq r_0, \alpha + \beta \leq r - r_0 \}. \]

Define the following norm
\[ |f, \Omega|_r = \max_{(\alpha, \beta) \in \Gamma_r} |\partial_1^\alpha \partial_2^\beta f, \Omega| + \max_{\alpha \geq 1, \beta \geq 1} |\partial_1^{\alpha+2} \partial_2^\beta f|, \]
where \( |f, \Omega| = \max_{(x,y) \in \Omega} \left( |f| + \left| \frac{\partial f}{\partial x} \right| + \left| x^k \frac{\partial f}{\partial y} \right| \right) \).

Theorem 5. Let \( f \) be a \( C^\infty \) solution of the equation (1) and \( \Psi \in G^* \). Then \( f \in G^* \). In particular, if \( \Psi \) is analytic in its arguments then so is \( f \).

Proof. Theorem 5 can be proved with the help of Proposition 3, Corollary 1, Lemmas 2-4. \( \square \)

Proposition 3. Assume that \( \Psi \in G^* \). Then there exist constants \( C, D \) such that for every \( H_0 \geq 1, H_1 \geq CH_0^{2k+3} \) if
\[ |f, \Omega|_d \leq H_0 H_1^{(d-r_0-2)}(d-r_0-2)!^s, \quad 0 \leq d \leq N+1, r_0 + 2 \leq N \]
then
\[ \max_{(x,y) \in \Omega} |\partial_1^\alpha \partial_2^\beta \Psi(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y})| \leq DH_0 H_1^{N-r_0-1}(N-r_0-1)!^s \]
for every \((\alpha, \beta) \in \Gamma_N+1\).

Corollary 1. Under the same hypotheses of Proposition 3 with \( d \leq N+1 \) replaced by \( d \leq N \), then
\[ \max_{x \in \Omega} |\partial_1^\alpha \partial_2^\beta \Psi(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y})| \leq D \left( |f, \Omega|_{N+1} + H_0 H_1^{N-r_0-1}(N-r_0-1)!^s \right) \]
for every \((\alpha, \beta) \in \Gamma_N+1\).

Since \( G_{k,\lambda} \) is elliptic if \( x \neq 0 \), it suffices to consider the case \((0,0) \in \Omega \) and \( \Omega \) is a small neighborhood of \((0,0)\). Let us define the distance
\[ \rho((u,v),(x,y)) = \begin{cases} \max \{|x^{k+1}-u^{k+1}|, (k+1)|y-v|\}, & \text{for } xu \geq 0 \\ \max \{|x^{k+1}+u^{k+1}|, (k+1)|y-v|\}, & \text{for } xu \leq 0 . \end{cases} \]
For two sets $S_1, S_2$, the distance between them is defined as

$$
\rho(S_1, S_2) = \inf_{(x, y) \in S_1, (u, v) \in S_2} \rho((x, y), (u, v)).
$$

Let $V^T(T \leq 1)$ be the cube with edges of size (in the $\rho$ metric) $2T$ which are parallel to the coordinate axes and centered at $(0, 0)$. Denote by $V_\delta^T$ the sub-cube which is homothetic with $V^T$ and such that the distance between its boundary and the boundary of $V^T$ is $\delta$. We shall prove by induction that if $T$ is small enough then there exist constants $H_0, H_1$ with $H_1 \geq CH_0^{2k+3}$ such that

\begin{equation}
(6) |f, V_\delta^T|_n \leq H_0 \quad \text{for} \quad 0 \leq n \leq 6k + 4,
\end{equation}

and

\begin{equation}
(7) |f, V_\delta^T|_n \leq H_0 \left( \frac{H_1}{\delta} \right)^{n-r_0-2} (n-r_0-2)!^\delta := Q_{n-1}
\end{equation}

for $n \geq 6k + 4$, and $\delta$ sufficiently small. Hence the desired conclusion follows. (6) follows easily from the $C^\infty$ smoothness assumption on $f$. Assume that (7) holds for $n = N$. We shall prove it for $n = N + 1$. Put $\delta' = \delta(1-1/N), \delta'' = \delta(1-4/N)$. Fix $(x, y) \in V_\delta^T$ and then define $\sigma = \rho((x, y), \partial V^T)$ and $\tilde{\sigma} = \sigma/N$. Let $V_\sigma(x, y)$ denote the cube with center at $(x, y)$ and edges of length $2\tilde{\sigma}$ which are parallel to the coordinate axes, and $S_\sigma(x, y)$ the boundary of $V_\sigma(x, y)$. Note that $\sigma \geq \delta$, and $V_\sigma(x, y) \subset V_\delta^T$. Let $E_1, E_3, E_2, E_4$ be edges of $S_\sigma(x, y)$ which are parallel to $Ox(0y)$ respectively. We have to estimate $\max_{(x, y) \in V_\delta^T} \sum_{(\alpha, \beta, \gamma) \in \Gamma} |(\partial_{12}^\alpha f(x, y))|$ for all $(\alpha, \beta, \gamma) \in \Xi_1, (\alpha_1, \beta_1) \in \Gamma_{N+1}$.

**Lemma 2.** Assume that $(\alpha, \beta, \gamma) \in \Xi_1$ and $(\alpha_1, \beta_1) \in \Gamma_{N+1}$. Then if $\alpha_1 \geq 1, \beta_1 \geq 1$ there exists a constant $C$ such that

\begin{equation}
\max_{(x, y) \in V_\delta^T} \sum_{(\alpha, \beta, \gamma) \in \Gamma} |(\partial_{12}^\alpha f(x, y))| \leq C \left( T^{1/\delta} |f, V_\delta^T|_{N+1} + Q_N \left( T^{1/\delta} + \frac{1}{H_1} \right) \right).
\end{equation}

**Lemma 3.** Assume that $(\alpha, \beta, \gamma) \in \Xi_1$. Then there exists a constant $C$ such that

\begin{equation}
\max_{(x, y) \in V_\delta^T} \sum_{(\alpha, \beta, \gamma) \in \Gamma} |(\partial_2^{N+1} f(x, y))| \leq C \left( T^{1/\delta} |f, V_\delta^T|_{N+1} + Q_N \left( T^{1/\delta} + \frac{1}{H_1} \right) \right).
\end{equation}
Lemma 4. Assume that $(\alpha, \beta, \gamma) \in \Xi_1$. Then there exists a constant $C$ such that

$$\max_{(x,y)\in\mathcal{V}_T^\gamma} |\gamma \partial_{\alpha,\beta} (\partial_1^{N-r+1} f(x,y))| \leq C \left( T^{\frac{1}{k+1}} |f, \mathcal{V}_T^\gamma|_{N+1} + Q_N \left( T^{\frac{1}{k+1}} + \frac{1}{H_1} \right) \right).$$

Lemma 5. Assume that $(\alpha_1, \beta_1) \in \Gamma_{N+1} \setminus \Gamma_N$, $\alpha_1 \geq 1, \beta_1 \geq 1$. Then there exists a constant $C$ such that

$$\max_{(x,y)\in\mathcal{V}_t^\gamma} |(\partial_1^{\alpha_1+2} \partial_2^{\beta_1} f(x,y))| \leq C \left( T^{\tau_{+1}} |f, \mathcal{V}_T^\gamma|_{N+1} + Q_N \left( T^{\frac{1}{k+1}} + \frac{1}{H_1} \right) \right).$$

§ 3. Case $k$ is even.

A. First, we consider the case $\lambda = 2N(k+1)$, where $N$ is an integer. In this case we will prove a similar result as in §2 by establishing the explicit uniform fundamental solutions of $G_{k,2N(k+1)}$. Let us maintain the notations used for $p, A_+, A_-, M, F_{k,\lambda}, \ldots$ from the very beginning of the paper (now, of course, with an even $k$). If $(u,v) \neq (0,v)$ is fixed then the real parts of $A_+, A_-$ change sign when $(x, y)$ passes through $(-u,v)$. Therefore, $M = A_+^{-\frac{k+\lambda}{2k+2}} A_-^{-\frac{k-\lambda}{2k+2}}$ may have singularities alone the half-line $(x, v)$ with $x \leq -u$ for an arbitrary complex number $\lambda$. But if $\lambda = 2N(k+1)$ then it is not difficult to see that $M = A_+^{-\frac{k+\lambda}{2k+2}} A_-^{-\frac{k-\lambda}{2k+2}}$ is smooth alone the half-line $(x, v)$ with $x < -u$, that is $M(\cdot, \cdot, u, v) \in C^\infty(\mathbb{R}^2 \setminus \{(u, v), (-u, v)\})$. Moreover, when $k$ is even and $u \neq 0$ we have $-\infty \leq p \leq 1$. More precisely, $p \to 1$ when $(x,y) \to (u,v)$, and $p \to -\infty$ when $(x,y) \to (-u,v)$. If $N < 0$ and $p \to -\infty$ then from the asymptotic expansions of hypergeometric functions (see [2], p. 63 ) we should choose the expressions for constants $C_1, C_2$ as in the beginning of the paper (with $\lambda$ replaced by $2N(k+1)$). And we will have $F_{k,2N(k+1)}(\cdot, \cdot, u, v) \in C^\infty(\mathbb{R}^2 \setminus (u,v))$, with

$$F_{k,2N(k+1)}(-u,v,u,v) = 0.$$ 

Similar conclusions hold for $F_{k,2N(k+1)}(x,y,u,v)$ when $N > 0$. If $N = 0$ then $F_{k,0}(\cdot, \cdot, u, v) \in C^\infty(\mathbb{R}^2 \setminus (u,v))$, with

$$F_{k,0}(-u,v,u,v) = -\cot \frac{k\pi}{2k+2}. $$
**Theorem 6.** Let $\Psi \in G^s$. Assume that $m \geq 2k^2 + 6k + 5$, $\lambda = 2N(k + 1)$, and $f$ is a $\mathbb{H}_{loc}^m(\Omega)$ solution of the equation (1). Then $f \in G^s$. In particular, if $\Psi$ is analytic in its arguments then so is $f$.

**Proof.** Almost all the arguments used for the case when $k$ is odd can be applied here. Therefore, we only give the sketch of the proof. Instead of the distance $\rho$ in §2 we use the following metric

$$\tilde{\rho}((u, v), (x, y)) = \max \{|x^{k+1} - u^{k+1}|, |y - v|\}. \Box$$

**B.** In this sub-section we will present some computations for finding the fundamental solutions of $G_{k, \lambda}$ with source at the origin $(0, 0)$ for $\lambda$ other than the values $2N(k + 1)$ considered in sub-section A. Make the following change of variables

$$x = \rho|\sin \theta|^\frac{1}{k+1}\text{sign}(\sin \theta), y = \frac{\rho^{k+1}}{k+1}\cos \theta, \theta \in (-\pi, \pi).$$

Then $G_{k, \lambda}$ will be transformed into

$$\text{sign}(\sin \theta)|\sin \theta|^{\frac{1}{k+1}} \left( \sin \theta \frac{\partial^2}{\partial \rho^2} + (k + 1)^2 \rho^{-2} \sin \theta \frac{\partial^2}{\partial \theta^2} + (i\lambda \cos \theta + (k + 1) \sin \theta) \rho^{-1} \frac{\partial}{\partial \rho} + (k + 1) \rho^{-2}(k \cos \theta - i\lambda \sin \theta) \frac{\partial}{\partial \theta} \right).$$

If we seek the fundamental solution in the form $F_{k, \lambda}(x, y) = \rho^{-k}F(\theta)$ then $F(\theta)$ must satisfy the following equation

$$(k + 1)^2 \sin \theta F''(\theta) + (k + 1)(k \cos \theta - i\lambda \sin \theta)F'(\theta) - ik\lambda \cos \theta F(\theta) = 0. \quad (8)$$

The general solutions of (8) are

$$F(\theta) = \left( C_3 + C_4 \int_{0}^{\theta} |\sin s|^{-\frac{k}{k+1}} e^{-\frac{i\lambda s}{k+1}} ds \right) e^{\frac{i\lambda \theta}{k+1}},$$

where $C_3$ and $C_4$ are some complex constants. Among all these solutions, we are interested in finding a non-trivial periodic solution. When $\lambda = 2N(k + 1)$ this case was considered in sub-section A – the periodic solution is $F(\theta) = e^{\frac{i\lambda \theta}{k+1}}$, and the function $F_{k, \lambda}(x, y) = \rho^{-k}F(\theta)$ serves as a fundamental solution. When
\[ \lambda = (2N + 1)(k + 1) \] then the periodic solution again is \( F(\theta) = e^{\frac{i\lambda \theta}{k+1}} \). But in this case, we have \( F_{k, \lambda}(x, y) = \rho^{-k}F(\theta) \) is a non-smooth solution of the equation \( G_{k, \lambda}f(x, y) = 0 \) (see [3]); hence, hypoellipticity for \( G_{k, \lambda} \) fails in this case. If \( \lambda \neq 2N(k + 1) \) and \( \lambda \neq (2N + 1)(k + 1) \) then we should choose

\[
C_3 = \frac{iC_4 \left( e^{\frac{i\pi \lambda}{k+1}} \int_0^\pi |\sin s|^{-\frac{k}{k+1}} e^{-\frac{i\lambda s}{k+1}} ds + e^{-\frac{i\pi \lambda}{k+1}} \int_{-\pi}^0 |\sin s|^{-\frac{k}{k+1}} e^{-\frac{i\lambda s}{k+1}} ds \right)}{2 \sin \frac{\pi \lambda}{k+1}}
\]

to obtain the only periodic solution. In this case, the function \( F_{k, \lambda}(x, y) = \rho^{-k}F(\theta) \) will be our desired fundamental solution.

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