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Gevrey Regularity of Solutions of Semilinear Hypoelliptic Equations on the Plane

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§1. Introduction.

In this note we discuss the Gevrey regularity (in particular, the analyticity) of solutions of semilinear elliptic degenerate equations of Grushin's type on \mathbb{R}^2 . Most of the results will appear in [1]. Some results are new and they are presented here for the first time. We confine ourself with consideration of a model equation. Precisely, we will consider the following equation

$$(1) \quad G_{k,\lambda}f + \Psi\left(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}\right) = 0 \text{ in a domain } \Omega \subset \mathbb{R}^2,$$

where

$$G_{k,\lambda} = \frac{\partial^2}{\partial x^2} + x^{2k} \frac{\partial^2}{\partial y^2} + i\lambda x^{k-1} \frac{\partial}{\partial y}$$

with $(x, y) \in \Omega \subset \mathbb{R}^2, \lambda \in \mathbb{C}, i = \sqrt{-1}$ and $k \in \mathbb{Z}_+, \Omega$ is a bounded domain in \mathbb{R}^2 . Let us define the following quantities

$$R = (x^{k+1} + u^{k+1})^2 + (k+1)^2(y-v)^2, p = \frac{4x^{k+1}u^{k+1}}{R},$$

$$A_+ = x^{k+1} + u^{k+1} + i(k+1)(y-v), A_- = x^{k+1} + u^{k+1} - i(k+1)(y-v),$$

$$M = A_+^{-\frac{k+\lambda}{2k+2}} A_-^{-\frac{k-\lambda}{2k+2}},$$

here we take $z_1^{z_2} = e^{z_2 \ln z_1}$ for $z_1, z_2 \in \mathbb{C}$ and if $z_1 = re^{i\varphi}, -\pi < \varphi \leq \pi$ then $\ln z_1 = \ln r + i\varphi$. First, we will find the uniform fundamental solution of $G_{k,\lambda}$, that is

$$G_{k,\lambda}F_{k,\lambda}(x, y, u, v) = \delta(x-u, y-v),$$

in the following form

$$F_{k,\lambda}(x, y, u, v) = F(p)M.$$

After some computations we arrive at

$$\begin{aligned} G_{k,\lambda}F_{k,\lambda} &= 16(k+1)^2 u^{2k+2} x^{2k} \left[(u^{k+1} - x^{k+1})^2 + (k+1)^2(y-v)^2 \right] \times \\ &\quad \times MR^{-3}F''(p) + 4(k+1)x^{k-1}u^{k+1} [k(x^{2k+2} + u^{2k+2} + (k+1)^2(y-v)^2) \\ &\quad - (6k+4)x^{k+1}u^{k+1}] MR^{-2}F'(p) + (\lambda^2 - k^2)x^{k-1}u^{k+1} MR^{-1}F(p). \end{aligned}$$

[1] N. M. Tri, To appear in J. Math. Sci. Univ. Tokyo.

Therefore, if $F(p)$ satisfies the following hypergeometric equation

$$(2) \quad p(1-p)F''(p) + [c - (1+a+b)p]F'(p) - abF(p) = 0,$$

with $a = \frac{k+\lambda}{2k+2}$, $b = \frac{k-\lambda}{2k+2}$, $c = \frac{k}{k+1}$, then formally we will have

$$G_{k,\lambda}F_{k,\lambda} = 0.$$

The general solutions of (2) are

$$F(p) = C_1 F\left(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\right) + C_2 p^{\frac{1}{k+1}} F\left(\frac{k+2+\lambda}{2k+2}, \frac{k+2-\lambda}{2k+2}, \frac{k+2}{k+1}, p\right),$$

where $F(a, b, c, p)$ is the Gauss hypergeometric function and C_1, C_2 are some complex constants [2].

§2. Case k is odd.

Since k is odd, we note that $0 \leq p \leq 1$. Moreover, $p = 1$ if and only if $x = \pm u \neq 0, y = v$. If $u = 0, v = 0$ then $p = 0$; therefore, from the result of [3]

$$G_{k,\lambda}F\left(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\right)M = -\frac{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k}{k+1}\right)}{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)}\delta(x, y)$$

we should choose

$$C_1 = -\frac{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)}{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k}{k+1}\right)}.$$

If $u \neq 0$ then the singularities of $F_{k,\lambda}(x, y, u, v)$ will be located at the one of $F(p)$. On the other hand, $F(p)$, with $0 \leq p \leq 1$, has singularity only when $p = 1$. As $p \rightarrow 1$ we have the following asymptotic expansions (see [2])

$$F\left(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\right) = -\frac{\Gamma\left(\frac{k}{k+1}\right)}{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)}\log(1-p) + O(1),$$

$$F\left(\frac{k+2+\lambda}{2k+2}, \frac{k+2-\lambda}{2k+2}, \frac{k+2}{k+1}, p\right) = -\frac{\Gamma\left(\frac{k+2}{k+1}\right)}{\Gamma\left(\frac{k+2+\lambda}{2k+2}\right)\Gamma\left(\frac{k+2-\lambda}{2k+2}\right)}\log(1-p) + O(1).$$

[2] H. Bateman, and A. Erdelyi, 1953, vol I, p. 74.

[3] N. M. Tri, J. Math. Sci. Univ. Tokyo, vol. 6, 1999, pp. 437-452.

We expect that $F_{k,\lambda}(x, y, u, v)$ has singularity only when $x = u, y = v$. Since $p^{\frac{1}{k+1}} = (4R^{-1})^{\frac{1}{k+1}}xu \rightarrow -1$ when $(x, y) \rightarrow (-u, v)$, we should choose

$$C_2 = -\frac{\Gamma\left(\frac{k+2+\lambda}{2k+2}\right)\Gamma\left(\frac{k+2-\lambda}{2k+2}\right)}{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k+2}{k+1}\right)}$$

such that $F(p)$ has no singularity at $x = -u, y = v$. Note that the following conditions

$$(3) \quad \lambda \neq \pm[2N(k+1) + k], \lambda \neq \pm[2N(k+1) + k + 2],$$

where N is a non-negative integer, guarantee that $C_1, C_2 < \infty$ and hence $F(p)$ has logarithm growth (if $u \neq 0$) at $(x, y) = (u, v)$.

Definition. The parameter λ is called admissible if λ satisfies the condition (3).

Therefore, if λ is admissible then we expect that the function $F(p)M$, or

$$F_{k,\lambda}(x, y, u, v) = -\frac{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)F\left(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\right)}{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k}{k+1}\right)A_+^{\frac{k+\lambda}{2k+2}}A_-^{\frac{k-\lambda}{2k+2}}} - \frac{xu\Gamma\left(\frac{k+2+\lambda}{2k+2}\right)\Gamma\left(\frac{k+2-\lambda}{2k+2}\right)F\left(\frac{k+2+\lambda}{2k+2}, \frac{k+2-\lambda}{2k+2}, \frac{k+2}{k+1}, p\right)}{2^{2-\frac{1}{k+1}}\pi\Gamma\left(\frac{k+2}{k+1}\right)A_+^{\frac{k+2+\lambda}{2k+2}}A_-^{\frac{k+2-\lambda}{2k+2}}},$$

will be our desired uniform fundamental solution. Indeed, we have

Theorem 1. Assume that λ is admissible. Then

$$G_{k,\lambda}F_{k,\lambda}(x, y, u, v) = \delta(x - u, y - v).$$

Remark 1. A similar expression for $F_{k,0}$ is also given in [4].

Let us denote $X'_1 = \frac{\partial}{\partial u} - iu^k \frac{\partial}{\partial v}$, $X'_2 = \frac{\partial}{\partial u} + iu^k \frac{\partial}{\partial v}$, and $G'_{k,\lambda} = X'_2 X'_1 + i(\lambda + k)u^{k-1} \frac{\partial}{\partial v}$. Noting that $F_{k,\lambda}(x, y, u, v) = F_{k,-\lambda}(u, v, x, y)$, from Theorem 1 we can easily deduce

[4] R. Beals, Journées Équations aux dérivées partielles, Saint-Jean-de-Monts, 1998, pp. 11-10

Proposition 1 (Representation formula). Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with piece-wise smooth boundary, $f \in C^2(\bar{\Omega})$ and λ is admissible then we have

$$(4) \quad f(x, y) = \int_{\Omega} F_{k,\lambda}(x, y, u, v) G'_{k,\lambda} f(u, v) dudv - \\ - \int_{\partial\Omega} F_{k,\lambda}(x, y, u, v) B'_1(f(u, v), k, -\lambda) ds + \int_{\partial\Omega} f(u, v) B'_2(F_{k,\lambda}(x, y, u, v), k) ds,$$

where

$$B'_1(f(u, v), k, -\lambda) = (\nu_1 - iu^k \nu_2) X'_2 f(u, v) - i(-\lambda + k) u^{k-1} \nu_2 f(u, v), \\ B'_2(F_{k,\lambda}(x, y, u, v), k) = (\nu_1 + iu^k \nu_2) X'_1 F_{k,\lambda}(x, y, u, v),$$

and $\nu = (\nu_1, \nu_2)$ is the unit outward normal vector on $\partial\Omega$.

Now, we re-state a well-known theorem on hypoellipticity of $G_{k,\lambda}$ as follows

Theorem 2. $G_{k,\lambda}$ is hypoelliptic if and only if the hypergeometric equation (2) has no bounded solution on the interval $[0, 1]$.

Proof. Here, with the help of $F_{k,\lambda}$, we give a proof, which is alternative to a well-known classical proof based on the theory of pseudo-differential operators. Suppose that $f \in C^2(\bar{\Omega})$ and $G_{k,\lambda} f(x, y) = h(x, y)$ where $h \in C^\infty(\bar{\Omega})$. Then we can express f through h as in (4), with $G'_{k,\lambda} f(u, v)$ replaced by $h(u, v)$. It is clear that the boundary integrals give $C^\infty(\Omega)$ functions. For the volume integral, we see that $\frac{\partial F_{k,\lambda}}{\partial y} = -\frac{\partial F_{k,\lambda}}{\partial v}$. Therefore, by integration by parts, we can differentiate the integral in x one time and in y as many times as we want to. And the resulting functions are continuous. We will complete the proof if we are able to show that if $f \in C^{n-1}(\Omega)$ then $f \in C^n(\Omega)$ for every positive integer n . This is the case because we already have $\frac{\partial^n f}{\partial y^n}$, $\frac{\partial^n f}{\partial y^{n-1} \partial x}$ and $\frac{\partial^{\alpha+\beta} u}{\partial y^\alpha \partial x^\beta}$, $\alpha + \beta \leq n - 1$ belong to $C(\Omega)$ from the above argument and assumption. We have to show that $\frac{\partial^n u}{\partial y^{n-2} \partial x^2}, \dots, \frac{\partial^n u}{\partial x^n} \in C(\Omega)$. Suppose that all the derivatives $\frac{\partial^n f}{\partial y^n}, \frac{\partial^n f}{\partial y^{n-1} \partial x}, \dots, \frac{\partial^n f}{\partial y^{n-j} \partial x^j}, 1 \leq j \leq n - 1$ are continuous. We shall prove that $\frac{\partial^n f}{\partial y^{n-j-1} \partial x^{j+1}} \in C(\Omega)$. Indeed, we have

$$(5) \quad \frac{\partial^2 f}{\partial x^2} = h - x^{2k} \frac{\partial^2 f}{\partial y^2} - i\lambda x^{k-1} \frac{\partial f}{\partial y}.$$

Therefore, differentiating $\frac{\partial^{n-2}}{\partial y^{n-j-1} \partial x^{j-1}}$ both sides of (5) gives

$$\begin{aligned} \frac{\partial^n f}{\partial y^{n-j-1} \partial x^{j+1}} &= \frac{\partial^{n-2} h}{\partial y^{n-j-1} \partial x^{j-1}} \\ &- \sum_{i=0}^j \binom{j}{i} 2k(2k-1) \cdots (2k-i+1) x^{2k-i} \frac{\partial^{n-i} f}{\partial y^{n-j+1} \partial x^{j-i-1}} \\ &- i\lambda \sum_{i=0}^j \binom{j}{i} (k-1)(k-2) \cdots (k-i) x^{k-i-1} \frac{\partial^{n-i-1} f}{\partial y^{n-j} \partial x^{j-i-1}} \in C(\Omega). \square \end{aligned}$$

Actually, a more detailed examination of the proof of Theorem 2 would show that the integral operators

$$\begin{aligned} K : h &\longrightarrow K(h)(x, y) = \int_{\Omega} F_{k,\lambda}(x, y, u, v) h(u, v) du dv, \\ {}^t K : h &\longrightarrow {}^t K(h)(x, y) = \int_{\Omega} F_{k,\lambda}(u, v, x, y) h(u, v) du dv \end{aligned}$$

map $C_0^\infty(\Omega)$ into $C^\infty(\Omega)$. In other words, K and ${}^t K$ are separately regular. Since $F_{k,\lambda}$ is a C^∞ function in the complement of the diagonal of $\Omega \times \Omega$, we conclude that K and ${}^t K$ are very regular.

Next, we introduce some notations

$$\Xi_t = \{(\alpha, \beta, \gamma) \in \mathbf{Z}_+^3 : \alpha + \beta \leq t, kt \geq \gamma \geq \alpha + (1+k)\beta - t\}.$$

For a function $f(x, y)$ on \mathbf{R}^2 , we write $\partial_1^\alpha f, \partial_2^\beta f, \partial_{1,2}^{\alpha,\beta} f, \gamma \partial_{\alpha,\beta} f$ for $\frac{\partial^\alpha f(x,y)}{\partial x^\alpha}$, $\frac{\partial^\beta f(x,y)}{\partial y^\beta}$, $\frac{\partial^{\alpha+\beta} f(x,y)}{\partial x^\alpha \partial y^\beta}$, $x^\gamma \frac{\partial^{\alpha+\beta} f(x,y)}{\partial x^\alpha \partial y^\beta}$, respectively. For $m \in \mathbf{Z}^+$, let us denote by $\mathbf{H}_{loc}^m(\Omega)$ the space of all function $f \in L_{loc}^2(\Omega)$ such that for any compact K of Ω we have $\sum_{(\alpha,\beta,\gamma) \in \Xi_m} \|\gamma \partial_{\alpha,\beta} f\|_{L^2(K)} < \infty$. Now we are in a position to formulate the main theorem of this section.

Theorem 3. *Assume that $m \geq 2k^2 + 6k + 5$. Let f be a $\mathbf{H}_{loc}^m(\Omega)$ solution of the equation (1) and $\Psi \in G^s$. Then $f \in G^s$. In particular, if Ψ is analytic in its arguments then so is f .*

Proof. The proof of Theorem 3 consists of Theorem 4 and Theorem 5 below. The proof follows the scheme : $f \in \mathbf{H}_{loc}^m \implies f \in C^\infty(\Omega) \implies f \in A(\Omega)$. \square

Theorem 4. *Let Ψ be a C^∞ -function of its arguments and $m \geq 2k^2 + 6k + 5$. Assume that $f \in \mathbf{H}_{loc}^m(\Omega)$ is a solution of the equation (1) then $f \in C^\infty(\Omega)$.*

Proof. Theorem 4 can be proved with the help of Proposition 2. \square

Proposition 2. Let $m \geq 2k^2 + 6k + 5$. Assume that $f \in \mathbb{H}_{loc}^m(\Omega)$. Then $\Psi(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}) \in \mathbb{H}_{loc}^{m-1}(\Omega)$.

Next, put $r_0 = 2k + 2$. For $r \in \mathbb{Z}_+$ let Γ_r denote the set of pairs of multi-indices (α, β) such that $\Gamma_r = \Gamma_r^1 \cup \Gamma_r^2$ where

$$\Gamma_r^1 = \{(\alpha, \beta) : \alpha \leq r_0, 2\alpha + \beta \leq r\}, \Gamma_r^2 = \{(\alpha, \beta) : \alpha \geq r_0, \alpha + \beta \leq r - r_0\}.$$

Define the following norm

$$|f, \Omega|_r = \max_{(\alpha, \beta) \in \Gamma_r} |\partial_1^\alpha \partial_2^\beta f, \Omega| + \max_{\substack{(\alpha, \beta) \in \Gamma_r \\ \alpha \geq 1, \beta \geq 1}} \max_{(x, y) \in \Omega} |\partial_1^{\alpha+2} \partial_2^\beta f|,$$

where $|f, \Omega| = \max_{(x, y) \in \Omega} \left(|f| + \left| \frac{\partial f}{\partial x} \right| + \left| x^k \frac{\partial f}{\partial y} \right| \right)$.

Theorem 5. Let f be a C^∞ solution of the equation (1) and $\Psi \in G^s$. Then $f \in G^s$. In particular, if Ψ is analytic in its arguments then so is f .

Proof. Theorem 5 can be proved with the help of Proposition 3, Corollary 1, Lemmas 2-4. \square

Proposition 3. Assume that $\Psi \in G^s$. Then there exist constants C, D such that for every $H_0 \geq 1, H_1 \geq CH_0^{2k+3}$ if

$$|f, \Omega|_d \leq H_0 H_1^{(d-r_0-2)} (d - r_0 - 2)!^s, \quad 0 \leq d \leq N + 1, r_0 + 2 \leq N$$

then

$$\max_{(x, y) \in \Omega} \left| \partial_1^\alpha \partial_2^\beta \Psi \left(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y} \right) \right| \leq DH_0 H_1^{N-r_0-1} (N - r_0 - 1)!^s$$

for every $(\alpha, \beta) \in \Gamma_{N+1}$.

Corollary 1. Under the same hypotheses of Proposition 3 with $d \leq N + 1$ replaced by $d \leq N$, then

$$\max_{x \in \Omega} \left| \partial_1^\alpha \partial_2^\beta \Psi \left(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y} \right) \right| \leq D \left(|f, \Omega|_{N+1} + H_0 H_1^{N-r_0-1} (N - r_0 - 1)!^s \right)$$

for every $(\alpha, \beta) \in \Gamma_{N+1}$.

Since $G_{k, \lambda}$ is elliptic if $x \neq 0$, it suffices to consider the case $(0, 0) \in \Omega$ and Ω is a small neighborhood of $(0, 0)$. Let us define the distance

$$\rho((u, v), (x, y)) = \begin{cases} \max \{ |x^{k+1} - u^{k+1}|, (k+1)|y - v| \}, & \text{for } xu \geq 0 \\ \max \{ x^{k+1} + u^{k+1}, (k+1)|y - v| \}, & \text{for } xu \leq 0. \end{cases}$$

For two sets S_1, S_2 , the distance between them is defined as

$$\rho(S_1, S_2) = \inf_{(x,y) \in S_1, (u,v) \in S_2} \rho((x,y), (u,v)).$$

Let $V^T (T \leq 1)$ be the cube with edges of size (in the ρ metric) $2T$ which are parallel to the coordinate axes and centered at $(0,0)$. Denote by V_δ^T the sub-cube which is homothetic with V^T and such that the distance between its boundary and the boundary of V^T is δ . We shall prove by induction that if T is small enough then there exist constants H_0, H_1 with $H_1 \geq CH_0^{2k+3}$ such that

$$(6) \quad |f, V_\delta^T|_n \leq H_0 \quad \text{for } 0 \leq n \leq 6k+4,$$

and

$$(7) \quad |f, V_\delta^T|_n \leq H_0 \left(\frac{H_1}{\delta} \right)^{n-r_0-2} (n-r_0-2)!^s := Q_{n-1}$$

for $n \geq 6k+4$, and δ sufficiently small. Hence the desired conclusion follows. (6) follows easily from the C^∞ smoothness assumption on f . Assume that (7) holds for $n = N$. We shall prove it for $n = N+1$. Put $\delta' = \delta(1-1/N)$, $\delta'' = \delta(1-4/N)$. Fix $(x,y) \in V_\delta^T$ and then define $\sigma = \rho((x,y), \partial V^T)$ and $\tilde{\sigma} = \sigma/N$. Let $V_{\tilde{\sigma}}(x,y)$ denote the cube with center at (x,y) and edges of length $2\tilde{\sigma}$ which are parallel to the coordinate axes, and $S_{\tilde{\sigma}}(x,y)$ the boundary of $V_{\tilde{\sigma}}(x,y)$. Note that $\sigma \geq \delta$, and $V_{\tilde{\sigma}}(x,y) \subset V_{\delta'}^T$. Let $E_1, E_3 (E_2, E_4)$ be edges of $S_{\tilde{\sigma}}(x,y)$ which are parallel to $Ox(Oy)$ respectively. We have to estimate $\max_{(x,y) \in V_\delta^T} |\gamma \partial_{\alpha,\beta} (\partial_1^{\alpha_1} \partial_2^{\beta_1} f)|$ for all $(\alpha, \beta, \gamma) \in \Xi_1$, $(\alpha_1, \beta_1) \in \Gamma_{N+1}$, and $\max_{(x,y) \in V_\delta^T} |(\partial_1^{2+\alpha_1} \partial_2^{\beta_1} f)|$ for all $(\alpha_1, \beta_1) \in \Gamma_{N+1}$, $\alpha_1 \geq 1, \beta_1 \geq 1$. Let us abbreviate $\frac{\partial^\alpha}{\partial u^\alpha}, \frac{\partial^\beta}{\partial v^\beta}, \frac{\partial^{\alpha+\beta}}{\partial u^\alpha \partial v^\beta}$ as $\partial_u^\alpha, \partial_v^\beta, \partial_u^\alpha \partial_v^\beta$, respectively.

Lemma 2. *Assume that $(\alpha, \beta, \gamma) \in \Xi_1$ and $(\alpha_1, \beta_1) \in \Gamma_{N+1}$. Then if $\alpha_1 \geq 1, \beta_1 \geq 1$ there exists a constant C such that*

$$\max_{(x,y) \in V_\delta^T} |\gamma \partial_{\alpha,\beta} (\partial_1^{\alpha_1} \partial_2^{\beta_1} f(x,y))| \leq C \left(T^{\frac{1}{k+1}} |f, V_{\delta'}^T|_{N+1} + Q_N \left(T^{\frac{1}{k+1}} + \frac{1}{H_1} \right) \right).$$

Lemma 3. *Assume that $(\alpha, \beta, \gamma) \in \Xi_1$. Then there exists a constant C such that*

$$\max_{(x,y) \in V_\delta^T} |\gamma \partial_{\alpha,\beta} (\partial_2^{N+1} f(x,y))| \leq C \left(T^{\frac{1}{k+1}} |f, V_{\delta''}^T|_{N+1} + Q_N \left(T^{\frac{1}{k+1}} + \frac{1}{H_1} \right) \right).$$

Lemma 4. Assume that $(\alpha, \beta, \gamma) \in \Xi_1$. Then there exists a constant C such that

$$\max_{(x,y) \in V_\delta^T} |\gamma \partial_{\alpha,\beta} (\partial_1^{N-r_0+1} f(x,y))| \leq C \left(T^{\frac{1}{k+1}} |f, V_{\delta'}^T|_{N+1} + Q_N \left(T^{\frac{1}{k+1}} + \frac{1}{H_1} \right) \right).$$

Lemma 5. Assume that $(\alpha_1, \beta_1) \in \Gamma_{N+1} \setminus \Gamma_N$, $\alpha_1 \geq 1, \beta_1 \geq 1$. Then there exists a constant C such that

$$\max_{(x,y) \in V_\delta^T} |(\partial_1^{\alpha_1+2} \partial_2^{\beta_1} f(x,y))| \leq C \left(T^{\frac{1}{k+1}} |f, V_{\delta''}^T|_{N+1} + Q_N \left(T^{\frac{1}{k+1}} + \frac{1}{H_1} \right) \right).$$

§3. Case k is even.

A. First, we consider the case $\lambda = 2N(k+1)$, where N is an integer. In this case we will prove a similar result as in §2 by establishing the explicit uniform fundamental solutions of $G_{k,2N(k+1)}$. Let us maintain the notations used for $p, A_+, A_-, M, F_{k,\lambda}, \dots$ from the very beginning of the paper (now, of course, with an even k). If $(u, v) \neq (0, v)$ is fixed then the real parts of A_+, A_- change sign when (x, y) passes through $(-u, v)$. Therefore, $M = A_+^{-\frac{k+\lambda}{2k+2}} A_-^{-\frac{k-\lambda}{2k+2}}$ may have singularities along the half-line (x, v) with $x \leq -u$ for an arbitrary complex number λ . But if $\lambda = 2N(k+1)$ then it is not difficult to see that $M = A_+^{-\frac{k+\lambda}{2k+2}} A_-^{-\frac{k-\lambda}{2k+2}}$ is smooth along the half-line (x, v) with $x < -u$, that is $M(\cdot, \cdot, u, v) \in C^\infty(\mathbb{R}^2 \setminus \{(u, v), (-u, v)\})$. Moreover, when k is even and $u \neq 0$ we have $-\infty \leq p \leq 1$. More precisely, $p \rightarrow 1$ when $(x, y) \rightarrow (u, v)$, and $p \rightarrow -\infty$ when $(x, y) \rightarrow (-u, v)$. If $N < 0$ and $p \rightarrow -\infty$ then from the asymptotic expansions of hypergeometric functions (see [2], p. 63) we should choose the expressions for constants C_1, C_2 as in the beginning of the paper (with λ replaced by $2N(k+1)$). And we will have $F_{k,2N(k+1)}(\cdot, \cdot, u, v) \in C^\infty(\mathbb{R}^2 \setminus (u, v))$, with

$$F_{k,2N(k+1)}(-u, v, u, v) = 0.$$

Similar conclusions hold for $F_{k,2N(k+1)}(x, y, u, v)$ when $N > 0$. If $N = 0$ then $F_{k,0}(\cdot, \cdot, u, v) \in C^\infty(\mathbb{R}^2 \setminus (u, v))$, with

$$F_{k,0}(-u, v, u, v) = -\frac{\cot \frac{k\pi}{2k+2}}{4u^k}.$$

Theorem 6. Let $\Psi \in G^s$. Assume that $m \geq 2k^2 + 6k + 5$, $\lambda = 2N(k + 1)$, and f is a $H_{loc}^m(\Omega)$ solution of the equation (1). Then $f \in G^s$. In particular, if Ψ is analytic in its arguments then so is f .

Proof. Almost all the arguments used for the case when k is odd can be applied here. Therefore, we only give the sketch of the proof. Instead of the distance ρ in §2 we use the following metric

$$\tilde{\rho}((u, v), (x, y)) = \max \{|x^{k+1} - u^{k+1}|, (k + 1)|y - v|\}. \square$$

B. In this sub-section we will present some computations for finding the fundamental solutions of $G_{k,\lambda}$ with source at the origin $(0, 0)$ for λ other than the values $2N(k + 1)$ considered in sub-section A. Make the following change of variables

$$x = \rho |\sin \theta|^{\frac{1}{k+1}} \text{sign}(\sin \theta), y = \frac{\rho^{k+1}}{k+1} \cos \theta, \theta \in (-\pi, \pi).$$

Then $G_{k,\lambda}$ will be transformed into

$$\begin{aligned} & \text{sign}(\sin \theta) |\sin \theta|^{\frac{k-1}{k+1}} \left(\sin \theta \frac{\partial^2}{\partial \rho^2} + (k+1)^2 \rho^{-2} \sin \theta \frac{\partial^2}{\partial \theta^2} + \right. \\ & \left. (i\lambda \cos \theta + (k+1) \sin \theta) \rho^{-1} \frac{\partial}{\partial \rho} + (k+1) \rho^{-2} (k \cos \theta - i\lambda \sin \theta) \frac{\partial}{\partial \theta} \right). \end{aligned}$$

If we seek the fundamental solution in the form $F_{k,\lambda}(x, y) = \rho^{-k} F(\theta)$ then $F(\theta)$ must satisfy the following equation

$$(8) \quad \begin{aligned} & (k+1)^2 \sin \theta F''(\theta) + (k+1)(k \cos \theta - i\lambda \sin \theta) F'(\theta) - \\ & - ik\lambda \cos \theta F(\theta) = 0. \end{aligned}$$

The general solutions of (8) are

$$F(\theta) = \left(C_3 + C_4 \int_0^\theta |\sin s|^{-\frac{k}{k+1}} e^{-\frac{i\lambda s}{k+1}} ds \right) e^{\frac{i\lambda \theta}{k+1}},$$

where C_3 and C_4 are some complex constants. Among all these solutions, we are interested in finding a non-trivial periodic solution. When $\lambda = 2N(k + 1)$ - this case was considered in sub-section A - the periodic solution is $F(\theta) = e^{\frac{i\lambda \theta}{k+1}}$, and the function $F_{k,\lambda}(x, y) = \rho^{-k} F(\theta)$ serves as a fundamental solution. When

$\lambda = (2N + 1)(k + 1)$ then the periodic solution again is $F(\theta) = e^{\frac{i\lambda\theta}{k+1}}$. But in this case, we have $F_{k,\lambda}(x, y) = \rho^{-k}F(\theta)$ is a non-smooth solution of the equation $G_{k,\lambda}f(x, y) = 0$ (see [3]); hence, hypoellipticity for $G_{k,\lambda}$ fails in this case. If $\lambda \neq 2N(k + 1)$ and $\lambda \neq (2N + 1)(k + 1)$ then we should choose

$$C_3 = \frac{iC_4 \left(e^{\frac{i\pi\lambda}{k+1}} \int_0^\pi |\sin s|^{-\frac{k}{k+1}} e^{-\frac{i\lambda s}{k+1}} ds + e^{-\frac{i\pi\lambda}{k+1}} \int_{-\pi}^0 |\sin s|^{-\frac{k}{k+1}} e^{-\frac{i\lambda s}{k+1}} ds \right)}{2 \sin \frac{\pi\lambda}{k+1}}$$

to obtain the only periodic solution. In this case, the function $F_{k,\lambda}(x, y) = \rho^{-k}F(\theta)$ will be our desired fundamental solution.

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