

Title	Gevrey Regularity of Solutions of Semilinear Hypoelliptic Equations on the Plane (Microlocal Analysis and Related Topics)
Author(s)	Tri, Nguyen Minh
Citation	数理解析研究所講究録 (2002), 1261: 140-149
Issue Date	2002-05
URL	http://hdl.handle.net/2433/42009
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Gevrey Regularity of Solutions of Semilinear Hypoelliptic Equations on the Plane

Nguyen Minh TRI

§1. Introduction.

In this note we discuss the Gevrey regularity (in particular, the analyticity) of solutions of semilinear elliptic degenerate equations of Grushin's type on \mathbb{R}^2 . Most of the results will appear in [1]. Some results are new and they are presented here for the first time. We confine ourself with consideration of a model equation. Precisely, we will consider the following equation

$$(1) \quad G_{k,\lambda}f + \Psi\left(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}\right) = 0 \text{ in a domain } \Omega \subset \mathbb{R}^2,$$

where

$$G_{k,\lambda} = \frac{\partial^2}{\partial x^2} + x^{2k} \frac{\partial^2}{\partial y^2} + i\lambda x^{k-1} \frac{\partial}{\partial y}$$

with $(x, y) \in \Omega \subset \mathbb{R}^2, \lambda \in \mathbb{C}, i = \sqrt{-1}$ and $k \in \mathbb{Z}_+, \Omega$ is a bounded domain in \mathbb{R}^2 . Let us define the following quantities

$$R = (x^{k+1} + u^{k+1})^2 + (k+1)^2(y-v)^2, p = \frac{4x^{k+1}u^{k+1}}{R},$$

$$A_+ = x^{k+1} + u^{k+1} + i(k+1)(y-v), A_- = x^{k+1} + u^{k+1} - i(k+1)(y-v),$$

$$M = A_+^{-\frac{k+\lambda}{2k+2}} A_-^{-\frac{k-\lambda}{2k+2}},$$

here we take $z_1^{z_2} = e^{z_2 \ln z_1}$ for $z_1, z_2 \in \mathbb{C}$ and if $z_1 = re^{i\varphi}, -\pi < \varphi \leq \pi$ then $\ln z_1 = \ln r + i\varphi$. First, we will find the uniform fundamental solution of $G_{k,\lambda}$, that is

$$G_{k,\lambda}F_{k,\lambda}(x, y, u, v) = \delta(x-u, y-v),$$

in the following form

$$F_{k,\lambda}(x, y, u, v) = F(p)M.$$

After some computations we arrive at

$$\begin{aligned} G_{k,\lambda}F_{k,\lambda} &= 16(k+1)^2 u^{2k+2} x^{2k} \left[(u^{k+1} - x^{k+1})^2 + (k+1)^2(y-v)^2 \right] \times \\ &\quad \times MR^{-3}F''(p) + 4(k+1)x^{k-1}u^{k+1} [k(x^{2k+2} + u^{2k+2} + (k+1)^2(y-v)^2) \\ &\quad - (6k+4)x^{k+1}u^{k+1}] MR^{-2}F'(p) + (\lambda^2 - k^2)x^{k-1}u^{k+1} MR^{-1}F(p). \end{aligned}$$

[1] N. M. Tri, To appear in J. Math. Sci. Univ. Tokyo.

Therefore, if $F(p)$ satisfies the following hypergeometric equation

$$(2) \quad p(1-p)F''(p) + [c - (1+a+b)p]F'(p) - abF(p) = 0,$$

with $a = \frac{k+\lambda}{2k+2}$, $b = \frac{k-\lambda}{2k+2}$, $c = \frac{k}{k+1}$, then formally we will have

$$G_{k,\lambda}F_{k,\lambda} = 0.$$

The general solutions of (2) are

$$F(p) = C_1 F\left(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\right) + C_2 p^{\frac{1}{k+1}} F\left(\frac{k+2+\lambda}{2k+2}, \frac{k+2-\lambda}{2k+2}, \frac{k+2}{k+1}, p\right),$$

where $F(a, b, c, p)$ is the Gauss hypergeometric function and C_1, C_2 are some complex constants [2].

§2. Case k is odd.

Since k is odd, we note that $0 \leq p \leq 1$. Moreover, $p = 1$ if and only if $x = \pm u \neq 0, y = v$. If $u = 0, v = 0$ then $p = 0$; therefore, from the result of [3]

$$G_{k,\lambda}F\left(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\right)M = -\frac{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k}{k+1}\right)}{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)}\delta(x, y)$$

we should choose

$$C_1 = -\frac{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)}{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k}{k+1}\right)}.$$

If $u \neq 0$ then the singularities of $F_{k,\lambda}(x, y, u, v)$ will be located at the one of $F(p)$. On the other hand, $F(p)$, with $0 \leq p \leq 1$, has singularity only when $p = 1$. As $p \rightarrow 1$ we have the following asymptotic expansions (see [2])

$$F\left(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\right) = -\frac{\Gamma\left(\frac{k}{k+1}\right)}{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)}\log(1-p) + O(1),$$

$$F\left(\frac{k+2+\lambda}{2k+2}, \frac{k+2-\lambda}{2k+2}, \frac{k+2}{k+1}, p\right) = -\frac{\Gamma\left(\frac{k+2}{k+1}\right)}{\Gamma\left(\frac{k+2+\lambda}{2k+2}\right)\Gamma\left(\frac{k+2-\lambda}{2k+2}\right)}\log(1-p) + O(1).$$

[2] H. Bateman, and A. Erdelyi, 1953, vol I, p. 74.

[3] N. M. Tri, J. Math. Sci. Univ. Tokyo, vol. 6, 1999, pp. 437-452.

We expect that $F_{k,\lambda}(x, y, u, v)$ has singularity only when $x = u, y = v$. Since $p^{\frac{1}{k+1}} = (4R^{-1})^{\frac{1}{k+1}}xu \rightarrow -1$ when $(x, y) \rightarrow (-u, v)$, we should choose

$$C_2 = -\frac{\Gamma\left(\frac{k+2+\lambda}{2k+2}\right)\Gamma\left(\frac{k+2-\lambda}{2k+2}\right)}{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k+2}{k+1}\right)}$$

such that $F(p)$ has no singularity at $x = -u, y = v$. Note that the following conditions

$$(3) \quad \lambda \neq \pm[2N(k+1) + k], \lambda \neq \pm[2N(k+1) + k + 2],$$

where N is a non-negative integer, guarantee that $C_1, C_2 < \infty$ and hence $F(p)$ has logarithm growth (if $u \neq 0$) at $(x, y) = (u, v)$.

Definition. The parameter λ is called admissible if λ satisfies the condition (3).

Therefore, if λ is admissible then we expect that the function $F(p)M$, or

$$F_{k,\lambda}(x, y, u, v) = -\frac{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)F\left(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\right)}{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k}{k+1}\right)A_+^{\frac{k+\lambda}{2k+2}}A_-^{\frac{k-\lambda}{2k+2}}} - \frac{xu\Gamma\left(\frac{k+2+\lambda}{2k+2}\right)\Gamma\left(\frac{k+2-\lambda}{2k+2}\right)F\left(\frac{k+2+\lambda}{2k+2}, \frac{k+2-\lambda}{2k+2}, \frac{k+2}{k+1}, p\right)}{2^{2-\frac{1}{k+1}}\pi\Gamma\left(\frac{k+2}{k+1}\right)A_+^{\frac{k+2+\lambda}{2k+2}}A_-^{\frac{k+2-\lambda}{2k+2}}},$$

will be our desired uniform fundamental solution. Indeed, we have

Theorem 1. Assume that λ is admissible. Then

$$G_{k,\lambda}F_{k,\lambda}(x, y, u, v) = \delta(x - u, y - v).$$

Remark 1. A similar expression for $F_{k,0}$ is also given in [4].

Let us denote $X'_1 = \frac{\partial}{\partial u} - iu^k \frac{\partial}{\partial v}$, $X'_2 = \frac{\partial}{\partial u} + iu^k \frac{\partial}{\partial v}$, and $G'_{k,\lambda} = X'_2 X'_1 + i(\lambda + k)u^{k-1} \frac{\partial}{\partial v}$. Noting that $F_{k,\lambda}(x, y, u, v) = F_{k,-\lambda}(u, v, x, y)$, from Theorem 1 we can easily deduce

[4] R. Beals, Journées Équations aux dérivées partielles, Saint-Jean-de-Monts, 1998, pp. 11-10

Proposition 1 (Representation formula). *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with piece-wise smooth boundary, $f \in C^2(\bar{\Omega})$ and λ is admissible then we have*

$$(4) \quad f(x, y) = \int_{\Omega} F_{k,\lambda}(x, y, u, v) G'_{k,\lambda} f(u, v) dudv - \\ - \int_{\partial\Omega} F_{k,\lambda}(x, y, u, v) B'_1(f(u, v), k, -\lambda) ds + \int_{\partial\Omega} f(u, v) B'_2(F_{k,\lambda}(x, y, u, v), k) ds,$$

where

$$B'_1(f(u, v), k, -\lambda) = (\nu_1 - iu^k \nu_2) X'_2 f(u, v) - i(-\lambda + k) u^{k-1} \nu_2 f(u, v), \\ B'_2(F_{k,\lambda}(x, y, u, v), k) = (\nu_1 + iu^k \nu_2) X'_1 F_{k,\lambda}(x, y, u, v),$$

and $\nu = (\nu_1, \nu_2)$ is the unit outward normal vector on $\partial\Omega$.

Now, we re-state a well-known theorem on hypoellipticity of $G_{k,\lambda}$ as follows

Theorem 2. *$G_{k,\lambda}$ is hypoelliptic if and only if the hypergeometric equation (2) has no bounded solution on the interval $[0, 1]$.*

Proof. Here, with the help of $F_{k,\lambda}$, we give a proof, which is alternative to a well-known classical proof based on the theory of pseudo-differential operators. Suppose that $f \in C^2(\bar{\Omega})$ and $G_{k,\lambda} f(x, y) = h(x, y)$ where $h \in C^\infty(\bar{\Omega})$. Then we can express f through h as in (4), with $G'_{k,\lambda} f(u, v)$ replaced by $h(u, v)$. It is clear that the boundary integrals give $C^\infty(\Omega)$ functions. For the volume integral, we see that $\frac{\partial F_{k,\lambda}}{\partial y} = -\frac{\partial F_{k,\lambda}}{\partial v}$. Therefore, by integration by parts, we can differentiate the integral in x one time and in y as many times as we want to. And the resulting functions are continuous. We will complete the proof if we are able to show that if $f \in C^{n-1}(\Omega)$ then $f \in C^n(\Omega)$ for every positive integer n . This is the case because we already have $\frac{\partial^n f}{\partial y^n}$, $\frac{\partial^n f}{\partial y^{n-1} \partial x}$ and $\frac{\partial^{\alpha+\beta} u}{\partial y^\alpha \partial x^\beta}$, $\alpha + \beta \leq n - 1$ belong to $C(\Omega)$ from the above argument and assumption. We have to show that $\frac{\partial^n u}{\partial y^{n-2} \partial x^2}, \dots, \frac{\partial^n u}{\partial x^n} \in C(\Omega)$. Suppose that all the derivatives $\frac{\partial^n f}{\partial y^n}, \frac{\partial^n f}{\partial y^{n-1} \partial x}, \dots, \frac{\partial^n f}{\partial y^{n-j} \partial x^j}, 1 \leq j \leq n - 1$ are continuous. We shall prove that $\frac{\partial^n f}{\partial y^{n-j-1} \partial x^{j+1}} \in C(\Omega)$. Indeed, we have

$$(5) \quad \frac{\partial^2 f}{\partial x^2} = h - x^{2k} \frac{\partial^2 f}{\partial y^2} - i\lambda x^{k-1} \frac{\partial f}{\partial y}.$$

Therefore, differentiating $\frac{\partial^{n-2}}{\partial y^{n-j-1} \partial x^{j-1}}$ both sides of (5) gives

$$\begin{aligned} \frac{\partial^n f}{\partial y^{n-j-1} \partial x^{j+1}} &= \frac{\partial^{n-2} h}{\partial y^{n-j-1} \partial x^{j-1}} \\ &- \sum_{i=0}^j \binom{j}{i} 2k(2k-1) \cdots (2k-i+1) x^{2k-i} \frac{\partial^{n-i} f}{\partial y^{n-j+1} \partial x^{j-i-1}} \\ &- i\lambda \sum_{i=0}^j \binom{j}{i} (k-1)(k-2) \cdots (k-i) x^{k-i-1} \frac{\partial^{n-i-1} f}{\partial y^{n-j} \partial x^{j-i-1}} \in C(\Omega). \square \end{aligned}$$

Actually, a more detailed examination of the proof of Theorem 2 would show that the integral operators

$$\begin{aligned} K : h &\longrightarrow K(h)(x, y) = \int_{\Omega} F_{k,\lambda}(x, y, u, v) h(u, v) du dv, \\ {}^t K : h &\longrightarrow {}^t K(h)(x, y) = \int_{\Omega} F_{k,\lambda}(u, v, x, y) h(u, v) du dv \end{aligned}$$

map $C_0^\infty(\Omega)$ into $C^\infty(\Omega)$. In other words, K and ${}^t K$ are separately regular. Since $F_{k,\lambda}$ is a C^∞ function in the complement of the diagonal of $\Omega \times \Omega$, we conclude that K and ${}^t K$ are very regular.

Next, we introduce some notations

$$\Xi_t = \{(\alpha, \beta, \gamma) \in \mathbf{Z}_+^3 : \alpha + \beta \leq t, kt \geq \gamma \geq \alpha + (1+k)\beta - t\}.$$

For a function $f(x, y)$ on \mathbf{R}^2 , we write $\partial_1^\alpha f, \partial_2^\beta f, \partial_{1,2}^{\alpha,\beta} f, \gamma \partial_{\alpha,\beta} f$ for $\frac{\partial^\alpha f(x,y)}{\partial x^\alpha}$, $\frac{\partial^\beta f(x,y)}{\partial y^\beta}$, $\frac{\partial^{\alpha+\beta} f(x,y)}{\partial x^\alpha \partial y^\beta}$, $x^\gamma \frac{\partial^{\alpha+\beta} f(x,y)}{\partial x^\alpha \partial y^\beta}$, respectively. For $m \in \mathbf{Z}^+$, let us denote by $\mathbf{H}_{loc}^m(\Omega)$ the space of all function $f \in L_{loc}^2(\Omega)$ such that for any compact K of Ω we have $\sum_{(\alpha,\beta,\gamma) \in \Xi_m} \|\gamma \partial_{\alpha,\beta} f\|_{L^2(K)} < \infty$. Now we are in a position to formulate the main theorem of this section.

Theorem 3. *Assume that $m \geq 2k^2 + 6k + 5$. Let f be a $\mathbf{H}_{loc}^m(\Omega)$ solution of the equation (1) and $\Psi \in G^s$. Then $f \in G^s$. In particular, if Ψ is analytic in its arguments then so is f .*

Proof. The proof of Theorem 3 consists of Theorem 4 and Theorem 5 below. The proof follows the scheme : $f \in \mathbf{H}_{loc}^m \implies f \in C^\infty(\Omega) \implies f \in A(\Omega)$. \square

Theorem 4. *Let Ψ be a C^∞ -function of its arguments and $m \geq 2k^2 + 6k + 5$. Assume that $f \in \mathbf{H}_{loc}^m(\Omega)$ is a solution of the equation (1) then $f \in C^\infty(\Omega)$.*

Proof. Theorem 4 can be proved with the help of Proposition 2. \square

Proposition 2. Let $m \geq 2k^2 + 6k + 5$. Assume that $f \in \mathbb{H}_{loc}^m(\Omega)$. Then $\Psi(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}) \in \mathbb{H}_{loc}^{m-1}(\Omega)$.

Next, put $r_0 = 2k + 2$. For $r \in \mathbb{Z}_+$ let Γ_r denote the set of pairs of multi-indices (α, β) such that $\Gamma_r = \Gamma_r^1 \cup \Gamma_r^2$ where

$$\Gamma_r^1 = \{(\alpha, \beta) : \alpha \leq r_0, 2\alpha + \beta \leq r\}, \Gamma_r^2 = \{(\alpha, \beta) : \alpha \geq r_0, \alpha + \beta \leq r - r_0\}.$$

Define the following norm

$$|f, \Omega|_r = \max_{(\alpha, \beta) \in \Gamma_r} |\partial_1^\alpha \partial_2^\beta f, \Omega| + \max_{\substack{(\alpha, \beta) \in \Gamma_r \\ \alpha \geq 1, \beta \geq 1}} \max_{(x, y) \in \Omega} |\partial_1^{\alpha+2} \partial_2^\beta f|,$$

where $|f, \Omega| = \max_{(x, y) \in \Omega} \left(|f| + \left| \frac{\partial f}{\partial x} \right| + \left| x^k \frac{\partial f}{\partial y} \right| \right)$.

Theorem 5. Let f be a C^∞ solution of the equation (1) and $\Psi \in G^s$. Then $f \in G^s$. In particular, if Ψ is analytic in its arguments then so is f .

Proof. Theorem 5 can be proved with the help of Proposition 3, Corollary 1, Lemmas 2-4. \square

Proposition 3. Assume that $\Psi \in G^s$. Then there exist constants C, D such that for every $H_0 \geq 1, H_1 \geq CH_0^{2k+3}$ if

$$|f, \Omega|_d \leq H_0 H_1^{(d-r_0-2)} (d - r_0 - 2)!^s, \quad 0 \leq d \leq N + 1, r_0 + 2 \leq N$$

then

$$\max_{(x, y) \in \Omega} \left| \partial_1^\alpha \partial_2^\beta \Psi \left(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y} \right) \right| \leq DH_0 H_1^{N-r_0-1} (N - r_0 - 1)!^s$$

for every $(\alpha, \beta) \in \Gamma_{N+1}$.

Corollary 1. Under the same hypotheses of Proposition 3 with $d \leq N + 1$ replaced by $d \leq N$, then

$$\max_{x \in \Omega} \left| \partial_1^\alpha \partial_2^\beta \Psi \left(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y} \right) \right| \leq D \left(|f, \Omega|_{N+1} + H_0 H_1^{N-r_0-1} (N - r_0 - 1)!^s \right)$$

for every $(\alpha, \beta) \in \Gamma_{N+1}$.

Since $G_{k, \lambda}$ is elliptic if $x \neq 0$, it suffices to consider the case $(0, 0) \in \Omega$ and Ω is a small neighborhood of $(0, 0)$. Let us define the distance

$$\rho((u, v), (x, y)) = \begin{cases} \max \{ |x^{k+1} - u^{k+1}|, (k+1)|y - v| \}, & \text{for } xu \geq 0 \\ \max \{ x^{k+1} + u^{k+1}, (k+1)|y - v| \}, & \text{for } xu \leq 0. \end{cases}$$

For two sets S_1, S_2 , the distance between them is defined as

$$\rho(S_1, S_2) = \inf_{(x,y) \in S_1, (u,v) \in S_2} \rho((x,y), (u,v)).$$

Let $V^T (T \leq 1)$ be the cube with edges of size (in the ρ metric) $2T$ which are parallel to the coordinate axes and centered at $(0,0)$. Denote by V_δ^T the sub-cube which is homothetic with V^T and such that the distance between its boundary and the boundary of V^T is δ . We shall prove by induction that if T is small enough then there exist constants H_0, H_1 with $H_1 \geq CH_0^{2k+3}$ such that

$$(6) \quad |f, V_\delta^T|_n \leq H_0 \quad \text{for } 0 \leq n \leq 6k+4,$$

and

$$(7) \quad |f, V_\delta^T|_n \leq H_0 \left(\frac{H_1}{\delta} \right)^{n-r_0-2} (n-r_0-2)!^s := Q_{n-1}$$

for $n \geq 6k+4$, and δ sufficiently small. Hence the desired conclusion follows. (6) follows easily from the C^∞ smoothness assumption on f . Assume that (7) holds for $n = N$. We shall prove it for $n = N+1$. Put $\delta' = \delta(1-1/N)$, $\delta'' = \delta(1-4/N)$. Fix $(x,y) \in V_{\delta'}^T$ and then define $\sigma = \rho((x,y), \partial V^T)$ and $\tilde{\sigma} = \sigma/N$. Let $V_{\tilde{\sigma}}(x,y)$ denote the cube with center at (x,y) and edges of length $2\tilde{\sigma}$ which are parallel to the coordinate axes, and $S_{\tilde{\sigma}}(x,y)$ the boundary of $V_{\tilde{\sigma}}(x,y)$. Note that $\sigma \geq \delta$, and $V_{\tilde{\sigma}}(x,y) \subset V_{\delta'}^T$. Let $E_1, E_3 (E_2, E_4)$ be edges of $S_{\tilde{\sigma}}(x,y)$ which are parallel to $Ox(Oy)$ respectively. We have to estimate $\max_{(x,y) \in V_{\delta'}^T} |\gamma \partial_{\alpha,\beta} (\partial_1^{\alpha_1} \partial_2^{\beta_1} f)|$ for all $(\alpha, \beta, \gamma) \in \Xi_1$, $(\alpha_1, \beta_1) \in \Gamma_{N+1}$, and $\max_{(x,y) \in V_{\delta'}^T} |(\partial_1^{2+\alpha_1} \partial_2^{\beta_1} f)|$ for all $(\alpha_1, \beta_1) \in \Gamma_{N+1}$, $\alpha_1 \geq 1, \beta_1 \geq 1$. Let us abbreviate $\frac{\partial^\alpha}{\partial u^\alpha}, \frac{\partial^\beta}{\partial v^\beta}, \frac{\partial^{\alpha+\beta}}{\partial u^\alpha \partial v^\beta}$ as $\partial_u^\alpha, \partial_v^\beta, \partial_u^\alpha \partial_v^\beta$, respectively.

Lemma 2. *Assume that $(\alpha, \beta, \gamma) \in \Xi_1$ and $(\alpha_1, \beta_1) \in \Gamma_{N+1}$. Then if $\alpha_1 \geq 1, \beta_1 \geq 1$ there exists a constant C such that*

$$\max_{(x,y) \in V_{\delta'}^T} |\gamma \partial_{\alpha,\beta} (\partial_1^{\alpha_1} \partial_2^{\beta_1} f(x,y))| \leq C \left(T^{\frac{1}{k+1}} |f, V_{\delta'}^T|_{N+1} + Q_N \left(T^{\frac{1}{k+1}} + \frac{1}{H_1} \right) \right).$$

Lemma 3. *Assume that $(\alpha, \beta, \gamma) \in \Xi_1$. Then there exists a constant C such that*

$$\max_{(x,y) \in V_{\delta'}^T} |\gamma \partial_{\alpha,\beta} (\partial_2^{N+1} f(x,y))| \leq C \left(T^{\frac{1}{k+1}} |f, V_{\delta''}^T|_{N+1} + Q_N \left(T^{\frac{1}{k+1}} + \frac{1}{H_1} \right) \right).$$

Lemma 4. Assume that $(\alpha, \beta, \gamma) \in \Xi_1$. Then there exists a constant C such that

$$\max_{(x,y) \in V_\delta^T} |\gamma \partial_{\alpha,\beta} (\partial_1^{N-r_0+1} f(x,y))| \leq C \left(T^{\frac{1}{k+1}} |f, V_{\delta'}^T|_{N+1} + Q_N \left(T^{\frac{1}{k+1}} + \frac{1}{H_1} \right) \right).$$

Lemma 5. Assume that $(\alpha_1, \beta_1) \in \Gamma_{N+1} \setminus \Gamma_N$, $\alpha_1 \geq 1, \beta_1 \geq 1$. Then there exists a constant C such that

$$\max_{(x,y) \in V_\delta^T} |(\partial_1^{\alpha_1+2} \partial_2^{\beta_1} f(x,y))| \leq C \left(T^{\frac{1}{k+1}} |f, V_{\delta''}^T|_{N+1} + Q_N \left(T^{\frac{1}{k+1}} + \frac{1}{H_1} \right) \right).$$

§3. Case k is even.

A. First, we consider the case $\lambda = 2N(k+1)$, where N is an integer. In this case we will prove a similar result as in §2 by establishing the explicit uniform fundamental solutions of $G_{k,2N(k+1)}$. Let us maintain the notations used for $p, A_+, A_-, M, F_{k,\lambda}, \dots$ from the very beginning of the paper (now, of course, with an even k). If $(u, v) \neq (0, v)$ is fixed then the real parts of A_+, A_- change sign when (x, y) passes through $(-u, v)$. Therefore, $M = A_+^{-\frac{k+\lambda}{2k+2}} A_-^{-\frac{k-\lambda}{2k+2}}$ may have singularities along the half-line (x, v) with $x \leq -u$ for an arbitrary complex number λ . But if $\lambda = 2N(k+1)$ then it is not difficult to see that $M = A_+^{-\frac{k+\lambda}{2k+2}} A_-^{-\frac{k-\lambda}{2k+2}}$ is smooth along the half-line (x, v) with $x < -u$, that is $M(\cdot, \cdot, u, v) \in C^\infty(\mathbb{R}^2 \setminus \{(u, v), (-u, v)\})$. Moreover, when k is even and $u \neq 0$ we have $-\infty \leq p \leq 1$. More precisely, $p \rightarrow 1$ when $(x, y) \rightarrow (u, v)$, and $p \rightarrow -\infty$ when $(x, y) \rightarrow (-u, v)$. If $N < 0$ and $p \rightarrow -\infty$ then from the asymptotic expansions of hypergeometric functions (see [2], p. 63) we should choose the expressions for constants C_1, C_2 as in the beginning of the paper (with λ replaced by $2N(k+1)$). And we will have $F_{k,2N(k+1)}(\cdot, \cdot, u, v) \in C^\infty(\mathbb{R}^2 \setminus (u, v))$, with

$$F_{k,2N(k+1)}(-u, v, u, v) = 0.$$

Similar conclusions hold for $F_{k,2N(k+1)}(x, y, u, v)$ when $N > 0$. If $N = 0$ then $F_{k,0}(\cdot, \cdot, u, v) \in C^\infty(\mathbb{R}^2 \setminus (u, v))$, with

$$F_{k,0}(-u, v, u, v) = -\frac{\cot \frac{k\pi}{2k+2}}{4u^k}.$$

Theorem 6. Let $\Psi \in G^s$. Assume that $m \geq 2k^2 + 6k + 5$, $\lambda = 2N(k + 1)$, and f is a $H_{loc}^m(\Omega)$ solution of the equation (1). Then $f \in G^s$. In particular, if Ψ is analytic in its arguments then so is f .

Proof. Almost all the arguments used for the case when k is odd can be applied here. Therefore, we only give the sketch of the proof. Instead of the distance ρ in §2 we use the following metric

$$\tilde{\rho}((u, v), (x, y)) = \max \{|x^{k+1} - u^{k+1}|, (k + 1)|y - v|\}. \square$$

B. In this sub-section we will present some computations for finding the fundamental solutions of $G_{k,\lambda}$ with source at the origin $(0, 0)$ for λ other than the values $2N(k + 1)$ considered in sub-section A. Make the following change of variables

$$x = \rho |\sin \theta|^{\frac{1}{k+1}} \text{sign}(\sin \theta), y = \frac{\rho^{k+1}}{k+1} \cos \theta, \theta \in (-\pi, \pi).$$

Then $G_{k,\lambda}$ will be transformed into

$$\begin{aligned} & \text{sign}(\sin \theta) |\sin \theta|^{\frac{k-1}{k+1}} \left(\sin \theta \frac{\partial^2}{\partial \rho^2} + (k+1)^2 \rho^{-2} \sin \theta \frac{\partial^2}{\partial \theta^2} + \right. \\ & \left. (i\lambda \cos \theta + (k+1) \sin \theta) \rho^{-1} \frac{\partial}{\partial \rho} + (k+1) \rho^{-2} (k \cos \theta - i\lambda \sin \theta) \frac{\partial}{\partial \theta} \right). \end{aligned}$$

If we seek the fundamental solution in the form $F_{k,\lambda}(x, y) = \rho^{-k} F(\theta)$ then $F(\theta)$ must satisfy the following equation

$$(8) \quad \begin{aligned} & (k+1)^2 \sin \theta F''(\theta) + (k+1)(k \cos \theta - i\lambda \sin \theta) F'(\theta) - \\ & - ik\lambda \cos \theta F(\theta) = 0. \end{aligned}$$

The general solutions of (8) are

$$F(\theta) = \left(C_3 + C_4 \int_0^\theta |\sin s|^{-\frac{k}{k+1}} e^{-\frac{i\lambda s}{k+1}} ds \right) e^{\frac{i\lambda \theta}{k+1}},$$

where C_3 and C_4 are some complex constants. Among all these solutions, we are interested in finding a non-trivial periodic solution. When $\lambda = 2N(k + 1)$ - this case was considered in sub-section A - the periodic solution is $F(\theta) = e^{\frac{i\lambda \theta}{k+1}}$, and the function $F_{k,\lambda}(x, y) = \rho^{-k} F(\theta)$ serves as a fundamental solution. When

$\lambda = (2N + 1)(k + 1)$ then the periodic solution again is $F(\theta) = e^{\frac{i\lambda\theta}{k+1}}$. But in this case, we have $F_{k,\lambda}(x, y) = \rho^{-k}F(\theta)$ is a non-smooth solution of the equation $G_{k,\lambda}f(x, y) = 0$ (see [3]); hence, hypoellipticity for $G_{k,\lambda}$ fails in this case. If $\lambda \neq 2N(k + 1)$ and $\lambda \neq (2N + 1)(k + 1)$ then we should choose

$$C_3 = \frac{iC_4 \left(e^{\frac{i\pi\lambda}{k+1}} \int_0^\pi |\sin s|^{-\frac{k}{k+1}} e^{-\frac{i\lambda s}{k+1}} ds + e^{-\frac{i\pi\lambda}{k+1}} \int_{-\pi}^0 |\sin s|^{-\frac{k}{k+1}} e^{-\frac{i\lambda s}{k+1}} ds \right)}{2 \sin \frac{\pi\lambda}{k+1}}$$

to obtain the only periodic solution. In this case, the function $F_{k,\lambda}(x, y) = \rho^{-k}F(\theta)$ will be our desired fundamental solution.

Address:

Institute of Mathematics, P. O. Box 631, Boho 10000, Hanoi, Vietnam

e-mail:triminh@thevinh.ncst.ac.vn

and

Division of Applied Mathematics, Korea Advanced Institute of Science and Technology,

373-1 Kusong-dong, Yusong-ku, Taejon 305-701, S. Korea

e-mail:triminh@amath.kaist.ac.kr