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Singular Solutions of the Briot-Bouquet Type
Partial Differential Equations

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1 Introduction

In this talk, we will study the following type of nonlinear singular first order partial differential equations:

\[ t \partial_t u = F(t, x, u, \partial_x u) \]  \hspace{1cm} (1.1)

where \((t, x) = (t, x_1, \ldots, x_n) \in \mathbb{C}_t \times \mathbb{C}_x^n, \partial_x u = (\partial_1 u, \ldots, \partial_n u), \partial_t = \frac{\partial}{\partial t}, \partial_i = \frac{\partial}{\partial x_i} \)

for \(i = 1, \ldots, n\), and \(F(t, x, u, v)\) with \(v = (v_1, \ldots, v_n)\) is a function defined in a polydisk \(\Delta\) centered at the origin of \(\mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_u \times \mathbb{C}_v^n\). Let us denote \(\Delta_0 = \Delta \cap \{t = 0, u = 0, v = 0\}\).

The assumptions are as follows:

- **(A1)** \(F(t, x, u, v)\) is holomorphic in \(\Delta\),
- **(A2)** \(F(0, x, 0, 0) = 0\) in \(\Delta_0\),
- **(A3)** \(\frac{\partial F}{\partial v_i}(0, x, 0, 0) = 0\) in \(\Delta_0\) for \(i = 1, \ldots, n\).

**Definition 1.1** ([2], [3]) *If the equation (1.1) satisfies (A1), (A2) and (A3) we say that the equation (1.1) is of Briot-Bouquet type with respect to \(t\).*

**Definition 1.2** ([2], [3]) *Let us define*

\[ \rho(x) = \frac{\partial F}{\partial u}(0, x, 0, 0), \]

*then the holomorphic function \(\rho(x)\) is called the characteristic exponent of the equation (1.1).*

Let us denote by

1. \(\mathcal{R}(\mathbb{C}\backslash\{0\})\) the universal covering space of \(\mathbb{C}\backslash\{0\}\),
2. \(S_\theta = \{t \in \mathcal{R}(\mathbb{C}\backslash\{0\}); |\arg t| < \theta\}\),
3. \(S(\epsilon(s)) = \{t \in \mathcal{R}(\mathbb{C}\backslash\{0\}); 0 < |t| < \epsilon(\arg t)\}\) for some positive-valued function \(\epsilon(s)\) defined and continuous on \(\mathbb{R}\),
4. \( D_R = \{ x \in \mathbb{C}^n; |x_i| < R \text{ for } i = 1, \ldots, n \} \),

5. \( \mathcal{C}\{x\} \) the ring of germs of holomorphic functions at the origin of \( \mathbb{C}^n \).

**Definition 1.3** We define the set \( \tilde{O}_+ \) of all functions \( u(t, x) \) satisfying the following conditions;

1. \( u(t, x) \) is holomorphic in \( S(\epsilon(s)) \times D_R \) for some \( \epsilon(s) \) and \( R > 0 \),
2. there is an \( a > 0 \) such that for any \( \theta > 0 \) and any compact subset \( K \) of \( D_R \)

\[
\max_{x \in K} |u(t, x)| = O(|t|^a) \quad \text{as} \quad t \to 0 \quad \text{in} \quad S_\theta.
\]

We know some results on the equation (1.1) of Briot-Bouquet type with respect to \( t \). We concern the following result. Gérard R. and Tahara H. studied in [2] the structure of holomorphic and singular solutions of the equation (1.1) and proved the following result;

**Theorem 1.4 (Gérard R. and Tahara H.)** If the equation (1.1) is of Briot-Bouquet type and \( \rho(0) \not\in \mathbb{N}^* = \{1, 2, 3, \ldots\} \) then we have;

1. (Holomorphic solutions) The equation (1.1) has a unique solution \( u_0(t, x) \) holomorphic near the origin of \( \mathbb{C} \times \mathbb{C}^n \) satisfying \( u_0(0, x) \equiv 0 \).
2. (Singular solutions) Denote by \( S_+ \) the set of all \( \tilde{O}_+ \)-solutions of (1.1).

\[
S_+ = \begin{cases} \{u_0(t, x)\} & \text{when } \text{Re}(0) \leq 0, \\
\{u_0(t, x)\} \cup \{U(\varphi); 0 \neq \varphi(x) \in \mathcal{C}\{x\}\} & \text{when } \text{Re}(0) > 0,
\end{cases}
\]

where \( U(\varphi) \) is an \( \tilde{O}_+ \)-solution of (1.1) having an expansion of the following form:

\[
U(\varphi) = \sum_{i \geq 1} u_i(x)t^i + \sum_{i+j+k \geq 1, j \geq 1} \varphi_{i,j,k}(x)t^{i+j+(j-1)}(\log t)^k, \quad \varphi_{0,1,0}(x) = \varphi(x).
\]

The purpose of this paper is to determine \( S_+ \) in the case \( \rho(0) \in \mathbb{N}^* \).

The main result of this paper is;

**Theorem 1.5** If the equation (1.1) is of Briot-Bouquet type and if \( \rho(0) = N \in \mathbb{N}^* \) and \( \rho(x) \not\equiv \rho(0) \), then

\[
S_+ = \{U(\varphi); \varphi(x) \in \mathcal{C}\{x\}\},
\]

where \( U(\varphi) \) is an \( \tilde{O}_+ \)-solution of (1.1) having an expansion of the following form:

\[
U(\varphi) = u_0^1(x)t + u_0^{e_0}(x)\phi_N(t, x) + \sum_{i+j+k \geq 1, j \geq 1, \beta \leq i+2j+2} u_{i,j,k}^{\beta}(x)t^{i+j+2(j-1)}(\log t)^k
\]

\[
+ w_{0,1,0}^0(x)t^{\rho(x)} + \sum_{i+j+k \geq 1, j \geq 1, \beta \leq i+2j+2} \sum_{k \leq i+2j+2} u_{i,j,k}^{\beta}(x)t^{i+j+(j-1)}(\log t)^k \Phi_N^\beta,
\]

\[
\rho(x) < 0.
\]
where $u^0_N(x) \equiv 0$, $w^0_{0,1,0}(x) = \varphi(x)$ is an arbitrary holomorphic function and the other coefficients $u^\beta_l(x)$, $w^\beta_{i,j,k}(x)$ are holomorphic functions determined by $w^0_{0,1,0}(x)$ and defined in a common disk, and

\[ l = (l_1, \ldots, l_n) \in \mathbb{N}^n, \quad |l| = l_1 + \cdots + l_n, \quad \beta = (\beta_l \in \mathbb{N}; \, l \in \mathbb{N}^n), \]
\[ |\beta| = \sum_{|l| \geq 0} \beta_l, \quad |\beta|_p = \sum_{|l| = p} \beta_l \text{ for } p \geq 0, \quad [\beta] = \sum_{|l| \geq 0} \beta_l, \]
\[ \Phi^\beta_N = \prod_{|l| \geq 0} \left( \frac{\partial_{z} \phi_N}{l!} \right)^{\beta_l}, \quad \partial_{z}^l = \partial_{z_1}^l \cdots \partial_{z_n}^l, \quad \phi_N(t, x) = \frac{t^{\rho(x)} - t^{N}}{\rho(x) - N}. \]

The following lemma will play an important role in the proof of Theorem 1.5.

At first, we define some notations. We set for $l \in \mathbb{N}^n$, $e_l = (\beta_k; k \in \mathbb{N}^n)$ with $\beta_l = 1$ and $\beta_k = 0$ for $k \neq l$ and for $p \in \{1, 2, \ldots, n\}$, $e(p) = (i_1, \ldots, i_n)$ with $i_p = 1$ and $i_q = 0$ for $q \neq p$, and define $l^1 < l^0$ is defined by $|l^1| < |l^0|$ and $l^1_i \leq l^0_i$ for $i = 1, \ldots, n$.

**Lemma 1.6** Let $\rho(x)$, $\phi_N$ and $\Phi^\beta_N$ be as in Theorem 1.5. Then we have:

1. $\partial_{x} \Phi^\beta_N = \sum_{|l| \geq 0} \beta_l (l_1 t + 1) \Phi^\beta_{N-e_l+e_l+e(p)}$ for $i = 1, \ldots, n$,
2. $t \partial_{t} \phi_N = \rho(x) \phi_N + t^N$,
3. $t \partial_{t} \Phi^\beta_N = |\beta| \rho(x) \Phi^\beta_N + \beta_0 t^N \Phi^\beta-e_0 + \sum_{|l| \geq 1} \sum_{|l| < l} \beta_l \frac{\partial^{p-1} \rho(x)}{p!-l!} \Phi^\beta_{N-e_0+e_l}$.

## 2 Construction of formal solutions in the case $\rho(0) = 1$

By [2] (Gérard-Tahara), if the equation (1.1) is of Briot-Bouquet type with respect to $t$, then it is enough to consider the following equation:

\[ Lu = t \partial_{t} u - \rho(x) u = a(x) t + G_2(x)(t, u, \partial_{x} u) \quad (2.1) \]

where $\rho(x)$ and $a(x)$ are holomorphic functions in a neighborhood of the origin, and the function $G_2(x)(t, X_0, X_1, \ldots, X_n)$ is a holomorphic function in a neighborhood of the origin in $\mathbb{C}_x^n \times \mathbb{C}_t \times \mathbb{C}_X_0 \times \mathbb{C}_X_1 \times \cdots \times \mathbb{C}_X_n$ with the following expansion:

\[ G_2(x)(t, X_0, X_1, \ldots, X_n) = \sum_{p+|\alpha| \geq 2} a_{p,\alpha}(x) t^p \{X_0\}^{\alpha_0} \{X_1\}^{\alpha_1} \cdots \{X_n\}^{\alpha_n} \]

and we may assume that the coefficients $\{a_{p,\alpha}(x)\}_{p+|\alpha| \geq 2}$ are holomorphic functions on $D_{R_0}$ for a sufficiently small $R_0 > 0$. Let $0 < R < R_0$. We put $A_{p,\alpha}(R) := \max_{x \in D_R} |a_{p,\alpha}(x)|$ for $p + |\alpha| \geq 2$. Then for $0 < r < R$

\[ \sum_{p+|\alpha| \geq 2} A_{p,\alpha}(R) \frac{R^p}{(R-r)^{|\alpha|-2}} X_0^{\alpha_0} X_1^{\alpha_1} \cdots X_n^{\alpha_n} \quad (2.2) \]
is convergent in a neighborhood of the origin.

In this section, we assume \( \rho(0) = 1 \) and \( \rho(x) \not\equiv 1 \) and we will construct formal solutions of the equation (2.1). In generally, we set \( u(t, x) = \sum_{i=1}^{N-1} u_i(x) t^i + t^{N-1} w(t, x) \), and we consider an equation for \( w(t, x) \).

**Proposition 2.1** If \( \rho(0) = 1 \) and \( \rho(x) \not\equiv 1 \), the equation (2.1) has a family of formal solutions of the form:

\[
    u = u_0^0(x) \phi_1 + \sum_{m \geq 2} \sum_{i+|\beta|=m \atop [\beta] \leq m-2} u_i^\beta(x) t^i \Phi_1^\beta
    + w_{0,1,0}^0(x) t^{\rho(x)} + \sum_{m \geq 2} \sum_{i+j+|\beta|=m,k \leq i+|\beta|_0+|\beta|_1 \atop j \geq 1,[\beta] \leq m-2} w_{i,j,k}^\beta(x) t^{i+j \rho(x)} \{\log t\}^k \Phi_1^\beta
\]

where \( w_{0,1,0}^0(x) \) is an arbitrary holomorphic function and the other coefficients \( u_i^\beta(x), w_{i,j,k}^\beta(x) \) are holomorphic functions determined by \( w_{0,1,0}^0(x) \) and defined in a common disk.

**Remark 2.2** By the relation \([\beta] \leq m - 2\) in summations of the above formal solution, we have \( \beta_l = 0 \) for any \( l \in \mathbb{N}^n \) with \(|l| \geq m\).

We define the following two sets \( U_m \) and \( W_m \) for \( m \geq 1 \) to prove Proposition 2.1.

**Definition 2.3** We denote by \( U_m \) the set of all functions \( u_m \) of the following forms:

\[
    u_1 = u_1^0(x) t + u_0^0(x) \phi_1,
    u_m = \sum_{i+|\beta|=m \atop [\beta] \leq m-2} u_i^\beta(x) t^i \Phi_1^\beta \quad \text{for } m \geq 2,
\]

and denote by \( W_m \) the set of all functions \( w_m \) of the following forms:

\[
    w_1 = w_{0,1,0}^0(x) t^{\rho(x)},
    w_m = \sum_{i+j+|\beta|=m \atop k \leq i+|\beta|_0+|\beta|_1 \atop j \geq 1,[\beta] \leq m-2} w_{i,j,k}^\beta(x) t^{i+j \rho(x)} \{\log t\}^k \Phi_1^\beta \quad \text{for } m \geq 2
\]

where \( u_i^\beta(x), w_{i,j,k}^\beta(x) \in \mathbb{C}\{x\} \).

We can rewrite the formal solution (2.3) as follows:

\[
    u = \sum_{m \geq 1} (u_m + w_m) \quad \text{where } u_m \in U_m, w_m \in W_m.
\]
Let us show important relations of \( u_m \) and \( w_m \) for \( m \geq 2 \). By Lemma 1.6, we have

\[
Lu_m = \sum_{i+j+|\beta|=m} \sum_{k \leq i+|\beta|} \sum_{l=0}^{m-2} \{i+(j+|\beta|-1)\rho(x)\} w_i^\beta(x) t^{i+j+\rho(x)} \{\log t\}^{k-1} \Phi_1^\beta
\]

\[
= k u_i^\beta(x) t^{i+j+\rho(x)} \{\log t\}^{k-1} \Phi_1^\beta + \sum_{|l|=1}^{m-1} \sum_{l<j} \beta_{l,j} \frac{\partial^{l-j}}{(l-j)!} u_i^\beta(x) t^{i+j+\rho(x)} \{\log t\}^{k-1} \Phi_1^\beta
\]

We show two lemmas.

**Lemma 2.4** If \( u_m \in U_m \) and \( w_m \in W_m \), then \( Lu_m \in U_m \) and \( Lw_m \in W_m \).

**Lemma 2.5** If \( u_m \in U_m \) and \( w_m \in W_m \), then the following relations hold for \( i, j = 1, \ldots, n \):

1. \( a(x) U_m \subset U_m \) and \( a(x) W_m \subset W_m \) for any holomorphic function \( a(x) \),
2. \( t U_m, \phi_1 U_m \subset U_{m+1} \) and \( t \phi_1 U_m, t \phi_1 W_m, \phi_1 W_m \subset W_{m+1} \),
3. \( u_m \times u_n, \partial_t u_m \times \partial_t u_n, \partial_t u_m \times u_n \in U_{m+n} \),
4. \( u_m \times u_n, \partial_t u_m \times \partial_t u_n, \partial_t u_m \times w_n \in W_{m+n} \),
5. \( u_m \times u_n, \partial_t u_m \times u_n, \partial_t u_m \times \partial_j w_n \in W_{m+n} \).

Let us show that \( u_m \) and \( w_m \) are determined inductively on \( m \geq 1 \). By substituting \( \sum_{m=1} u_m + w_m \) into (2.1), we have

\[
(1 - \rho(x)) u_1^0(x) + u_0^0(x) = a(x),
\]

and for \( m \geq 2 \)

\[
Lu_m = \sum_{p+|a| \geq 2} \sum_{p+|m_a| = m} a_{p,a}(x) t^p \prod_{h_0=1}^{\alpha_0} u_{m_{0,h_0}} \prod_{j=1}^{n} \prod_{h_j=1}^{\alpha_j} \partial_j u_{m_{j,h_j}},
\]

\[
Lw_m = \sum_{p+|a| \geq 2} \sum_{p+|m_a| = m} a_{p,a}(x) t^p \prod_{h_0=1}^{\alpha_0} (u_{m_{0,h_0}} + w_{m_{0,h_0}}) \prod_{j=1}^{n} \prod_{h_j=1}^{\alpha_j} \partial_j (u_{m_{j,h_j}} + w_{m_{j,h_j}})
\]

\[
- \sum_{p+|a| \geq 2} \sum_{p+|m_a| = m} a_{p,a}(x) t^p \prod_{h_0=1}^{\alpha_0} u_{m_{0,h_0}} \prod_{j=1}^{n} \prod_{h_j=1}^{\alpha_j} \partial_j u_{m_{j,h_j}},
\]

where \( |m_a| = \sum_{i=0}^{n} m_i(\alpha_i) \) and \( m_i(\alpha_i) = m_{i,1} + \cdots + m_{i,\alpha_i} \) for \( i = 0, 1, \ldots, n \).

We take any holomorphic function \( \varphi(x) \in \mathbb{C}\{x\} \) and put \( u_0^0(x) = \varphi(x) \), and by (2.6), we put \( u_1^0(x) = 0 \) and \( u_0^0(x) = a(x) \).
For $m \geq 2$, let us show that $u_m$ and $w_m$ are determined by induction. By Lemma 2.5, the right side of (2.7) belongs to $U_m$ and the right side of (2.8) belongs to $W_m$. Further by $m_{j,h_j} \geq 1$, we have $m_{j,h_j} < m$ for $h_j = 1, \ldots, \alpha_j$ and $j = 0, \ldots, n$. Then for $m \geq 2$, we compare with the coefficients of $t^i \Phi_1^\beta$ and $t^{i+j \rho(x)} \{\log t\}^k \Phi_1^\beta$ respectively for (2.7) and (2.8), then put

$$
\begin{align*}
&\{i + (|\beta| - 1)\rho(x)\} u_i^\beta (x) \\
&+ (\beta_0 + 1) u_{i-1}^\beta (x) + \sum_{|\rho_0| = 1}^{m-1} \sum_{0 \leq l < \rho_0} (\beta_0 + 1) \frac{\partial^l \rho(x)}{(l!)} u_i^{\rho_0 - l^1} (x) \\
&= f_i^\beta (\{a_{p,\alpha}\}_{2 \leq p + |\alpha| \leq m}, \{u_{i'}^\beta(x)\}_{i'+|\beta'|<m})
\end{align*}
(2.9)
$$

and

$$
\begin{align*}
&\{i + (j + |\beta| - 1)\rho(x)\} w_{i,j,k}^\beta (x) + (k + 1) w_{i,j,k+1}^\beta (x) \\
&+ (\beta_0 + 1) w_{i-1,j,k}^\beta (x) + \sum_{|\rho_0| = 1}^{m-1} \sum_{0 \leq l < \rho_0} (\beta_0 + 1) \frac{\partial^l \rho(x)}{(l!)} w_{i,j,k}^{\rho_0 - l} (x) \\
&= g_{i,j,k}^\beta (\{a_{p,\alpha}\}_{2 \leq p + |\alpha| \leq m}, \{u_{i'}^\beta(x)\}_{i'+|\beta'|<m}, \{w_{i,j',k'}^\beta(x)\}_{i'+j'+|\beta'|<m})
\end{align*}
(2.10)
$$

Hence we obtain Proposition 2.1. Q.E.D.

3 Convergence of the formal solutions in the case $\rho(0) = 1$

In this section, we show that the formal solution (2.3) converges in $\tilde{O}_+$. 

**Proposition 3.1** Let $\gamma$ satisfy $0 < \gamma < 1$ and let $\lambda$ be sufficiently large. Then for any sufficiently small $r > 0$ we have the following result;

For any $\theta > 0$ there is an $\epsilon > 0$ such that the formal solution (2.3) converges in the following region:

$$\{ (t, x) \in C_t \times C_x^n; |\eta(t, \lambda)t| < \epsilon, |\eta(t, \lambda)^2 t^{\rho(x)}| < \epsilon, |\eta(t, \lambda)^l t^\gamma| < \epsilon, t \in S_\theta \text{ and } x \in D_r \},$$

where $\eta(t, \lambda) = \max \{|(\log t)/\lambda|, 1\}$.

In this section, we put $w_{i,0,0}^\beta(x) = u_{i}^\beta(x)$ and $w_{i,0,k}^\beta(x) \equiv 0$ for $k \geq 1$ in the formal solution (2.3). Then the formal solution (2.3) is as follows:

$$u = u_{0,0,0}^\beta(x) \phi_1 + u_{0,1,0}(x) t^{\rho(x)} + \sum_{m \geq 2} \sum_{i+j+|\beta| = m} \sum_{k \leq i+j+|\beta|} \sum_{|\beta| \leq m-2} w_{i,j,k}^\beta(x) t^{i+j+\rho(x)} \{\log t\}^k \Phi_1^\beta.$$

(3.1)

Let us define the following set $V_m$ for (3.1).
Definition 3.2 We denote by $V_m$ the set of all the functions $v_m$ of the following forms:

$$v_m = u_m + w_m \quad \text{for} \quad u_m \in U_m \quad \text{and} \quad w_m \in W_m.$$  \hfill (3.2)

We define the following estimate for the function $v_m$.

Definition 3.3 For the function (3.2), we define

$$||v_1||_{r,c,\lambda} = ||v_1||_{r,c} := \frac{||w_{0,0,0}^0||_r}{c} + ||w_{0,1,0}^0||_r,$$

$$||v_m||_{r,c,\lambda} := \sum_{i+j+|\beta|=m \leq m-2} \sum_{|\beta|_0+|\beta|_1} \frac{||w_{i,j,k}^\beta||_r \lambda^k}{c^{<\beta>}}$$

for $m \geq 2$ \hfill (3.3)

for $c > 0$ and $\lambda > 0$, where

$$||w_{i,j,k}^\beta||_r = \max_{x \in D_r} |w_{i,j,k}^\beta(x)| \quad \text{and} \quad <\beta> = \sum_{|\beta|_0^\geq 0} (|\beta|+1)\beta_i.$$ We will make use of

Lemma 3.4 For a holomorphic function $f(x)$ on $D_{R_0}$, we have

$$||\partial_x^\alpha f||_R \leq \frac{\alpha!}{(R_0-R)^{|\alpha|}} ||f||_{R_0} \quad \text{for} \quad 0 < R < R_0.$$

Proof. By Cauchy’s integral formula, we have the desired result. Q.E.D

Lemma 3.5 If a holomorphic function $f(x)$ on $D_R$ satisfies

$$||f||_r \leq \frac{C}{(R-r)^p} \quad \text{for} \quad 0 < r < R$$

then we have

$$||\partial_i f||_r \leq \frac{C(p+1)}{(R-r)^{p+1}} \quad \text{for} \quad 0 < r < R, \quad i = 1, \ldots, n.$$ For the proof, see Hörmander ([5]lemma 5.1.3)

Let us show the following estimate for the function $L v_m$.

Lemma 3.6 Let $0 < R < R_0$. Then there exists a positive constant $\sigma$ such that for $m \geq 2$, if $v_m \in V_m$ we have

$$||Lv_m||_{r,c,\lambda} \geq \frac{\sigma}{2} m ||v_m||_{r,c,\lambda} \quad \text{for} \quad 0 < r \leq R$$

for sufficiently small $c > 0$ and sufficiently large $\lambda > 0$.  

---

\[178\]
Let us estimate the function $\partial_i v_m$.

**Definition 3.7** For the function $v_m \in V_m$ we define

$$D_p v_m := \sum_{i+j+|\beta|=m} \sum_{k \leq i+|\beta| + |\beta|_1 \leq m-2} \partial_p w^\beta_{i,j,k}(x) \{t^{i+jp(x)} \log t\}^k \Phi_1^\beta$$

for $p = 1, \ldots, n$.

**Lemma 3.8** If $v_m \in V_m$, then for $i = 1, \ldots, n$, we have

$$\|\partial_i v_m\|_{r,c,\lambda} \leq \|D_i v_m\|_{r,c,\lambda} + c_0 \lambda m \|v_m\|_{r,c,\lambda} + \frac{3m - 2}{c} \|v_m\|_{r,c,\lambda}$$

for $0 < r \leq R$.

(3.4)

Therefore by the relations (2.7), (2.8) and Lemma 3.8, we have the following lemma.

**Lemma 3.9** If $u = \sum_{m \geq 1} v_m$ is a formal solution of the equation (2.1) constructed in Section 2, we have the following inequality for $v_m$ ($m \geq 2$):

$$\|L v_m\|_{r,c,\lambda} \leq \sum_{p+|\alpha|\geq 2} \|a_{p,\alpha}\|_r \prod_{h_0=1}^{\alpha_0} \|v_{m_{0,h_0}}\|_{r,c,\lambda}$$

$$\times \prod_{i=1}^{n} \prod_{h_i=1}^{a_i} \{\|D_i v_{m_{i,h_i}}\|_{r,c,\lambda} + c_0 \lambda m_{i,h_i} \|v_{m_{i,h_i}}\|_{r,c,\lambda} + \frac{3m_{i,h_i} - 2}{c} \|v_{m_{i,h_i}}\|_{r,c,\lambda}\}.$$

Let us define a majorant equation to show that the formal solution (3.1) converges.

We take $A_1$ so that

$$\frac{\|w_{0,0,0}^0\|_R}{c} + \|v_{0,1,0}^0\|_R \leq A_1,$$

$$\frac{\|\partial_i w_{0,0,0}^0\|_R}{c} + \|\partial_i v_{0,1,0}^0\|_R \leq A_1$$

for $i = 1, \ldots, n$.

Then we consider the following equation:

$$\frac{\sigma}{2} \ Y = \frac{\sigma}{2} A_1 t_1$$

$$+ \frac{1}{R - r} \sum_{p+|\alpha|\geq 2} \frac{A_{p,a}(R)}{(R - r)^{p+|\alpha|-2}} t_1^p Y^{\alpha_0} \prod_{i=1}^{n} \left(e Y + c_0 \lambda Y + \frac{3}{c} Y\right)^{\alpha_i}.$$(3.5)
The equation (3.5) has a unique holomorphic solution $Y = Y(t_1)$ with $Y(0) = 0$ at $(Y, t_1) = (0, 0)$ by implicit function theorem. By an easy calculation, the solution $Y = Y(t_1)$ has the following form:

$$Y = \sum_{m \geq 1} Y_m t_1^m \text{ with } Y_m = \frac{C_m}{(R - r)^{m-1}}$$

where $Y_1 = C_1 = A_1$ and $C_m \geq 0$ for $m \geq 1$.

Then we have:

**Lemma 3.10** For $m \geq 1$, we have

$$m ||v_m||_{r, c, \lambda} \leq Y_m \text{ for } 0 < r < R.$$  (3.6)

Let us show that the formal solution (3.1) converges by using (3.6) in Lemma 3.10. We rewrite $v_m$ as follows:

$$v_m = \sum_{i+j+|\beta| = m} \sum_{k \leq \gamma + |\beta|} \frac{w_{\frac{\beta}{\lambda}}^i (x)}{c^{<\beta>}} \frac{\lambda^k}{t^{i+j|\rho(x)|}} \left( \frac{\log t}{\lambda} \right)^k \Psi_1^\beta,$$

where

$$\Psi_1^\beta = \prod_{|l| \geq 0} \left( c^{l+1} \frac{\partial_x \phi_1}{l!} \right)^{\beta_l}.$$  (3.7)

Firstly let us estimate (3.7). For $||\phi_1||_R$, we have the following lemma.

**Lemma 3.11** For any $\gamma$ with $0 < \gamma < 1$, there is an $R > 0$ such that

$$||\phi_1||_R = O (|t|^\gamma) \text{ as } t \to 0 \text{ in } S_\theta$$

holds for any $\theta > 0$.

By Lemma 3.11, there exists a positive constant $c_1$ such that

$$||\phi_1||_R \leq c_1 |t|^\gamma \text{ in } S_\theta.$$  (3.8)

By Lemma 3.4 and (3.8), we have

$$||\Psi_1^\beta||_r \leq \prod_{|\beta| \geq 0} \left( c^{l+1} \frac{\partial_x \phi_1}{l!} |t|^\gamma \right)^{\beta_l} = \left( \frac{c}{R - r} \right)^{<\beta>} \left( c_1 (R - r) |t|^\gamma \right)^{|\beta|}$$

for $0 < r < R < R_0$ in $S_\theta$.

Let us estimate $t^{i+j\rho(x)} \left( \frac{\log t}{\lambda} \right)^k \Psi_1^\beta$.

We put $\eta(t, \lambda) = \max \left\{ \left| \frac{\log t}{\lambda} \right|, 1 \right\}$, $c_2 = \max \left\{ \frac{c}{R - r}, 1 \right\}$ and $c_3 = c_1 (R - r)$. Since
we have $|\beta| \leq m - 2 < m = i + j + |\beta| < \beta > \leq 2|\beta| + |\beta| \leq i + j + 3|\beta|$ and $k \leq i + |\beta|_0 + |\beta|_1 + 2(j - 1) \leq i + |\beta| + 2j$, we obtain
\[
\left\| t^{i+j\rho(x)} \left( \frac{\log t}{\lambda} \right)^k \right\| < \left\{ |c_2 \eta(t, \lambda) t| \right\}^i \left\{ \left| (c_2)^3 c_3 \eta(t, \lambda) t^\gamma \right| \right\}^j \leq i + j + 3|\beta| < \beta > \leq 2|\beta| + |\beta| \leq i + j + 3|\beta|
\]
in $S_\theta$. For any sufficiently small $\epsilon > 0$, there exists a sufficiently small $\delta > 0$ such that for any $t \in S_\theta$ with $0 < |t| < \delta$ we have
\[
|c_2 \eta(t, \lambda) t| < \epsilon, \quad ||c_2 \eta(t, \lambda)^2 t^\rho(z)||_r < \epsilon, \quad |(c_2)^3 c_3 \eta(t, \lambda) t^\gamma| < \epsilon.
\]
Then by Lemma 3.10, we have
\[
||u||_r \leq \sum_{m \geq 1} Y_m \epsilon^m
\]
for sufficiently small $|t|$ in $S_\theta$. Hence the formal solution (3.1) converges for $x \in D_r$ and sufficiently small $|t|$ in $S_\theta$. Q.E.D.

References


