数理解析研究所講究録 1261 巻 2002 年 182-191

Modified Elastic Wave Equations on Riemannian Manifolds and Kähler Manifolds

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The physical generalization of the elastic wave equation on a Riemannian manifold does not necessarily admit any decomposition of solutions into longitudinal wave solutions and transverse wave solutions. However we will show that some modified systems of equations on Riemannian manifolds have good properties as to such decompositions. That is, we introduce some geometrically invariant systems of differential equations on any Riemannian manifolds and also on any Kähler manifolds, which have the same characteristic varieties with the physical generalizations of the elastic wave equations. Further we prove the local decomposition theorems of distribution solutions for those systems. In particular, the solutions of our systems on Kähler manifolds are decomposed into 4 solutions with different propagation speeds.

Definition 1. Let $\bigwedge^{(p)} T^*M$ be a vector bundle of p-differential forms. Let $\mathcal{E}_M^{(p)}$ be a sheaf of p-forms with C^∞ coefficients, and $\mathcal{D}b_M^{(p)}$ a sheaf of p-forms with distribution coefficients. In this article, distributions do not mean the dual space of $C_0^\infty(M)$. Our distributions behave as "functions" for coordinate transformations.

Definition 2. We denote by $\widetilde{\mathcal{E}}_M^{(p)}$, $\widetilde{\mathcal{D}b}_M^{(p)}$ the sheaves of sections of $\mathcal{E}_{\widetilde{M}}^{(p)}$, $\mathcal{D}b_{\widetilde{M}}^{(p)}$ which do not include the covariant vector dt . It comes to this that for $\alpha \in \widetilde{\mathcal{D}b}_M^{(p)}$, $\beta \in \widetilde{\mathcal{E}}_M^{(p)}$, we get $\langle dt, \alpha\rangle^* = 0$, $\langle dt, \beta\rangle^* = 0$.

Definition 3. The inner products $\langle\cdot,\cdot\rangle^* : \bigwedge^{(p)} T_x^*M \times \bigwedge^{(p)} T_x^*M \to \mathbb{R}$, $\langle\cdot,\cdot\rangle : \bigwedge^{(1)} T_x^*M \times \bigwedge^{(1)} T_xM \to \mathbb{R}$ are defined as follows. We choose a positive orthonormal system $(\omega^1, \cdots, \omega^n)$ of T_x^*M; that is, there is a positive number α

such that $\omega^1 \wedge \cdots \wedge \omega^n = \alpha\Omega_x > 0$. Then for

$$\phi = \sum_{1\le i_1<\cdots<i_p\le n} \phi_{i_1\cdots i_p}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p},$$

$$\psi = \sum_{1\le i_1<\cdots<i_p\le n} \psi_{i_1\cdots i_p}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p},$$

we define

$$\langle\phi,\psi\rangle^* := \sum_{1\le i_1<\cdots<i_p\le n} \phi_{i_1\cdots i_p}\psi_{i_1\cdots i_p},$$

and for

$$\sigma = \sum_{1\le i\le n} \sigma_i dx^i, \qquad \tau = \sum_{1\le i\le n} \tau^i\partial_i\ ,$$

we define

$$\langle\sigma,\tau\rangle := \sum_{1\le i\le n} \sigma_i\tau^i.$$

Definition 4. We denote by $d : \mathcal{D}b_M^{(p)} \longrightarrow \mathcal{D}b_M^{(p+1)}$ the exterior differential operator which acts on $\mathcal{D}b_M^{(p)}$ as a sheaf morphism. Then the following formulas are well-known:

$$\begin{cases} 1.\ d(\phi\pm\psi) = d\phi\pm d\psi & (\phi,\psi\in\mathcal{D}b_M^{(p)}),\\ 2.\ d(\phi\wedge\psi) = d\phi\wedge\psi + (-1)^p\phi\wedge d\psi & (\phi\in\mathcal{E}_M^{(p)},\psi\in\mathcal{D}b_M^{(q)}),\\ 3.\ d(d\phi) = 0 & (\phi\in\mathcal{D}b_M^{(p)}),\\ 4.\ \text{For } f\in\mathcal{D}b_M^{(0)},\ df := \sum \dfrac{\partial f}{\partial x_j}dx^j \in \mathcal{D}b_M^{(1)}. \end{cases}$$

Here $0 \le p \le n$. If $p = n$, $d\phi = 0$ holds.

Definition 5. The vector bundle isomorphism $* : \bigwedge T^*M \to \bigwedge T^*M$ is defined below:

$$\begin{cases} 1.\ * : \bigwedge^{(p)} T_x^*M \mapsto \bigwedge^{(n-p)} T_x^*M \quad \text{is a linear map},\\ 2.\ *\,(\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}) = (-1)^{(i_1-1)+\cdots+(i_p-p)}\omega^{j_1}\wedge\cdots\wedge\omega^{j_{n-p}},\\ \qquad \text{for any permutation } (i_1,\cdots,i_p,j_1,\cdots,j_{n-p}) \text{ of } (1,\cdots,n). \end{cases}$$

Here $(i_1\cdots i_p)$ and $(j_1\cdots j_{n-p})$ are indices satisfying

$$\begin{cases} 1.\ (i_1\cdots i_p j_1\cdots j_{n-p}) \text{ is a permitation of } (1\cdots n),\\ 2.\ 1\le i_1<\cdots<i_p\le n,\ 1\le j_1<\cdots<j_{n-p}\le n. \end{cases}$$

Remark 6. The definition above does not depend on the choice of the positive orthonormal system $\{\omega^1, \cdots, \omega^n\}$.

Proposition 7. *We set* $\phi, \psi \in \bigwedge^{(p)} T_x^* M$. *Then we obtain*

$$\begin{cases} 1.\ \phi \wedge *\psi = (*\phi) \wedge \psi = \langle \phi, \psi \rangle^* \ \omega^1 \wedge \cdots \wedge \omega^n, \\ 2.\ *1 = \omega^1 \wedge \cdots \wedge \omega^n = \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^n, \\ 3.\ *\phi = (-1)^{(i_1-1)+\cdots+(i_p-p)} \sqrt{g}\, g^{i_1 j_1} \cdots g^{i_p j_p} \phi_{i_1 \cdots i_p}\, dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}} \\ \qquad \in \bigwedge^{(n-p)} T_x^* M. \end{cases}$$

Here $g = \det(g_{\lambda\kappa})$.

Let $U \subset M$ be an open subset. We set $\alpha^{(p)} \in \mathcal{D}b_M^{(p)}(U)$, $\beta^{(p)} \in \mathcal{E}_M^{(p)}(U)$. We suppose that $\beta^{(p)}$ has a compact support in U. Then the following integral is well-defined.

$$(\alpha^{(p)}, \beta^{(p)}) := \int_M \langle \alpha^{(p)}, \beta^{(p)} \rangle^* \ \omega^1 \wedge \cdots \wedge \omega^n.$$

Definition 8. We set $\alpha^{(p)} \in \mathcal{D}b_M^{(p)}$, $\beta^{(p-1)} \in \mathcal{E}_M^{(p-1)}$. We suppose $\beta^{(p-1)}$ has a compact support. Then the sheaf morphism $\delta : \mathcal{D}b_M^{(p)} \to \mathcal{D}b_M^{(p-1)}$ is defined as

$$(\delta\alpha^{(p)}, \beta^{(p-1)}) = (\alpha^{(p)}, d\beta^{(p-1)}).$$

Hence we have

$$\delta = (-1)^{n(p-1)+1} * d * .$$

Definition 9. Let $\mathfrak{X}_s^r$ be the sheaf of $\bigotimes^r T_x M \otimes \bigotimes^s T_x^* M$-valued C^∞ functions, and $\mathcal{D}b_s^r$ the sheaf of $\bigotimes^r T_x M \otimes \bigotimes^s T_x^* M$-valued distributions. Then, the sheaf morphisms $\nabla : \mathfrak{X}_s^r \to \mathfrak{X}_{s+1}^r$, $\mathcal{D}b_s^r \to \mathcal{D}b_{s+1}^r$ are defined as follows:

$$\begin{cases} 1.\ \text{For } a(x) \in \mathfrak{X}_0^0, & \text{we have} \quad \nabla a(x) = \dfrac{\partial a}{\partial x^j} dx^j. \\ 2.\ \text{For } \dfrac{\partial}{\partial x^j} \in \mathfrak{X}_0^1, & \text{we have} \quad \nabla\left(\dfrac{\partial}{\partial x^j}\right) = \Gamma_j{}^i{}_k \dfrac{\partial}{\partial x^i} \otimes dx^k. \\ 3.\ \text{For } dx^j \in \mathfrak{X}_1^0, & \text{we have} \quad \nabla\left(dx^j\right) = -\Gamma_i{}^j{}_k dx^i \otimes dx^k. \\ 4.\ \text{For } e \in \mathfrak{X}_s^r, f \in \mathfrak{X}_{s'}^{r'}, & \text{we have} \quad \nabla(e \otimes f) = (\nabla e) \otimes f + e \otimes \nabla f. \end{cases}$$

Here,

$$\left\{ \Gamma_i{}^j{}_k = g^{jl} \Gamma_{ilk} = g^{jl} \cdot \frac{1}{2} \left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^l} \right) \right\}$$

are the Riemann-Christoffel symbols.

Proposition 10. *We set*

$$e = e^{r}_{i_1\cdots i_s} dx^{i_1} \otimes \cdots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}} \in \mathfrak{X}^r_s.$$

Then we have

$$\nabla e = \left(\partial_k e^{r}_{i_1\cdots i_s} + e^{q}_{i_1\cdots i_s} \Gamma_q{}^r{}_k + e^{r}_{i_1\cdots i_{p-1} q i_{p+1}\cdots i_s} \Gamma_{i_p}{}^q{}_k\right) \times dx^k \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}}.$$

Then we call the following *the covariant differentiation* :

$$\nabla_k e = \left(\partial_k e^{r}_{i_1\cdots i_s} + e^{q}_{i_1\cdots i_s} \Gamma_q{}^r{}_k + e^{r}_{i_1\cdots i_{p-1} q i_{p+1}\cdots i_s} \Gamma_{i_p}{}^q{}_k\right) \times dx^{i_1} \otimes \cdots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}}.$$

For

$$u = \sum_{1 \le i_1 < \cdots < i_p \le n} u_{i_1\cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \in \widetilde{\mathcal{D}b}^{(p)}_M,$$

we define an operator P_{R} for $\widetilde{\mathcal{D}b}^{(p)}_M$ on M $(1 \le p \le n-1)$, where the coefficients $\{u_{i_1\cdots i_p}\}$ are supposed to be alternating with respect to $(i_1 \cdots i_p)$.

Definition 11. We define sheaf-morphisms $P_{\mathrm{R}} : \widetilde{\mathcal{D}b}^{(p)}_M \longrightarrow \widetilde{\mathcal{D}b}^{(p)}_M$ by

$$P_{\mathrm{R}}\, u := \rho \frac{\partial^2}{\partial t^2} u + (\lambda + 2\mu) d\delta u + \mu \delta d u,$$

where the density constant ρ and the Làme constants λ, μ are positive.
For $p = 1$, this equation is the covariant form of $P_{\mathrm{R}}\, u^i$.

When $p = 0$ or n, $P_{\mathrm{R}}\, u = 0$ reduces to a wave equation. Therefore we suppose $1 \le p \le n-1$.
and

For $u \in \widetilde{\mathcal{D}b}^{(p)}_M$, we define equations $\mathfrak{M}^{\mathrm{R}}$, $\mathfrak{M}^{\mathrm{R}}_1$, $\mathfrak{M}^{\mathrm{R}}_2$, $\mathfrak{M}^{\mathrm{R}}_0$ below:

$$\mathfrak{M}^{\mathrm{R}} : \quad P_{\mathrm{R}}\, u = 0,$$

$$\mathfrak{M}^{\mathrm{R}}_1 : \begin{cases} P_{\mathrm{R}}\, u = 0, \\ du = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha \Delta) u = 0, \\ du = 0, \end{cases}$$

$$\mathfrak{M}_2^{\mathrm{R}}:\begin{cases} P_{\mathrm{R}}\, u=0, \\ \delta u=0, \end{cases} \Longleftrightarrow \begin{cases} (\partial_t^2+\beta\Delta)u=0, \\ \delta u=0, \end{cases}$$

$$\mathfrak{M}_0^{\mathrm{R}}:\begin{cases} P_{\mathrm{R}}\, u=0, \\ du=0, \\ \delta u=0, \end{cases} \Longleftrightarrow \begin{cases} \partial_t^2 u=0, \\ du=0, \\ \delta u=0. \end{cases}$$

Here, $\alpha=(\lambda+2\mu)/\rho$, $\beta=\mu/\rho$ and $\Delta=d\delta+\delta d:\widetilde{\mathcal{D}b}_M^{(p)}\to\widetilde{\mathcal{D}b}_M^{(p)}$ is the Laplacian.

Further we define subsheaves $\mathcal{S}ol(\mathfrak{M}^{\mathrm{R}};p)$, $\mathcal{S}ol(\mathfrak{M}_1^{\mathrm{R}};p)$, $\mathcal{S}ol(\mathfrak{M}_2^{\mathrm{R}};p)$, $\mathcal{S}ol(\mathfrak{M}_0^{\mathrm{R}};p)$ of $\widetilde{\mathcal{D}b}_M^{(p)}$ as follows:

$$\mathcal{S}ol(\mathfrak{M}^{\mathrm{R}};p):=\left\{u\in\widetilde{\mathcal{D}b}_M^{(p)}\;\middle|\;u \text{ satisfies } \mathfrak{M}^{\mathrm{R}}\right\},$$
$$\mathcal{S}ol(\mathfrak{M}_1^{\mathrm{R}};p):=\left\{u\in\widetilde{\mathcal{D}b}_M^{(p)}\;\middle|\;u \text{ satisfies } \mathfrak{M}_1^{\mathrm{R}}\right\},$$
$$\mathcal{S}ol(\mathfrak{M}_2^{\mathrm{R}};p):=\left\{u\in\widetilde{\mathcal{D}b}_M^{(p)}\;\middle|\;u \text{ satisfies } \mathfrak{M}_2^{\mathrm{R}}\right\},$$
$$\mathcal{S}ol(\mathfrak{M}_0^{\mathrm{R}};p):=\left\{u\in\widetilde{\mathcal{D}b}_M^{(p)}\;\middle|\;u \text{ satisfies } \mathfrak{M}_0^{\mathrm{R}}\right\}.$$

Then, we have the theorem below.

Theorem 12. *For any germ* $u\in\mathcal{S}ol(\mathfrak{M}^{\mathrm{R}};p)\Big|_{(\mathring{t},\mathring{x})}$*, there exist some germs* $u_j\in\mathcal{S}ol(\mathfrak{M}_j^{\mathrm{R}};p)\Big|_{(\mathring{t},\mathring{x})}$ $(j=1,2)$ *such that* $u=u_1+u_2$.

Further, the equation $u=u_1+u_2=0$ *implies* $u_1,u_2\in\mathcal{S}ol(\mathfrak{M}_0^{\mathrm{R}};p)\Big|_{(\mathring{t},\mathring{x})}$. *Equivalently, we have the following exact sequence:*

$$0\longrightarrow\mathcal{S}ol(\mathfrak{M}_0^{\mathrm{R}};p)\longrightarrow\mathcal{S}ol(\mathfrak{M}_1^{\mathrm{R}};p)\oplus\mathcal{S}ol(\mathfrak{M}_2^{\mathrm{R}};p)\longrightarrow\mathcal{S}ol(\mathfrak{M}^{\mathrm{R}};p)\longrightarrow 0.$$

However, a distribution solution u *of* $P_{\mathrm{org}}\, u=0$ *does not necessarily admit any decomposition of solutions above.*

Remark 13. The meaning of the decomposition above is stated for the decomposition $u^i=u_1^i+u_2^i\in\mathcal{D}b_0^1$ satisfying the conditions below:

$$\nabla_i\, {u_1}^i=0,\quad \nabla^i {u_2}^j-\nabla^j {u_2}^i=0.$$

Let X be an n-dimensional complex manifold with a Hermitian metric, and $\bigwedge^{(q,r)}T^*X$ a vector bundle of (q,r)-type differential forms on X. Let $\mathcal{E}_X^{(q,r)}$

be a sheaf of (q,r)-forms on X with C^∞ coefficients, and $\mathcal{D}b_X^{(q,r)}$ a sheaf of (q,r)-currents on X. We define $\mathcal{E}_{\tilde{X}}^{(q,r)}$, $\mathcal{D}b_{\tilde{X}}^{(q,r)}$, $\widetilde{\mathcal{E}}_X^{(q,r)}$ and $\widetilde{\mathcal{D}b}_X^{(q,r)}$ similarly.

Definition 14. We denote by $\partial : \mathcal{D}b_X^{(q,r)} \to \mathcal{D}b_X^{(q+1,r)}$ the exterior differential operator which acts on $\mathcal{D}b_X^{(q,r)}$ as a sheaf morphism and $\overline{\partial} : \mathcal{D}b_X^{(q,r)} \to \mathcal{D}b_X^{(q,r+1)}$ the conjugate exterior differential operator. For a section

$$\phi = \phi_{i_1 \cdots i_q \bar{j}_1 \cdots \bar{j}_r}\, dz^{i_1} \wedge \cdots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_r} \quad \text{of } \mathcal{D}b_X^{(q,r)},$$

the following formulas are well-known:

$$\begin{cases} d\phi &= (\partial + \overline{\partial})\phi, \\[2mm] \partial\phi &= \dfrac{\partial \phi}{\partial z^k}\, dz^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_r} \in \mathcal{D}b_X^{(q+1,r)}, \\[2mm] \overline{\partial}\phi &= \dfrac{\partial \phi}{\partial \bar{z}^k}\, d\bar{z}^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_r} \in \mathcal{D}b_X^{(q,r+1)}. \end{cases}$$

Definition 15. The linear operator $*$ induces vector bundle isomorphisms $\bigwedge^{(q,r)} T^*X \longrightarrow \bigwedge^{(n-r,n-q)} T^*X$. Hence we have sheaf-morphisms $* : \mathcal{D}b_X^{(q,r)} \longrightarrow \mathcal{D}b_X^{(n-r,n-q)}$ on X as follows: For

$$u = u_{i_1 \cdots i_q \bar{j}_1 \cdots \bar{j}_r}\, dz^{i_1} \wedge \cdots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_r} \in \mathcal{D}b_X^{(q,r)},$$

we get

$$\begin{aligned} *u = (-1)^{qn+(i_1-1)+\cdots+(i_q-q)+(j_1-1)+\cdots+(j_r-r)} &\sqrt{g}\, g^{i_1\bar{k}_1} \cdots g^{i_q\bar{k}_q} g^{\bar{j}_1 l_1} \cdots g^{\bar{j}_r l_r} \\ &\times u_{i_1 \cdots i_q \bar{j}_1 \cdots \bar{j}_r} dz^{l_1} \wedge \cdots \wedge dz^{l_{n-r}} \wedge d\bar{z}^{k_1} \wedge \cdots \wedge d\bar{z}^{k_{n-q}} \\ \in \mathcal{D}b_X^{(n-r,n-q)}. \end{aligned}$$

Let $U \subset X$ be an open subset. We set $\alpha^{(q,r)} \in \mathcal{D}b_X^{(q,r)}(U)$, $\beta^{(q,r)} \in \mathcal{E}_X^{(q,r)}(U)$. We suppose that $\beta^{(q,r)}$ has a compact support in U. Then the following integral is well-defined.

$$(\alpha^{(q,r)}, \beta^{(q,r)}) := \int_X \langle \alpha^{(q,r)}, \beta^{(q,r)} \rangle^* \, \omega^1 \wedge \cdots \wedge \omega^n \wedge \overline{\omega}^1 \wedge \cdots \wedge \overline{\omega}^n.$$

Definition 16. We set $\alpha^{(q,r)} \in \mathcal{D}b_X^{(q,r)}$, $\beta^{(q-1,r)} \in \mathcal{E}_X^{(q-1,r)}$, and $\gamma^{(q,r-1)} \in \mathcal{E}_X^{(q,r-1)}$. We suppose $\beta^{(q-1,r)}$ and $\gamma^{(q,r-1)}$ have compact supports. Then sheaf morphisms $\overline{\vartheta} : \mathcal{D}b_X^{(q,r)} \to \mathcal{D}b_X^{(q-1,r)}$ and $\vartheta : \mathcal{D}b_X^{(q,r)} \to \mathcal{D}b_X^{(q,r-1)}$ are defined as

$$(\overline{\vartheta}\alpha^{(q,r)}, \beta^{(q-1,r)}) = (\alpha^{(q,r)}, \partial\beta^{(q-1,r)}),$$
$$(\vartheta\alpha^{(q,r)}, \gamma^{(q,r-1)}) = (\alpha^{(q,r)}, \overline{\partial}\gamma^{(q,r-1)}).$$

Then they satisfy the following equations:

$$\begin{cases} \delta &= \overline{\vartheta} + \vartheta, \\ \overline{\vartheta} &= - * \overline{\partial} *, \\ \vartheta &= - * \partial * . \end{cases}$$

Definition 17. We define sheaf-morphisms $P_{\mathbb{C}}$, $P_{\mathbb{C}}^* : \widetilde{\mathcal{D}b}_X^{(q,r)} \longrightarrow \widetilde{\mathcal{D}b}_X^{(q,r)}$ on $\widetilde{X}$ which are similar to $P_{\mathbb{R}}$:

$$P_{\mathbb{C}} = \frac{\partial^2}{\partial t^2} + \alpha_1 \partial\overline{\vartheta} + \alpha_2 \overline{\vartheta}\partial,$$
$$P_{\mathbb{C}}^* = \frac{\partial^2}{\partial t^2} + \alpha_3 \overline{\partial}\vartheta + \alpha_4 \vartheta\overline{\partial},$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are positive constants.

Then, we get the theorems below.

Theorem 18. *The system of the partial differential equations $P_{\mathbb{C}}\, u = 0$ for $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}$ on $\widetilde{X}$ is of weakly hyperbolic type, and any distribution solution u is locally decomposed into a sum $u = u_1 + u_2$ of 2 solutions u_1 and u_2 satisfying the following conditions:*

$$\partial u_1 = 0, \quad \overline{\vartheta} u_2 = 0.$$

In particular, each u_j satisfies the following wave equation with propagation speed $\sqrt{\alpha_j}$, respectively:

$$\frac{\partial^2}{\partial t^2} u_j + \alpha_j \Box u_j = 0, \quad (j = 1, 2).$$

Here, $\Box = \partial\overline{\vartheta} + \overline{\vartheta}\partial$ is a complex Laplace-Beltrami operator.

Theorem 19. *The system of the partial differential equations $P_{\mathbb{C}}^*\, u = 0$ for $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}$ on $\widetilde{X}$ is of weakly hyperbolic type, and any distribution solution u is* *locally decomposed into a sum* $u = u_1 + u_2$ *of* 2 *solutions* u_3 *and* u_4 *satisfying the following conditions:*

$$\overline{\partial} u_3 = 0, \quad \vartheta u_4 = 0.$$

In particular, each u_j *satisfies the following wave equation with propagation speed* $\sqrt{\alpha_j}$ *, respectively:*

$$\frac{\partial^2}{\partial t^2} u_j + \alpha_j \overline{\Box} u_j = 0, \quad (j = 3, 4).$$

Here, $\overline{\Box} = \overline{\partial}\overline{\vartheta} + \overline{\vartheta}\overline{\partial}$ *is also a complex Laplace-Beltrami operator.*

Now we assume that X is a Kähler manifold. Then the following equations for operators on $\widetilde{\mathcal{D}b}_X^{(q,r)}$ are well-known:

$$\begin{cases} \Box = \overline{\Box} = \frac{1}{2}\Delta, \\ \partial\vartheta + \vartheta\partial = 0, \quad \overline{\partial}\,\overline{\vartheta} + \overline{\vartheta}\,\overline{\partial} = 0, \\ \partial\overline{\partial} = -\overline{\partial}\partial, \quad \vartheta\overline{\vartheta} = -\overline{\vartheta}\vartheta. \end{cases}$$

Definition 20. We define sheaf-morphisms $P_K : \widetilde{\mathcal{D}b}_X^{(q,r)} \longrightarrow \widetilde{\mathcal{D}b}_X^{(q,r)}$ on $\widetilde{X}$ by

$$P_K = \frac{\partial^2}{\partial t^2} + \alpha_1 \partial\overline{\vartheta} + \alpha_2 \overline{\vartheta}\partial + \alpha_3 \overline{\partial}\vartheta + \alpha_4 \vartheta\overline{\partial}.$$

Here, $\alpha_1, \alpha_2, \alpha_3$ and α_4 are positive coefficients.

When $q, r = 0$ or n, $P_K\, u = 0$ reduces to a wave equation. When $q = 0, n$ or $r = 0, n$, P_K stands for P_C^* or P_C , respectively. Therefore, we suppose $1 \leq q, r \leq n - 1$.

For $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}$, we define equations $\mathfrak{M}^K$, $\mathfrak{M}_1^K$, $\mathfrak{M}_2^K$, $\mathfrak{M}_3^K$, $\mathfrak{M}_4^K$, $\mathfrak{M}_{13}^K$, $\mathfrak{M}_{24}^K$, $\mathfrak{M}_{12}^K$, $\mathfrak{M}_{34}^K$, $\mathfrak{M}_0^K$ below:

$$\mathfrak{M}^K \; : \quad P_K\, u = 0,$$

$$\mathfrak{M}_1^K : \begin{cases} P_K\, u = 0, \\ \partial u = 0, \\ \overline{\partial} u = 0, \end{cases} \iff \begin{cases} \left(\partial_t^2 + \dfrac{\alpha_1 + \alpha_3}{2}\Delta\right) u = 0, \\ \partial u = 0, \\ \overline{\partial} u = 0, \end{cases}$$

$$\mathfrak{M}_2^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0, \\ \bar{\vartheta} u = 0, \\ \bar{\partial} u = 0, \end{cases} \iff \begin{cases} \left(\partial_t^2 + \dfrac{\alpha_2 + \alpha_3}{2} \Delta\right) u = 0, \\ \bar{\vartheta} u = 0, \\ \bar{\partial} u = 0, \end{cases}$$

$$\mathfrak{M}_3^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0, \\ \partial u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} \left(\partial_t^2 + \dfrac{\alpha_1 + \alpha_4}{2} \Delta\right) u = 0, \\ \partial u = 0, \\ \vartheta u = 0, \end{cases}$$

$$\mathfrak{M}_4^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0, \\ \bar{\vartheta} u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} \left(\partial_t^2 + \dfrac{\alpha_2 + \alpha_4}{2} \Delta\right) u = 0, \\ \bar{\vartheta} u = 0, \\ \vartheta u = 0, \end{cases}$$

$$\mathfrak{M}_{13}^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0, \\ \partial u = 0, \\ \bar{\partial} u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ \partial u = 0, \\ \bar{\partial} u = 0, \\ \vartheta u = 0, \end{cases}$$

$$\mathfrak{M}_{24}^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0, \\ \bar{\vartheta} u = 0, \\ \bar{\partial} u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ \bar{\vartheta} u = 0, \\ \bar{\partial} u = 0, \\ \vartheta u = 0, \end{cases}$$

$$\mathfrak{M}_{12}^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0, \\ \bar{\partial} u = 0, \\ \bar{\vartheta} u = 0, \\ \partial u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ \bar{\partial} u = 0, \\ \bar{\vartheta} u = 0, \\ \partial u = 0, \end{cases}$$

$$\mathfrak{M}_{34}^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0, \\ \vartheta u = 0, \\ \bar{\vartheta} u = 0, \\ \partial u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ \vartheta u = 0, \\ \bar{\vartheta} u = 0, \\ \partial u = 0, \end{cases}$$

$$\mathfrak{M}_0^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0, \\ \partial u = 0, \\ \bar{\partial} u = 0, \\ \vartheta u = 0, \\ \bar{\vartheta} u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ \partial u = 0, \\ \bar{\partial} u = 0, \\ \vartheta u = 0, \\ \bar{\vartheta} u = 0. \end{cases}$$

Further we define subsheaves $\mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_1^{\mathrm{K}}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_2^{\mathrm{K}}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_3^{\mathrm{K}}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_4^{\mathrm{K}}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_{13}^{\mathrm{K}}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_{24}^{\mathrm{K}}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_{12}^{\mathrm{K}}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_{34}^{\mathrm{K}}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_0^{\mathrm{K}}; q, r)$ of $\widetilde{\mathcal{D}b}_X^{(q,r)}$ as the sheaves of $\widetilde{\mathcal{D}b}_X^{(q,r)}$-solutions, respectively.

Then, we have the theorem below.

Theorem 21. *For any germ* $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}; q, r)$*, there exist some germs* $u_j \in \mathcal{S}ol(\mathfrak{M}_j^{\mathrm{K}}; q, r)$ $(j = 1, 2, 3, 4)$ *such that* $u = u_1 + u_2 + u_3 + u_4$.

Further, we get

$$u = u_1 + u_2 + u_3 + u_4 = 0 \Longleftrightarrow \begin{cases} u_1 = u_{12} - u_{13}, \\ u_2 = u_{24} - u_{12}, \\ u_3 = u_{13} - u_{34}, \\ u_4 = u_{34} - u_{24}. \end{cases}$$

Here, $u_{ij} \in \mathcal{S}ol(\mathfrak{M}_{ij}^{\mathrm{K}}; q, r)$ $\big((i, j) = (1, 2), (1, 3), (2, 4), (3, 4)\big)$. *Equivalently, we have the following exact sequence:*

$$0 \longrightarrow \mathcal{S}ol(\mathfrak{M}_0^{\mathrm{K}}; q, r) \longrightarrow \bigoplus_{(i,j)} \mathcal{S}ol(\mathfrak{M}_{ij}^{\mathrm{K}}; q, r) \longrightarrow \bigoplus \mathcal{S}ol(\mathfrak{M}_i^{\mathrm{K}}; q, r) \longrightarrow \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}; q, r) \longrightarrow 0.$$