数理解析研究所講究録 1261 巻 2002 年 172-181



# Singular Solutions of the Briot-Bouquet Type Partial Differential Equations

Caritas College, Hiroshi Yamazawa
カリタス女子短期大学, 山澤浩司

## 1 Introduction

In this talk, we will study the following type of nonlinear singular first order partial differential equations:

$$t\partial_t u = F(t, x, u, \partial_x u) \tag{1.1}$$

where $(t, x) = (t, x_1, \ldots, x_n) \in \mathbf{C}_t \times \mathbf{C}_x^n$, $\partial_x u = (\partial_1 u, \ldots, \partial_n u)$, $\partial_t = \dfrac{\partial}{\partial t}$, $\partial_i = \dfrac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$, and $F(t, x, u, v)$ with $v = (v_1, \ldots, v_n)$ is a function defined in a polydisk $\Delta$ centered at the origin of $\mathbf{C}_t \times \mathbf{C}_x^n \times \mathbf{C}_u \times \mathbf{C}_v^n$. Let us denote $\Delta_0 = \Delta \cap \{t = 0, u = 0, v = 0\}$.

The assumptions are as follows:

(A1) $F(t, x, u, v)$ is holomorphic in $\Delta$,
(A2) $F(0, x, 0, 0) = 0$ in $\Delta_0$,
(A3) $\dfrac{\partial F}{\partial v_i}(0, x, 0, 0) = 0$ in $\Delta_0$ for $i = 1, \ldots, n$.

**Definition 1.1** ([2], [3]) *If the equation* (1.1) *satisfies* (A1), (A2) *and* (A3) *we say that the equation* (1.1) *is of Briot-Bouquet type with respect to t.*

**Definition 1.2** ([2], [3]) *Let us define*

$$\rho(x) = \frac{\partial F}{\partial u}(0, x, 0, 0),$$

*then the holomorphic function* $\rho(x)$ *is called the characteristic exponent of the equation* (1.1).

Let us denote by

1. $\mathcal{R}(\mathbf{C}\backslash\{0\})$ the universal covering space of $\mathbf{C}\backslash\{0\}$,
2. $S_\theta = \{t \in \mathcal{R}(\mathbf{C}\backslash\{0\});\ |\arg t| < \theta\}$,
3. $S(\epsilon(s)) = \{t \in \mathcal{R}(\mathbf{C}\backslash\{0\});\ 0 < |t| < \epsilon(\arg t)\}$ for some positive-valued function $\epsilon(s)$ defined and continuous on $\mathbf{R}$,

4. $D_R = \{x \in \mathbf{C}^n;\ |x_i| < R \text{ for } i = 1, \ldots, n\}$,

5. $\mathbf{C}\{x\}$ the ring of germs of holomorphic functions at the origin of $\mathbf{C}^n$.

**Definition 1.3** *We define the set $\tilde{\mathcal{O}}_+$ of all functions $u(t,x)$ satisfying the following conditions;*
1. *$u(t,x)$ is holomorphic in $S(\epsilon(s)) \times D_R$ for some $\epsilon(s)$ and $R > 0$,*
2. *there is an $a > 0$ such that for any $\theta > 0$ and any compact subset $K$ of $D_R$*

$$\max_{x \in K} |u(t,x)| = O(|t|^a) \quad as \quad t \to 0 \quad in \quad S_\theta.$$

We know some results on the equation (1.1) of Briot-Bouquet type with respect to $t$. We concern the following result. Gérard R. and Tahara H. studied in [2] the structure of holomorphic and singular solutions of the equation (1.1) and proved the following result;

**Theorem 1.4 (Gérard R. and Tahara H.)** *If the equation* (1.1) *is of Briot-Bouquet type and $\rho(0) \notin \mathbf{N}^* = \{1, 2, 3, \ldots\}$ then we have;*
(1) (*Holomorphic solutions*) *The equation* (1.1) *has a unique solution $u_0(t,x)$ holomorphic near the origin of $\mathbf{C} \times \mathbf{C}^n$ satisfying $u_0(0,x) \equiv 0$.*
(2) (*Singular solutions*) *Denote by $S_+$ the set of all $\tilde{\mathcal{O}}_+$-solutions of* (1.1).

$$S_+ = \begin{cases} \{u_0(t,x)\} & \text{when} \quad Re\rho(0) \leq 0, \\ \{u_0(t,x)\} \cup \{U(\varphi); 0 \neq \varphi(x) \in \mathbf{C}\{x\}\} & \text{when} \quad Re\rho(0) > 0, \end{cases}$$

*where $U(\varphi)$ is an $\tilde{\mathcal{O}}_+$-solution of* (1.1) *having an expansion of the following form:*

$$U(\varphi) = \sum_{i \geq 1} u_i(x) t^i + \sum_{i+2j \geq k+2, j \geq 1} \varphi_{i,j,k}(x) t^{i+j\rho(x)} (\log t)^k, \quad \varphi_{0,1,0}(x) = \varphi(x).$$

The purpose of this paper is to determine $S_+$ in the case $\rho(0) \in \mathbf{N}^*$.

The main result of this paper is;

**Theorem 1.5** *If the equation* (1.1) *is of Briot-Bouquet type and if $\rho(0) = N \in \mathbf{N}^*$ and $\rho(x) \not\equiv \rho(0)$, then*

$$S_+ = \{U(\varphi);\ \varphi(x) \in \mathbf{C}\{x\}\},$$

*where $U(\varphi)$ is an $\tilde{\mathcal{O}}_+$-solution of* (1.1) *having an expansion of the following form:*

$$U(\varphi) = u_1^0(x) t + u_0^{e_0}(x) \phi_N(t,x) + \sum_{\substack{i+|\beta| \geq 2, |\beta| < \infty \\ [\beta] \leq i+|\beta|-2}} u_i^\beta(x) t^i \Phi_N^\beta$$
$$+ w_{0,1,0}^0(x) t^{\rho(x)} + \sum_{\substack{i+j+|\beta| \geq 2, |\beta| < \infty \\ j \geq 1, [\beta] \leq i+j+|\beta|-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_N^\beta,$$

*where* $u_N^0(x) \equiv 0$, $w_{0,1,0}^0(x) = \varphi(x)$ *is an arbitrary holomorphic function and the other coefficients* $u_i^\beta(x)$, $w_{i,j,k}^\beta(x)$ *are holomorphic functions determined by* $w_{0,1,0}^0(x)$ *and defined in a common disk, and*

$$l = (l_1, \dots, l_n) \in \mathbf{N}^n,\ |l| = l_1 + \cdots + l_n,\ \beta = (\beta_l \in \mathbf{N};\ l \in \mathbf{N}^n),$$
$$|\beta| = \sum_{|l|\geq 0} \beta_l,\ |\beta|_p = \sum_{|l|=p} \beta_l \text{ for } p \geq 0,\ [\beta] = \sum_{|l|\geq 2} (|l|-1)\beta_l,$$
$$\Phi_N^\beta = \prod_{|l|\geq 0} \left(\frac{\partial_x^l \phi_N}{l!}\right)^{\beta_l},\ \partial_x^l = \partial_1^{l_1} \cdots \partial_n^{l_n},\ \phi_N(t,x) = \frac{t^{\rho(x)} - t^N}{\rho(x) - N}.$$

The following lemma will play an important role in the proof of Theorem 1.5.

At first, we define some notations. We set for $l \in \mathbf{N}^n$, $e_l = (\beta_k;\ k \in \mathbf{N}^n)$ with $\beta_l = 1$ and $\beta_k = 0$ for $k \neq l$ and for $p \in \{1, 2, \dots, n\}$, $e(p) = (i_1, \dots, i_n)$ with $i_p = 1$ and $i_q = 0$ for $q \neq p$, and define $l^1 < l^0$ is defined by $|l^1| < |l^0|$ and $l_i^1 \leq l_i^0$ for $i = 1, \dots, n$.

**Lemma 1.6** *Let* $\rho(x)$, $\phi_N$ *and* $\Phi_N^\beta$ *be as in Theorem 1.5. Then we have;*
1. $\partial_p \Phi_N^\beta = \sum_{|l|\geq 0} \beta_l (l_p + 1) \Phi_N^{\beta - e_l + e_{l+e(p)}}$ *for* $i = 1, \dots, n$,
2. $t\partial_t \phi_N = \rho(x)\phi_N + t^N$,
3. $t\partial_t \Phi_N^\beta = |\beta|\rho(x)\Phi_N^\beta + \beta_0 t^N \Phi_N^{\beta - e_0} + \sum_{|l^0|\geq 1} \sum_{l^1 < l^0} \beta_{l^0} \frac{\partial_x^{(l^0 - l^1)!}\rho(x)}{l^0 - l^1} \Phi_N^{\beta - e_{l^0} + e_{l^1}}$.

# 2 Construction of formal solutions in the case $\rho(0) = 1$

By [2] (Gérard-Tahara), if the equation (1.1) is of Briot-Bouquet type with respect to $t$, then it is enough to consider the following equation:

$$Lu = t\partial_t u - \rho(x)u = a(x)t + G_2(x)(t, u, \partial_x u) \tag{2.1}$$

where $\rho(x)$ and $a(x)$ are holomorphic functions in a neighborhood of the origin, and the function $G_2(x)(t, X_0, X_1, \dots, X_n)$ is a holomorphic function in a neighborhood of the origin in $\mathbf{C}_x^n \times \mathbf{C}_t \times \mathbf{C}_{X_0} \times \mathbf{C}_{X_1} \times \cdots \times \mathbf{C}_{X_n}$ with the following expansion:

$$G_2(x)(t, X_0, X_1, \dots, X_n) = \sum_{p+|\alpha|\geq 2} a_{p,\alpha}(x) t^p \{X_0\}^{\alpha_0} \{X_1\}^{\alpha_1} \cdots \{X_n\}^{\alpha_n}$$

and we may assume that the coefficients $\{a_{p,\alpha}(x)\}_{p+|\alpha|\geq 2}$ are holomorphic functions on $D_{R_0}$ for a sufficiently small $R_0 > 0$. Let $0 < R < R_0$. We put $A_{p,\alpha}(R) := \max_{x\in D_R} |a_{p,\alpha}(x)|$ for $p + |\alpha| \geq 2$. Then for $0 < r < R$

$$\sum_{p+|\alpha|\geq 2} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} t^p X_0^{\alpha_0} X_1^{\alpha_1} \times \cdots \times X_n^{\alpha_n} \tag{2.2}$$

is convergent in a neighborhood of the origin.

In this section, we assume $\rho(0) = 1$ and $\rho(x) \not\equiv 1$ and we will construct formal solutions of the equation (2.1). In generally, we set $u(t,x) = \sum_{i=1}^{N-1} u_i(x)t^i + t^{N-1}w(t,x)$, and we consider an equation for $w(t,x)$.

**Proposition 2.1** *If $\rho(0) = 1$ and $\rho(x) \not\equiv 1$, the equation (2.1) has a family of formal solutions of the form:*

$$
\begin{aligned}
u = u_0^{eo}(x)\phi_1 &+ \sum_{m\geq 2} \sum_{\substack{i+|\beta|=m \\ [\beta]\leq m-2}} u_i^\beta(x) t^i \Phi_1^\beta \\
&+ w_{0,1,0}^0(x) t^{\rho(x)} + \sum_{m\geq 2} \sum_{\substack{i+j+|\beta|=m \\ j\geq 1, [\beta]\leq m-2}} \sum_{\substack{k\leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta
\end{aligned} \tag{2.3}
$$

*where $w_{0,1,0}^0(x)$ is an arbitrary holomorphic function and the other coefficients $u_i^\beta(x)$, $w_{i,j,k}^\beta(x)$ are holomorphic functions determined by $w_{0,1,0}^0(x)$ and defined in a common disk.*

**Remark 2.2** *By the relation $[\beta] \leq m-2$ in summations of the above formal solution, we have $\beta_l = 0$ for any $l \in \mathbf{N}^n$ with $|l| \geq m$.*

We define the following two sets $U_m$ and $W_m$ for $m \geq 1$ to prove Proposition 2.1.

**Definition 2.3** *We denote by $U_m$ the set of all functions $u_m$ of the following forms:*

$$
\begin{aligned}
u_1 &= u_1^0(x)t + u_0^{eo}(x)\phi_1, \\
u_m &= \sum_{\substack{i+|\beta|=m \\ [\beta]\leq m-2}} u_i^\beta(x) t^i \Phi_1^\beta \quad \text{for } m \geq 2,
\end{aligned} \tag{2.4}
$$

*and denote by $W_m$ the set of all functions $w_m$ of the following forms:*

$$
\begin{aligned}
w_1 &= w_{0,1,0}^0(x) t^{\rho(x)}, \\
w_m &= \sum_{\substack{i+j+|\beta|=m \\ j\geq 1, [\beta]\leq m-2}} \sum_{\substack{k\leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta \quad \text{for } m \geq 2
\end{aligned} \tag{2.5}
$$

*where $u_i^\beta(x)$, $w_{i,j,k}^\beta(x) \in \mathbf{C}\{x\}$.*

We can rewrite the formal solution (2.3) as follows:

$$
u = \sum_{m\geq 1} (u_m + w_m) \quad \text{where } u_m \in U_m,\ w_m \in W_m.
$$

Let us show important relations of $u_m$ and $w_m$ for $m \geq 2$. By Lemma 1.6,we have

$$
\begin{aligned}
Lw_m \;=\; & \sum_{\substack{i+j+|\beta|=m \\ j\geq 1,[\beta]\leq m-2}} \sum_{\substack{k\leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \Big\{\{i+(j+|\beta|-1)\rho(x)\}w^{\beta}_{i,j,k}(x)t^{i+j\rho(x)}\{\log t\}^k\Phi_1^{\beta} \\
+ \;& kw^{\beta}_{i,j,k}(x)t^{i+j\rho(x)}\{\log t\}^{k-1}\Phi_1^{\beta} + \beta_0 w^{\beta}_{i,j,k}(x)t^{i+j\rho(x)+1}\{\log t\}^k\Phi_1^{\beta-e_0} \\
& + \sum_{|l^0|=1}^{m-1}\sum_{l^1<l^0}\beta_{l^0}\frac{\partial_x^{l^0-l^1}\rho(x)}{(l^0-l^1)!}w^{\beta}_{i,j,k}(x)t^{i+j\rho(x)}\{\log t\}^k\Phi_1^{\beta-e_{l^0}+e_{l^1}}\Big\}.
\end{aligned}
$$

We show two lemmas.

**Lemma 2.4** *If $u_m \in U_m$ and $w_m \in W_m$, then $Lu_m \in U_m$ and $Lw_m \in W_m$.*

**Lemma 2.5** *If $u_m \in U_m$ and $w_m \in W_m$, then the following relations hold for $i,j=1,\dots,n$*

1. *$a(x)U_m \subset U_m$ and $a(x)W_m \subset W_m$ for any holomorphic function $a(x)$,*
2. *$tU_m$, $\phi_1 U_m \subset U_{m+1}$ and $t^{\rho(x)}U_m$, $tW_m$, $t^{\rho(x)}W_m$, $\phi_1 W_m \subset W_{m+1}$,*
3. *$u_m \times u_n$, $\partial_i u_m \times \partial_j u_n$, $\partial_i u_m \times u_n \in U_{m+n}$,*
4. *$w_m \times w_n$, $\partial_i w_m \times \partial_j w_n$, $\partial_i w_m \times w_n$, $\in W_{m+n}$,*
5. *$u_m \times w_n$, $\partial_i u_m \times w_n$, $u_m \times \partial_j w_n$, $\partial_i u_m \times \partial_j w_n \in W_{m+n}$.*

Let us show that $u_m$ and $w_m$ are determined inductively on $m \geq 1$. By substituting $\sum_{m\geq 1}(u_m+w_m)$ into (2.1), we have

$$
(1-\rho(x))u_1^0(x) + u_0^{e_0}(x) = a(x), \tag{2.6}
$$

and for $m \geq 2$

$$
Lu_m = \sum_{\substack{p+|\alpha|\geq 2 \\ p+|m_n|=m}} a_{p,\alpha}(x)t^p \prod_{h_0=1}^{\alpha_0} u_{m_{0,h_0}} \prod_{j=1}^{n}\prod_{h_j=1}^{\alpha_j}\partial_j u_{m_{j,h_j}}, \tag{2.7}
$$

$$
\begin{aligned}
Lw_m = & \sum_{\substack{p+|\alpha|\geq 2 \\ p+|m_n|=m}} a_{p,\alpha}(x)t^p \prod_{h_0=1}^{\alpha_0}(u_{m_{0,h_0}}+w_{m_{0,h_0}}) \prod_{j=1}^{n}\prod_{h_j=1}^{\alpha_j}\partial_j(u_{m_{j,h_j}}+w_{m_{j,h_j}}) \\
& - \sum_{\substack{p+|\alpha|\geq 2 \\ p+|m_n|=m}} a_{p,\alpha}(x)t^p \prod_{h_0=1}^{\alpha_0} u_{m_0,h_0} \prod_{j=1}^{n}\prod_{h_j=1}^{\alpha_j}\partial_j u_{m_{j,h_j}},
\end{aligned} \tag{2.8}
$$

where $|m_n| = \sum_{i=0}^{n} m_i(\alpha_i)$ and $m_i(\alpha_i) = m_{i,1}+\cdots+m_{i,\alpha_i}$ for $i=0,1,\dots,n$.

We take any holomorphic function $\varphi(x) \in \mathbf{C}\{x\}$ and put $w^0_{0,1,0}(x) = \varphi(x)$, and by (2.6), we put $u_1^0(x) \equiv 0$ and $u_0^{e_0}(x) = a(x)$.

For $m \geq 2$, let us show that $u_m$ and $w_m$ are determined by induction. By Lemma 2.5, the right side of (2.7) belongs to $U_m$ and the right side of (2.8) belongs to $W_m$. Further by $m_{j,h_j} \geq 1$, we have $m_{j,h_j} < m$ for $h_j = 1, \dots, \alpha_j$ and $j = 0, \dots, n$. Then for $m \geq 2$, we compare with the coefficients of $t^i \Phi_1^\beta$ and $t^{i+j\rho(x)}\{\log t\}^k \Phi_1^\beta$ respectively for (2.7) and (2.8), then put

$$\begin{aligned}
&\{i + (|\beta| - 1)\rho(x)\}u_i^\beta(x) \\
&+ (\beta_0 + 1)u_{i-1}^{\beta+e_0}(x) + \sum_{|l^0|=1}^{m-1} \sum_{0 \leq l^1 < l^0} (\beta_{l^0} + 1)\frac{\partial_x^{l^0-l^1}\rho(x)}{(l^0 - l^1)!} u_i^{\beta+e_{l^0}-e_{l^1}}(x) \\
&= f_i^\beta(\{a_{p,\alpha}\}_{2 \leq p+|\alpha| \leq m},\ \{u_{i'}^{\beta'}(x)\}_{i'+|\beta'|<m})
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&\{i + (j + |\beta| - 1)\rho(x)\}w_{i,j,k}^\beta(x) + (k+1)w_{i,j,k+1}^\beta(x) \\
&+ (\beta_0 + 1)w_{i-1,j,k}^{\beta+e_0}(x) + \sum_{|l^0|=1}^{m-1} \sum_{0 \leq l^1 < l^0} (\beta_{l^0} + 1)\frac{\partial_x^{l^0-l^1}\rho(x)}{(l^0 - l^1)!} w_{i,j,k}^{\beta+e_{l^0}-e_{l^1}}(x) \\
&= g_{i,j,k}^\beta(\{a_{p,\alpha}\}_{2 \leq p+|\alpha| \leq m},\ \{u_{i'}^{\beta'}(x)\}_{i'+|\beta'|<m},\ \{w_{i',j',k'}^{\beta'}(x)\}_{i'+j'+|\beta'|<m}).
\end{aligned} \tag{2.10}$$

Hence we obtain Proposition 2.1. Q.E.D.

# 3 Convergence of the formal solutions in the case $\rho(0) = 1$

In this section, we show that the formal solution (2.3) converges in $\widetilde{\mathcal{O}}_+$.

**Proposition 3.1** *Let $\gamma$ satisfy $0 < \gamma < 1$ and let $\lambda$ be sufficiently large. Then for any sufficiently small $r > 0$ we have the following result;*

*For any $\theta > 0$ there is an $\epsilon > 0$ such that the formal solution (2.3) converges in the following region:*

$$\begin{aligned}
\{(t, x) \in \mathbf{C}_t \times \mathbf{C}_x^n;\ &|\eta(t,\lambda)t| < \epsilon,\ |\eta(t,\lambda)^2 t^{\rho(x)}| < \epsilon, \\
&|\eta(t,\lambda)t^\gamma| < \epsilon,\ t \in S_\theta \text{ and } x \in D_r\},
\end{aligned}$$

*where $\eta(t, \lambda) = \max\{|(\log t)/\lambda|,\ 1\}$.*

In this section, we put $w_{i,0,0}^\beta(x) = u_i^\beta(x)$ and $w_{i,0,k}^\beta(x) \equiv 0$ for $k \geq 1$ in the formal solution (2.3). Then the formal solution (2.3) is as follows:

$$\begin{aligned}
u = &w_{0,0,0}^{e_0}(x)\phi_1 + w_{0,1,0}^0(x)t^{\rho(x)} \\
&+ \sum_{m \geq 2} \sum_{\substack{i+j+|\beta|=m \\ [\beta] \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)}\{\log t\}^k \Phi_1^\beta.
\end{aligned} \tag{3.1}$$

Let us define the following set $V_m$ for (3.1).

**Definition 3.2** *We denote by $V_m$ the set of all the functions $v_m$ of the following forms:*

$$v_m = u_m + w_m \quad for \quad u_m \in U_m \quad and \quad w_m \in W_m. \tag{3.2}$$

We define the following estimate for the function $v_m$.

**Definition 3.3** *For the function* (3.2), *we define*

$$\begin{aligned} ||v_1||_{r,c,\lambda} &= ||v_1||_{r,c} := \frac{||w^{e_0}_{0,0,0}||_r}{c} + ||w^0_{0,1,0}||_r, \\ ||v_m||_{r,c,\lambda} &:= \sum_{\substack{i+j+|\beta|=m \\ [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \frac{||w^\beta_{i,j,k}||_r \lambda^k}{c^{<\beta>}} \quad for \quad m \ge 2 \end{aligned} \tag{3.3}$$

*for $c > 0$ and $\lambda > 0$, where*

$$||w^\beta_{i,j,k}||_r = \max_{x\in D_r} |w^\beta_{i,j,k}(x)| \text{ and } <\beta> = \sum_{|l|\ge 0} (|l|+1)\beta_l.$$

We will make use of

**Lemma 3.4** *For a holomorphic function $f(x)$ on $D_{R_0}$, we have*

$$||\partial^\alpha_x f||_R \le \frac{\alpha!}{(R_0 - R)^{|\alpha|}} ||f||_{R_0} \quad for \quad 0 < R < R_0.$$

Proof. By Cauchy's integral formula, we have the desired result. Q.E.D

**Lemma 3.5** *If a holomorphic function $f(x)$ on $D_R$ satisfies*

$$||f||_r \le \frac{C}{(R-r)^p} \quad for \quad 0 < r < R$$

*then we have*

$$||\partial_i f||_r \le \frac{Ce(p+1)}{(R-r)^{p+1}} \quad for \quad 0 < r < R, \quad i = 1, \dots, n.$$

For the proof, see Hörmander ([5]lemma 5.1.3)

Let us show the following estimate for the function $Lv_m$.

**Lemma 3.6** *Let $0 < R < R_0$. Then there exists a positive constant $\sigma$ such that for $m \ge 2$, if $v_m \in V_m$ we have*

$$||Lv_m||_{r,c,\lambda} \ge \frac{\sigma}{2} m ||v_m||_{r,c,\lambda} \quad for \quad 0 < r \le R$$

*for sufficiently small $c > 0$ and sufficiently large $\lambda > 0$.*

Let us estimate the function $\partial_i v_m$.

**Definition 3.7** *For the function $v_m \in V_m$ we define*

$$D_p v_m := \sum_{\substack{i+j+|\beta|=m \\ [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \partial_p w^{\beta}_{i,j,k}(x) t^{i+j\rho(x)}\{\log t\}^k \Phi_1^{\beta}$$

*for $p = 1, \dots, n$.*

**Lemma 3.8** *If $v_m \in V_m$, then for $i = 1, \dots, n$, we have*

$$||\partial_i v_m||_{r,c,\lambda} \le ||D_i v_m||_{r,c,\lambda} + c_0\lambda m ||v_m||_{r,c,\lambda} + \frac{3m-2}{c}||v_m||_{r,c,\lambda} \quad \text{for} \quad 0 < r \le R. \tag{3.4}$$

Therefore by the relations (2.7), (2.8) and Lemma 3.8, we have the following lemma.

**Lemma 3.9** *If $u = \sum_{m\ge 1} v_m$ is a formal solution of the equation* (2.1) *constructed in Section 2, we have the following inequality for $v_m$ $(m \ge 2)$:*

$$\begin{aligned} ||Lv_m||_{r,c,\lambda} \le & \sum_{\substack{p+|\alpha|\ge 2 \\ p+|m_n|=m}} ||a_{p,\alpha}||_r \prod_{h_0=1}^{\alpha_0} ||v_{m_{0,h_0}}||_{r,c,\lambda} \\ & \times \prod_{i=1}^{n} \prod_{h_i=1}^{\alpha_i} \{||D_i v_{m_{i,h_i}}||_{r,c,\lambda} + c_0\lambda m_{i,h_i} ||v_{m_{i,h_i}}||_{r,c,\lambda} + \frac{3m_{i,h_i}-2}{c}||v_{m_{i,h_i}}||_{r,c,\lambda}\}. \end{aligned}$$

Let us define a majorant equation to show that the formal solution (3.1) converges.
We take $A_1$ so that

$$\frac{||w^{eo}_{0,0,0}||_R}{c} + ||w^0_{0,1,0}||_R \le A_1,$$
$$\frac{||\partial_i w^{eo}_{0,0,0}||_R}{c} + ||\partial_i w^0_{0,1,0}||_R \le A_1$$

for $i = 1, \dots, n$.

Then we consider the following equation:

$$\begin{aligned} \frac{\sigma}{2}Y = & \frac{\sigma}{2}A_1 t_1 \\ & + \frac{1}{R-r}\sum_{p+|\alpha|\ge 2} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} t_1{}^p Y^{\alpha_0} \prod_{i=1}^{n}\left(eY + c_0\lambda Y + \frac{3}{c}Y\right)^{\alpha_i}. \end{aligned} \tag{3.5}$$

The equation (3.5) has a unique holomorphic solution $Y = Y(t_1)$ with $Y(0) = 0$ at $(Y, t_1) = (0,0)$ by implicit function theorem. By an easy calculation, the solution $Y = Y(t_1)$ has the following form:

$$Y = \sum_{m\geq 1} Y_m {t_1}^m \text{ with } Y_m = \frac{C_m}{(R-r)^{m-1}}$$

where $Y_1 = C_1 = A_1$ and $C_m \geq 0$ for $m \geq 1$.
Then we have;

**Lemma 3.10** *For $m \geq 1$, we have*

$$m||v_m||_{r,c,\lambda} \leq Y_m \quad \text{for} \quad 0 < r < R. \tag{3.6}$$

Let us show that the formal solution (3.1) converges by using (3.6) in Lemma 3.10. We rewrite $v_m$ as follows:

$$v_m = \sum_{\substack{i+j+|\beta|=m \\ [\beta]\leq m-2}} \sum_{\substack{k\leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \frac{w^{\beta}_{i,j,k}(x)\lambda^k}{c^{<\beta>}} t^{i+j\rho(x)} \left(\frac{\log t}{\lambda}\right)^k \Psi^{\beta}_1,$$

where

$$\Psi^{\beta}_1 = \prod_{|l|\geq 0} \left(c^{|l|+1}\frac{\partial^l_x \phi_1}{l!}\right)^{\beta_l}. \tag{3.7}$$

Firstly let us estimate (3.7). For $||\phi_1||_R$, we have the following lemma.

**Lemma 3.11** *For any $\gamma$ with $0 < \gamma < 1$, there is an $R > 0$ such that*

$$||\phi_1||_R = O\left(|t|^{\gamma}\right) \text{ as } t \to 0 \text{ in } S_{\theta}$$

*holds for any $\theta > 0$.*

By Lemma 3.11, there exists a positive constant $c_1$ such that

$$||\phi_1||_R \leq c_1|t|^{\gamma} \quad \text{in} \quad S_{\theta}. \tag{3.8}$$

By Lemma 3.4 and (3.8), we have

$$||\Psi^{\beta}_1||_r \leq \prod_{|l|\geq 0} \left(c^{|l|+1}\frac{c_1}{(R-r)^{|l|}}|t|^{\gamma}\right)^{\beta_l} = \left(\frac{c}{R-r}\right)^{<\beta>} (c_1(R-r)|t|^{\gamma})^{|\beta|} \tag{3.9}$$

for $0 < r < R < R_0$ in $S_{\theta}$.

Let us estimate $t^{i+j\rho(x)}\left(\dfrac{\log t}{\lambda}\right)^k \Psi^{\beta}_1$.

We put $\eta(t,\lambda) = \max\left\{\left|\frac{\log t}{\lambda}\right|,\ 1\right\}$, $c_2 = \max\left\{\frac{c}{R-r}, 1\right\}$ and $c_3 = c_1(R-r)$. Since we have $[\beta] \leq m-2 < m = i+j+|\beta|$, $< \beta > \leq 2|\beta| + [\beta] \leq i+j+3|\beta|$ and $k \leq i + |\beta|_0 + |\beta|_1 + 2(j-1) \leq i + |\beta| + 2j$, we obtain

$$\left\| t^{i+j\rho(x)} \left(\frac{\log t}{\lambda}\right)^k \Psi_1^\beta \right\|_r \leq \{|c_2\eta(t,\lambda)t|\}^i \left\{||c_2\eta(t,\lambda)^2 t^{\rho(x)}||_r\right\}^j \left\{|(c_2)^3 c_3 \eta(t,\lambda)t^\gamma|\right\}^{|\beta|}$$

in $S_\theta$. For any sufficiently small $\epsilon > 0$, there exists a sufficiently small $\delta > 0$ such that for any $t \in S_\theta$ with $0 < |t| < \delta$ we have

$$|c_2\eta(t,\lambda)t| < \epsilon, \ ||c_2\eta(t,\lambda)^2 t^{\rho(x)}||_r < \epsilon, \ |(c_2)^3 c_3 \eta(t,\lambda)t^\gamma| < \epsilon.$$

Then by Lemma 3.10, we have

$$||u||_r \leq \sum_{m\geq 1} Y_m \epsilon^m \tag{3.10}$$

for sufficiently small $|t|$ in $S_\theta$. Hence the formal solution (3.1) converges for $x \in D_r$ and sufficiently small $|t|$ in $S_\theta$. Q.E.D.

# References


[1] Briot, Ch. and Bouquet, J. Cl., *Recherches sur les propriétés des fonctions définies par des équations différentielles*, J. Ecole Polytech., 21(1856), 133–197.

[2] Gérard, R. and Tahara, H., *Holomorphic and Singular Solutions of Nonlinear Singular First Order Partial Differential Equations*, Publ. RIMS, Kyoto Univ., 26(1990), 979–1000.

[3] ———, *Singular Nonlinear Partial Differential Equations*, Vieweg, 1996.

[4] Hill, E., *Ordinary differential equations in the complex domain*, John Wiley and Sons, 1976.

[5] Hörmander, L., *Linear partial differential operators*, Springer, 1963.

[6] Kimura T., *Ordinary differential equations*, Iwanami Shoten, 1977 (in Japanese).

[7] Yamane, H., *Nonlinear Singular First Order Partial Differential Equations Whose Characteristic Exponent Takes a Positive Integral Value*, Publ. RIMS, Kyoto Univ., 33(1997), 801–811.