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Fuchsian PDE with applications to normal forms of resonant vector fields \(^\dagger\)

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Motivations

Let \( t \in \mathbb{C} \) or \( t \in \mathbb{C} \). We consider Fuchsian ordinary differential equations \( P = p(t \frac{d}{dt}) \), where \( p(\zeta) \) us an polynomial of one variable. We call \( p(\zeta) \) an incidencial polynomial of \( P \). We consider the solvability of the equation \( Pu = f(t) \), where \( f(t) \) is analytic at the origin \( t = 0 \).

If the "non-log condition"

\[
(1) \quad p(\zeta) \neq 0 \quad \text{for} \quad \zeta = 0, 1, 2, \ldots
\]

is fulfilled \( Pu = f \) has an analytic solution. Indeed, the solution is constructed by a method of indeterminate coefficients if we expand \( u \) in Taylor series.

Now, let us consider the case where a "non-log condition" is not fulfilled. For the sake of simplicity, we consider under the condition

\[
(2) \quad \exists \zeta_0 \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}, \quad p(\zeta_0) = 0.
\]

**Remark.** If there exists \( \zeta_1 \in \mathbb{Z} \) such that \( p(\zeta_1) = 0 \) Condition (2) is a special case where the difference of characteristic exponents have integral difference.

By Frobenius theorem, the fundamental solutions contain a function of the form \( t^\lambda \log t \), where \( \lambda \) is a certain constant. It follows that the solution \( u \) of \( Pu = f(t) \) is singular, or \( u \) has finite differentiability.

**Question**

What happens in the case of nonlinear partial differential equations of Fuchs type?

In order to answer to this question, we first introduce a class of so-called

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Fuchsian partial differential equations which appear from geometric problems.

We also cite related works by Tahara, Mandai, Yamane and Y. Mazawa.

Vector fields with an isolated singular point

We consider

\[
\mathcal{X}(x) = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j}, \quad x = (x_1, \ldots, x_n),
\]

where \(a_j(x)\) is a smooth function of \(x\). We assume

\[
\mathcal{X}(0) = 0,
\]

and the origin \(x = 0\) is an isolated singular point of \(\mathcal{X}\).

We want to linearize \(\mathcal{X}(x)\) by a coordinate change

\[
x = y + v(y), \quad v = O(|y|^2).
\]

We write

\[
\mathcal{X}(x) = x\Lambda \frac{\partial}{\partial x} + R(x) \frac{\partial}{\partial x} \equiv X(x) \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1'}, \ldots, \frac{\partial}{\partial x_n}\right)
\]

\[
X(x) = x\Lambda + R(x),
\]

where

\[
R(x) = (R_1(x), \ldots, R_n(x)), \quad R(x) = O(|x|^2),
\]

and \(\Lambda\) is an \(n \times n\) constant matrix.

Noting that

\[
X(x) \frac{\partial}{\partial x} = X(y + v(y)) \frac{\partial y}{\partial x} \frac{\partial}{\partial y} = X(y + v(y)) \left(\frac{\partial x}{\partial y}\right)^{-1} \frac{\partial}{\partial y},
\]

the linearizability condition implies

\[
X(y + v)(1 + \partial_y v)^{-1} = y\Lambda.
\]

It follows that

\[
(y + v)\Lambda + R(y + v) = y\Lambda(1 + \partial_y v) = y\Lambda + y\Lambda\partial_y v.
\]
Therefore $v$ solves the so-called homology equation

(*) \[ \mathcal{L}v \equiv y\Lambda\partial_y v - v\Lambda = R(y + v(y)), \quad v = (v_1, \ldots, v_n). \]

Therefore we have

Eq. (*) has a solution $v$ if and only if $\mathcal{X}$ is linearized by a coordinate change $x = y + v(y)$.

Expression of a homology equation

We calculate the form of $\mathcal{L}$ in case $\Lambda$ is a diagonal matrix. Namely we assume

\[
\Lambda = \begin{pmatrix}
\lambda_1 & 0 \\
& \ddots & \ddots \\
& 0 & \lambda_n
\end{pmatrix}.
\]

Because we have

\[ y\Lambda\partial_y = \sum_{k=1}^{n} \lambda_k y_k \frac{\partial}{\partial y_k} \]

we have

\[
\mathcal{L}v = \begin{pmatrix}
\sum \lambda_k y_k \frac{\partial}{\partial y_k} - \lambda_1 & 0 \\
& \ddots & \ddots \\
& 0 & \sum \lambda_k y_k \frac{\partial}{\partial y_k} - \lambda_n
\end{pmatrix} \begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{pmatrix}.
\]

Remark. The homology equation (*) is a special case of totally characteristic Fuchsian PDE. (cf. Tahara [4]). We also cite Shirai [3].

Non-log condition and a non resonant condition

For simplicity, we consider the above example. The indicial polynomial is defined by

\[
\sum_{k=1}^{n} \lambda_k \zeta_k - \lambda_j, \quad (j = 1, \ldots, n).
\]

Non resonant condition

$\mathcal{L}$ is said to be non-resonant if

\[
\sum_{k=1}^{n} \lambda_k \alpha_k - \lambda_j \neq 0 \quad \text{for} \quad \forall \alpha \in (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n, |\alpha| \geq 2.
\]
Non resonant condition implies the existence of a formal solution. Indeed, we have
\[ \mathcal{L}\left(\sum_{\alpha}v_{\alpha}y^{\alpha}\right) = \sum_{\alpha}(\sum_{k=1}^{n}\lambda_{k}\alpha_{k} - \Lambda)v_{\alpha}y^{\alpha}. \]

Hence \( \mathcal{L}^{-1} \) exist on a set of formal power series if a non-resonant condition is fulfilled. It should be noted that a non-resonance condition is a non-log condition.

Two theorems concerning the solvability of homology equations

As to the solvability of (\(*\)), probably the first result was obtained by Poincaré in 19th century. He introduced a so-called Poincaré condition. Then the middle of 20th century, Siegel introduced a Siegel condition and he essentially showed the solvability of (\(*\)) under a Siegel condition. On the other hand, in the real domain, Sternberg showed the solvability of (\(*\)) in a class of smooth functions without any diophantine condition. He essentially assumed the nonresonance condition. As to the resonant case, Hartman showed the solvability of (\(*\)) in a class of continuous functions. Our result is closely related to Hartman’s theorem.

**Sternberg’s theorem**

*Suppose that a hyperbolic condition*

\[(14) \quad \text{Re} \lambda_{k} \neq 0, \quad k = 1, \ldots, n\]

is fulfilled. Moreover, assume that a non resonant condition is satisfied. Then, Eq. (\*) has a smooth solution.

Sternberg’s theorem shows the solvability of (\*) under non-log condition.

**Grobman- Hartman’s theorem**

*If the hyperbolicity condition is satisfied, Eq. (\*) has a continuous solution.*

**Remark.** A continuous solution of (\*) is defined by a weak solution. Hartman’s theorem treats the case where a non-log condition is not satisfied.

The Object of the Study
We consider the case where a non-log condition is not satisfied. The typical example is a volume preserving vector field, \( \lambda_1 + \cdots + \lambda_n = 0 \). We want to solve (*) in a real domain in a class of finitely differentiable functions, which corresponds to Hartman's theorem in a \( C^\ell \) class. This is closely related to the construction of a singular solution in a complex domain.

**Remark.** Geometrically, the resonance does not vanish under a formal change of variables. Because the solvability of (*) implies that the change of variables \( x = y + v(y) \) linearizes the given vector field, the resonance also vanish under the change of variables. This implies that Eq. (*) does not necessarily have a formal solution.

**Heuristic Statement of Results - \( C^\ell \) Hartman theorem**

For the sake of simplicity, we will state the special case of our theorem.

**Theorem** Assume that \( \lambda_k \ (k = 1, \ldots, n) \) are nonzero real number, (a hyperbolicity). Then Eq. (*) has a \( C^\ell \) solution for a certain \( \ell \geq 0 \) determined by an indicial polynomials.

**Idea of the Proof.**

**Why Picard's iteration does not work?**

Firstly, we note the loss of derivatives of \( \mathcal{L}^{-1} \). In fact, even if there exists \( \mathcal{L}^{-1} \), we have a loss of derivatives. In order to see this, let us consider

\[
\left( t \frac{d}{dt} - \lambda \right) u = g(t), \quad \text{Re} \lambda < 0.
\]

The solution is given by

\[
u(t) = \int_0^1 \sigma^{-\lambda-1} g(\sigma t) d\sigma.
\]

Clearly, we do not gain the derivatives.

On the other hand, in order to define the right-hand side of (*), \( R(y + v) \) one needs derivatives of \( v \). Indeed, Sobolev's embedding theorem implies: if \( 0 \leq m < k - n/p < m + 1 \) and \( 0 \leq \alpha < k - m - n/p < 1 \), it follows that \( W^{k,p}(\Omega) \subset C^{m,\alpha}(\Omega) \).

Here \( W^{k,p}(\Omega) \) is the space of distributions whose derivatives up to order
$k$ is in the Lebesgue space $L^p(\Omega)$. $C^{m,\alpha}(\Omega)$ is a Hölder space, namely the set of functions with derivatives up to $m$ has Hölder exponent $\alpha$. Therefore the iteration scheme $v = \mathcal{L}^{-1}R(y + v)$ does not seem to converge.

In view of this we need to employ a Nash-Moser scheme, a rapidly convergent iteration scheme.

**Rapidly Convergent Iteration Scheme**

1. We need a smoothing operator which has not a smoothing effect transversal to the singular locus of the equation, $y_j = 0$, $(j = 1, \ldots, n)$.

2. The crucial step of the Nash-Moser iteration scheme is to solve a linearized equation. The linearized equation of (*) at $v = w$ is given by

$$\mathcal{L}v - \nabla R(y + w)v.$$

We note that $w$ is singular or does not have regularity. The solvability of linear Fuchsian partial differential equations with singular coefficients seems open.

In order to handle these problems we use a Mellin transform, and a Nash-Moser iteration scheme of tangential type.

**Statement of the Theorem**

**Mellin Transform**

Let $N \geq 1$ be an integer. Let $f(x) = (f_1(x), \ldots, f_N(x))$ be an integrable function on $\mathbb{R}^n_+$, and let us define a Mellin transform $\hat{f}(\zeta)$ ($\zeta \in \Gamma + i\mathbb{R}^n$) by

$$\hat{f}(\zeta) = \int_{\mathbb{R}^n_+} f(x)x^{\zeta - e}dx, \quad e = (1, \ldots, 1).$$

The inverse Mellin transform is given by

$$f(x) = M^{-1}(\hat{f})(\zeta) = \frac{1}{(2\pi i)^n} \int_{\mathbb{R}^n} \hat{f}(\eta + i\xi)x^{-\eta - i\xi}d\xi, \quad x > 0, \quad j = 1, \ldots, n,$$

where $\eta \in \Gamma$ is chosen so that the integral converges.

**Definition of a function space**

Let $\sigma \geq 0$, and let $\Gamma \subset \mathbb{R}^n$ be an open set. We define $H_\sigma \equiv H_{\sigma,\Gamma}$ as the set of holomorphic vector-valued functions

$$v(\zeta) = (v_1(\zeta), \ldots, v_N(\zeta)), \quad \zeta = \eta + i\xi \in \Gamma + i\mathbb{R}^n$$
such that

\[ \|v\|_{\sigma,\Gamma} := \sup_{\eta \in \Gamma} \int_{\mathbb{R}^n} \langle \zeta \rangle^\sigma |v(\zeta)| d\xi < \infty, \]

where

\[ \langle \zeta \rangle = 1 + \sum_{j=1}^{n} |\zeta_j|, \quad |v(\zeta)| = \left( \sum_{j=1}^{N} |v_j(\zeta)|^2 \right)^{1/2}. \]

The space \( H_{\sigma,\Gamma} \) is a Banach space with the norm \( \| \cdot \|_{\sigma,\Gamma} \).

Let \( \mathcal{H}_{\sigma,\Gamma} \) be the inverse Mellin transform of \( H_{\sigma,\Gamma} \). The norm of \( \mathcal{H}_{\sigma,\Gamma} \) is defined by

\[ \|u\|_{\mathcal{H}_{\sigma,\Gamma}} \equiv \|u\|_{\sigma,\Gamma} := \|M(u)\|_{H_{\sigma,\Gamma}}. \]

We define an incidencial polynomial by

\[ p(\zeta) = -\sum_{j=1}^{n} \zeta_j \lambda_j I - \Lambda, \]

where \( I \) is an identity matrix.

We say that \( R \in \mathcal{H}_{\nu,\Gamma} \) at the origin if \( \exists \psi \in C_0^\infty(\mathbb{R}^n) \) being identically equal to 1 in some neighborhood of the origin such that

\[ M(\psi R) \in \mathcal{H}_{\nu,\Gamma}. \]

Then we have

\textbf{Theorem} Suppose that there exist \( C > 0 \) and an open bounded set \( \Gamma, 0 \in \Gamma \subset \mathbb{R}^n \) such that

\[ |p(\eta + i\xi)| > C > 0, \quad \forall \eta \in \Gamma, \forall \xi \in \mathbb{R}^n. \]

Let \( \sigma \geq 1 \) be an integer. Then there exists \( \nu \geq 0 \) such that, if

\[ R \in \mathcal{H}_{\nu,\Gamma} \quad \text{and} \quad \nabla R_j \in \mathcal{H}_{\nu,\Gamma}, \quad j = 1, \ldots, n \]

at the origin, Eq. (*) has a solution \( v \in \mathcal{H}_{\sigma,\Gamma'} \) for every \( \Gamma' \subset \subset \Gamma \).

\textbf{Remark} The set \( \Gamma \) determine the vanishing order of \( v \in \mathcal{H}_{\sigma,\Gamma'} \). Hence \( \Gamma \) expresses the smoothness up to the set \( y_j = 0 \) \((j = 1, \ldots, n)\), because we have the interior regularity, \( x_j > 0 \) \(j = 1, \ldots, n\).

In order to construct a solution in some neighborhood of the origin, we construct solutions in the domain \( \pm y_j \geq 0 \) \((j = 1, \ldots, n)\). Then we patch up these solutions.
Further extensions

We will briefly mention how the above theorem is extended to more general systems. We consider $N$ ($N \geq 1$) system of equations for the unknown vector $v = (v_1, \ldots, v_N)$

$$p_j(\delta)u_j + a_j(x, \delta^\alpha u; |\alpha| \leq s) = 0, \quad j = 1, \ldots, N,$$

where $1 \leq s \leq m$ are integers and $\delta_j = \partial/\partial x_j$,

$$\delta^\alpha = \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n},$$

and $p_j(\zeta)$ is a polynomial of $\zeta$. The nonlinear term $a_j(x, z)$, $z = (z^j_\alpha)$ is supposed to be real-valued and smooth in $\mathbb{R}^n \times \Omega$, where $\Omega$ is a neighborhood of the origin $z = 0$.

Then we have the same assertion as to the above theorem.

**Example.** We consider Monge-Ampère operator

$$M(u) = u_{xx}u_{yy} - u_{xy}^2 + ku_{xy} + cu$$

in some neighborhood of the origin $(x, y) \in \mathbb{R}^2$. Here $k$ is a real constant and $c$ is a complex constant.

Let $u_0 = x^2y^2$ and $f_0 = M(u_0)$. We want to solve

$$M(u_0 + v) = f_0(x, y) + g(x, y), \quad \text{in } \mathbb{R}^2,$$

where $g(x, y)$ is a given function. This equation is related to find a surface with a prescribed Gaussian curvature. The general theory does not apply this equation because of the degeneracy of $u_0$.

The incidencial polynomial is given by

$$p(\zeta) = -2\zeta_1(\zeta_1 + 1) - 2\zeta_2(\zeta_2 + 1) - (k - 8)\zeta_1\zeta_2 - c.$$

Our theorem shows that If $4 < k < 12$ and $c = iK$, $K \gg 1$ there exists a solution $v$ of the above equation.

**Proof of the Theorem**

**Definition of a smoothing operator in $\mathcal{H}_{s, \Gamma}$**

Let $\phi \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ near the origin $x = 0$. Let $N \geq 1$ and let $\ell \geq 1$ be positive integers. We set

$$\psi_N(\zeta) = \exp\left(\frac{1}{N^{2\tau}} \sum_{j=1}^{n} \zeta_j^{2\tau}\right),$$
and define
\[
\chi_{N}^{\ell} = \int_{\mathbb{R}^{n}} \left\{ \psi_{N}(\zeta) \left( e^{-\sigma\zeta/N} - \sum_{\nu=1}^{\ell} \left( -\frac{\sigma\zeta}{N} \right)^{\nu} \frac{1}{\nu!} \right) + (1 - \psi_{N}(\zeta)) e^{-\sigma\zeta/N} \right\} \, dc
\]

where \( \tau \) is an odd integer such that \( 2\tau \geq \ell \). We can easily see that \( \chi_{N}^{\ell}(\zeta) \) is an entire function of \( \zeta \) and real,

\[
\overline{\chi_{N}^{\ell}(\zeta)} = \chi_{N}^{\ell}(\overline{\zeta}).
\]

We define a smoothing operator \( S_{N} \) by
\[
S_{N}v := M^{-1}(\chi_{N+1}^{\ell}(\zeta)\hat{v}(\zeta)), \quad v \in \mathcal{H}_{s,\Gamma},
\]
where \( \hat{v}(\zeta) \) denotes the Mellin transform of \( v \), and \( M^{-1} \) denotes the inverse Mellin transform.

**Proof of the Theorem**

Let \( 1 < \tau < 2 \) and \( d > 1 \) be the constants chosen later. Let \( S_{k} \) \((k = 0, 1, 2, \ldots)\) be a smoothing operator defined above with \( N + 1 = \mu_{k} := d^{r^{k}} \). We define
\[
G(v) = Lv - R(y + v).
\]

Let \( L_{w} \) be the linearized operator of \( G \) at \( v = w \). We define \( g_{0} = G(0) \).

**Iterative scheme**

We construct an approximate sequence \( \{w_{k}\} \) by
\[
w_{0} = 0, \quad w_{k+1} = w_{k} + S_{k}\rho_{k}, \quad L_{w_{k}}\rho_{k} = g_{k}, \quad g_{k} = -G(w_{k}), \quad k = 0, 1, 2, \ldots
\]

**Estimates**

There exist \( \exists \nu, \exists \kappa \) and \( \exists c > 0, \nu > \kappa > 1 \) such that
\[
\|g_{k}\|_{0,\Gamma} \leq c\mu_{k}^{-\kappa}d^{\kappa}\|g_{0}\|_{\nu+1,\Gamma}.
\]

If we can show this estimate we see that \( g_{k} \to 0 \) as \( k \to \infty \) and that \( \{w_{k}\} \) is a Cauchy sequence. It follows that \( w := \lim_{k}w_{k} \) satisfies \( G(w) = 0 \).

**Step 1 A priori estimate of \( w_{k} \).**

There exists \( C > 0 \) independent of \( k \) such that, for \( j = 1, \ldots, k + 1 \)
\[
\|w_{j}\|_{\ell,\Gamma} \leq Cd^{\kappa}\|g_{0}\|_{\nu+1,\Gamma}, \quad \text{if} \ \ell < \kappa + s,
\]
\[ \|w_j\|_{\ell,\Gamma} \leq C\mu_{j-1}^{\ell+1-\kappa-s}d^\kappa\|g_0\|_{\nu+1,\Gamma}, \quad \text{if } \ell \geq \kappa + s. \]

**Step 2** *A priori estimate of \( g_k \)*

There exists \( C > 0 \) independent of \( k \) such that

\[ \|g_k\|_{\nu,\Gamma} \leq Cd^\kappa\|g_0\|_{\nu+1,\Gamma}(1 + \mu_k^{(\nu+m+n+2-\kappa-s)/\tau}). \]

Using these estimates we can show the desired estimate. The constants \( \tau \) and \( d \) are determined by the equation.

**References**


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