Fuchsian PDE with applications to normal forms of resonant vector fields

Motivations

Let \( t \in \mathbb{C} \) or \( t \in \mathbb{C} \). We consider Fuchsian ordinary differential equations \( P \equiv p(t \frac{d}{dt}) \), where \( p(\zeta) \) us an polynomial of one variable. We call \( p(\zeta) \) an incidential polynomial of \( P \). We consider the solvability of the equation \( Pu = f(t) \), where \( f(t) \) is analytic at the origin \( t = 0 \).

If the "non-log condition"

\[
(1) \quad p(\zeta) \neq 0 \quad \text{for} \quad \zeta = 0, 1, 2, \ldots
\]

is fulfilled \( Pu = f \) has an analytic solution. Indeed, the solution is constructed by a method of indeterminate coefficients if we expand \( u \) in Taylor series.

Now, let us consider the case where a "non-log condition" is not fulfilled. For the sake of simplicity, we consider under the condition

\[
(2) \quad \exists \zeta_0 \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}, \quad p(\zeta_0) = 0.
\]

Remark. If there exists \( \zeta_1 \in \mathbb{Z} \) such that \( p(\zeta_1) = 0 \) Condition (2) is a special case where the difference of characteristic exponents have integral difference.

By Frobenius theorem, the fundamental solutions contain a function of the form \( t^\lambda \log t \), where \( \lambda \) is a certain constant. It follows that the solution \( u \) of \( Pu = f(t) \) is singular, or \( u \) has finite differentiability.

Question

What happens in the case of nonlinear partial differential equations of Fuchs type?

In order to answer to this question, we first introduce a class of so-called

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Fuchsian partial differential equations which appear from geometric problems.

We also cite related works by Tahara, Mandai, Yamane and Yamaazawa.

**Vector fields with an isolated singular point**

We consider

$$\mathcal{X}(x) = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j}, \quad x = (x_1, \ldots, x_n),$$

where $a_j(x)$ is a smooth function of $x$. We assume

$$\mathcal{X}(0) = 0,$$

and the origin $x = 0$ is an isolated singular point of $\mathcal{X}$.

We want to linearize $\mathcal{X}(x)$ by a coordinate change

$$x = y + v(y), \quad v = O(|y|^2).$$

We write

$$\mathcal{X}(x) = x \Lambda \frac{\partial}{\partial x} + R(x) \frac{\partial}{\partial x} \equiv X(x) \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial x} = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)$$

$$X(x) = x \Lambda + R(x),$$

where

$$R(x) = (R_1(x), \ldots, R_n(x)), \quad R(x) = O(|x|^2),$$

and $\Lambda$ is an $n \times n$ constant matrix.

Noting that

$$X(x) \frac{\partial}{\partial x} = X(y + v(y)) \frac{\partial y}{\partial x} \frac{\partial}{\partial y} = X(y + v(y)) \left( \frac{\partial x}{\partial y} \right)^{-1} \frac{\partial}{\partial y},$$

the linearizability condition implies

$$X(y + v)(1 + \partial_y v)^{-1} = y \Lambda.$$

It follows that

$$(y + v) \Lambda + R(y + v) = y \Lambda (1 + \partial_y v) = y \Lambda + y \Lambda \partial_y v.$$
Therefore $v$ solves the so-called *homology equation*

\[
\mathcal{L} v \equiv y \Lambda \partial_y v - v \Lambda = R(y + v(y)), \quad v = (v_1, \ldots, v_n).
\]

Therefore we have

*Eq. (*) has a solution $v$ if and only if $\mathcal{X}$ is linearized by a coordinate change $x = y + v(y)$.*

**Expression of a homology equation**

We calculate the form of $\mathcal{L}$ in case $\Lambda$ is a diagonal matrix. Namely we assume

\[
\Lambda = \begin{pmatrix}
\lambda_1 & 0 \\
\vdots & \ddots \\
0 & \cdots & \lambda_n
\end{pmatrix}.
\]

Because we have

\[
y \Lambda \partial_y = \sum_{k=1}^{n} \lambda_k y_k \frac{\partial}{\partial y_k}
\]

we have

\[
\mathcal{L} v = \begin{pmatrix}
\sum \lambda_k y_k \frac{\partial}{\partial y_k} - \lambda_1 & 0 \\
\vdots & \ddots \\
0 & \sum \lambda_k y_k \frac{\partial}{\partial y_k} - \lambda_n
\end{pmatrix}
\begin{pmatrix}
v_1 \\
\vdots \\
v_n
\end{pmatrix}.
\]

**Remark.** The homology equation (*) is a special case of totally characteristic Fuchsian PDE. (cf. Tahara [4]). We also cite Shirai [3].

**Non-log condition and a non resonant condition**

For simplicity, we consider the above example. The indicial polynomial is defined by

\[
\sum_{k=1}^{n} \lambda_k \zeta_k - \lambda_j, \quad (j = 1, \ldots, n).
\]

**Non resonant condition**

$\mathcal{L}$ is said to be non-resonant if

\[
\sum_{k=1}^{n} \lambda_k \alpha_k - \lambda_j \neq 0 \quad \text{for } \forall \alpha \in (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n, |\alpha| \geq 2.
\]
Non resonant condition implies the existence of a formal solution. Indeed, we have

\[ \mathcal{L}(\sum_{\alpha}v_{\alpha}y^{\alpha}) = \sum_{\alpha} \left( \sum_{k=1}^{n} \lambda_{k}\alpha_{k} - \Lambda \right) v_{\alpha}y^{\alpha}. \]

Hence \( \mathcal{L}^{-1} \) exist on a set of formal power series if a non-resonant condition is fulfilled. It should be noted that a non-resonance condition is a non log condition.

Two theorems concerning the solvability of homology equations

As to the solvability of (*), probably the first result was obtained by Poincaré in 19th century. He introduced a so-called *Poincaré condition*. Then the middle of 20th century, Siegel introduced a Siegel condition and he essentially showed the solvability of (*) under a Siegel condition. On the other hand, in the real domain, Sternberg showed the solvability of (*) in a class of smooth functions without any diophantine condition. He essentially assumed the nonresonance condition. As to the resonant case, Hartman showed the solvability of (*) in a class of continuous functions. Our result is closely related to Hartman’s theorem.

**Sternberg’s theorem**

*Suppose that a hyperbolic condition*

\[ (14) \quad \text{Re}\lambda_{k} \neq 0, \quad k = 1, \ldots, n \]

*is fulfilled. Moreover, assume that a non resonant condition is satisfied. Then, Eq. (*) has a smooth solution.*

Sternberg’s theorem shows the solvability of (*) under non-log condition.

**Grobman- Hartman’s theorem**

*If the hyperbolicity condition is satisfied, Eq. (*) has a continuous solution.*

**Remark.** A continuous solution of (*) is defined by a weak solution. Hartman’s theorem treats the case where a non-log condition is not satisfied.

The Object of the Study
We consider the case where a non-log condition is not satisfied. The typical example is a volume preserving vector field, $\lambda_1 + \cdots + \lambda_n = 0$. We want to solve (*) in a real domain in a class of finitely differentiable functions, which corresponds to Hartman's theorem in a $C^\ell$ class. This is closely related to the construction of a singular solution in a complex domain.

**Remark.** Geometrically, the resonance does not vanish under a formal change of variables. Because the solvability of (*) implies that the change of variables $x = y + v(y)$ linearizes the given vector field, the resonance also vanish under the change of variables. This implies that Eq. (*) does not necessarily have a formal solution.

Heuristic Statement of Results - $C^\ell$ Hartman theorem -

For the sake of simplicity, we will state the special case of our theorem.

**Theorem** Assume that $\lambda_k$ ($k = 1, \ldots, n$) are nonzero real number, (a hyperbolicity). Then Eq. (*) has a $C^\ell$ solution for a certain $\ell \geq 0$ determined by an indicial polynomials.

Idea of the Proof.

**Why Picard's iteration does not work ?**

Firstly, we note the loss of derivatives of $\mathcal{L}^{-1}$. In fact, even if there exists $\mathcal{L}^{-1}$, we have a loss of derivatives. In order to see this, let us consider

$$\left( t \frac{d}{dt} - \lambda \right) u = g(t), \quad \text{Re}\lambda < 0.$$ 

The solution is given by

$$u(t) = \int_0^1 \sigma^{-\lambda-1}g(\sigma t)d\sigma.$$ 

Clearly, we do not gain the derivatives.

On the other hand, in order to define the right-hand side of (*), $R(y + v)$ one needs derivatives of $v$. Indeed, Sobolev's embedding theorem implies:

if $0 \leq m < k - n/p < m + 1$ and $0 \leq \alpha < k - m - n/p < 1$, it follows that $W^{k,p}(\Omega) \hookrightarrow C^{m,\alpha}(\Omega)$.

Here $W^{k,p}(\Omega)$ is the space of distributions whose derivatives up to order
$k$ is in the Lebesgue space $L^p(\Omega)$. $C^{m,\alpha}(\Omega)$ is a Hölder space, namely the set of functions with derivatives up to $m$ has Hölder exponent $\alpha$. Therefore the iteration scheme $v = L^{-1}R(y + v)$ does not seem to converge.

In view of this we need to employ a Nash-Moser scheme, a rapidly convergent iteration scheme.

Rapidly Convergent Iteration Scheme

1. We need a smoothing operator which has not a smoothing effect transversal to the singular locus of the equation, $y_j = 0, (j = 1, \ldots, n)$.

2. The crucial step of the Nash-Moser iteration scheme is to solve a linearized equation. The linearized equation of (*) at $v = w$ is given by

$$Lv - \nabla R(y + w)v.$$ 

We note that $w$ is singular or does not have regularity. The solvability of linear Fuchsian partial differential equations with singular coefficients seems open.

In order to handle these problems we use a Mellin transform, and a Nash-Moser iteration scheme of tangential type.

Statement of the Theorem

Mellin Transform
Let $N \geq 1$ be an integer. Let $f(x) = (f_1(x), \ldots, f_N(x))$ be an integrable function on $\mathbb{R}_+^n$, and let us define a Mellin transform $\hat{f}(\zeta)$ ($\zeta \in \Gamma + i\mathbb{R}^n$) by

$$\hat{f}(\zeta) = \int_{\mathbb{R}_+^n} f(x)x^{\zeta - e}dx, \quad e = (1, \ldots, 1).$$

The inverse Mellin transform is given by

$$f(x) = M^{-1}(\hat{f})(\zeta) = \frac{1}{(2\pi i)^n} \int_{\mathbb{R}^n} \hat{f}(\eta + i\xi)x^{-\eta - i\xi}d\xi, \quad x_j > 0, j = 1, \ldots, n,$

where $\eta \in \Gamma$ is chosen so that the integral converges.

Definition of a function space
Let $\sigma \geq 0$, and let $\Gamma \subset \mathbb{R}^n$ be an open set. We define $H_\sigma \equiv H_{\sigma,\Gamma}$ as the set of holomorphic vector-valued functions

$$v(\zeta) = (v_1(\zeta), \ldots, v_N(\zeta)), \quad \zeta = \eta + i\xi \in \Gamma + i\mathbb{R}^n$$
such that
\[ \|v\|_{\sigma, \Gamma} := \sup_{\eta \in \Gamma} \int_{\mathbb{R}^n} \langle \zeta \rangle^\sigma |v(\zeta)| d\xi < \infty, \]
where
\[ \langle \zeta \rangle = 1 + \sum_{j=1}^{n} |\zeta_j|, \quad |v(\zeta)| = \left( \sum_{j=1}^{N} |v_j(\zeta)|^2 \right)^{1/2}. \]
The space \( H_{\sigma, \Gamma} \) is a Banach space with the norm \( \| \cdot \|_{\sigma, \Gamma} \).

Let \( \mathcal{H}_{\sigma, \Gamma} \) be the inverse Mellin transform of \( H_{\sigma, \Gamma} \). The norm of \( \mathcal{H}_{\sigma, \Gamma} \) is defined by
\[ \|u\|_{\mathcal{H}_{\sigma, \Gamma}} \equiv \|u\|_{\sigma, \Gamma} := \|M(u)\|_{H_{\sigma, \Gamma}}. \]

We define an incidencial polynomial by
\[ p(\zeta) = -\sum_{j=1}^{n} \zeta_j \lambda_j I - \Lambda, \]
where \( I \) is an identity matrix.

We say that \( R \in \mathcal{H}_{\nu, \Gamma} \) at the origin if \( \exists \psi \in C_0^\infty(\mathbb{R}^n) \) being identically equal to 1 in some neighborhood of the origin such that
\[ M(\psi R) \in \mathcal{H}_{\nu, \Gamma}. \]

Then we have

**Theorem** Suppose that there exist \( C > 0 \) and an open bounded set \( \Gamma, 0 \in \Gamma \subset \mathbb{R}^n \) such that
\[ |p(\eta + i \xi)| > C > 0, \quad \forall \eta \in \Gamma, \forall \xi \in \mathbb{R}^n. \]

Let \( \sigma \geq 1 \) be an integer. Then there exists \( \nu \geq 0 \) such that, if
\[ R \in \mathcal{H}_{\nu, \Gamma} \quad \text{and} \quad \nabla R_j \in \mathcal{H}_{\nu, \Gamma}, \quad j = 1, \ldots, n \]
at the origin, Eq. (*) has a solution \( v \in \mathcal{H}_{\sigma, \Gamma'} \) for every \( \Gamma' \subset \subset \Gamma \).

**Remark** The set \( \Gamma \) determine the vanishing order of \( v \in \mathcal{H}_{\sigma, \Gamma'} \). Hence \( \Gamma \) expresses the smoothness up to the set \( y_j = 0 \) \( (j = 1, \ldots, n) \), because we have the interior regularity, \( x_j > 0 \) \( j = 1, \ldots, n \).

In order to construct a solution in some neighborhood of the origin, we construct solutions in the domain \( \pm y_j \geq 0 \) \( (j = 1, \ldots, n) \). Then we patch up these solutions.
Further extensions

We will briefly mention how the above theorem is extended to more general systems. We consider $N$ ($N \geq 1$) system of equations for the unknown vector $v = (v_1, \ldots, v_N)$

$$p_j(\delta)u_j + a_j(x, \delta^\alpha u; |\alpha| \leq s) = 0, \quad j = 1, \ldots, N,$$

where $1 \leq s \leq m$ are integers and $\delta_j = \partial/\partial x_j$,

$\delta^\alpha = \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n}$,

and $p_j(\zeta)$ is a polynomial of $\zeta$. The nonlinear term $a_j(x, z)$, $z = (z_\alpha^j)$ is supposed to be real-valued and smooth in $\mathbb{R}^n \times \Omega$, where $\Omega$ is a neighborhood of the origin $z = 0$.

Then we have the same assertion as to the above theorem.

Example. We consider Monge-Ampère operator

$$M(u) = u_{xx}u_{yy} - u_{xy}^2 + kxyu_{xy} + cu$$

in some neighborhood of the origin $(x, y) \in \mathbb{R}^2$. Here $k$ is a real constant and $c$ is a complex constant.

Let $u_0 = x^2y^2$ and $f_0 = M(u_0)$. We want to solve

$$M(u_0 + v) = f_0(x, y) + g(x, y), \quad \text{in } \mathbb{R}^2,$$

where $g(x, y)$ is a given function. This equation is related to find a surface with a prescribed Gaussian curvature. The general theory does not apply this equation because of the degeneracy of $u_0$.

The incidenial polynomial is given by

$$p(\zeta) = -2\zeta_1(\zeta_1 + 1) - 2\zeta_2(\zeta_2 + 1) - (k - 8)\zeta_1\zeta_2 - c.$$

Our theorem shows that If $4 < k < 12$ and $c = iK$, $K \gg 1$ there exists a solution $v$ of the above equation.

Proof of the Theorem

Definition of a smoothing operator in $\mathcal{H}_{s, \Gamma}$

Let $\phi \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ near the origin $x = 0$. Let $N \geq 1$ and let $\ell \geq 1$ be positive integers. We set

$$\psi_N(\zeta) = \exp \left( \frac{1}{N^{2\tau}} \sum_{j=1}^{n} \zeta_j^{2\tau} \right),$$
and define
\[ \chi_N^\ell = \int_{\mathbb{R}^n} \left\{ \psi_N(\zeta) \left( e^{-\sigma \zeta/N} - \sum_{\nu=1}^{\ell} \left( -\frac{\sigma \zeta}{N} \right)^\nu \frac{1}{\nu!} \right) + (1 - \psi_N(\zeta)) e^{-\sigma \zeta/N} \right\} \, dc \]

where \( \tau \) is an odd integer such that \( 2\tau \geq \ell \). We can easily see that \( \chi_N^\ell(\zeta) \) is an entire function of \( \zeta \) and real,
\[ \overline{\chi_N^\ell(\zeta)} = \chi_N^\ell(\overline{\zeta}) \]

We define a smoothing operator \( S_N \) by
\[ S_N v := M^{-1}(\chi_{N+1}^\ell(\zeta) \hat{v}(\zeta)), \quad v \in \mathcal{H}_{s,\Gamma} \]
where \( \hat{v}(\zeta) \) denotes the Mellin transform of \( v \), and \( M^{-1} \) denotes the inverse Mellin transform.

**Proof of the Theorem**

Let \( 1 < \tau < 2 \) and \( d > 1 \) be the constants chosen later. Let \( S_k \) \( (k = 0, 1, 2, \ldots) \) be a smoothing operator defined above with \( N + 1 = \mu_k := d^{\tau^k} \). We define
\[ G(v) = Lv - R(y + v). \]
Let \( L_w \) be the linearized operator of \( G \) at \( v = w \). We define \( g_0 = G(0) \).

**Iterative scheme**

We construct an approximate sequence \( \{w_k\} \) by
\[ w_0 = 0, \quad w_{k+1} = w_k + S_k \rho_k, \quad L_w \rho_k = g_k, \quad g_k = -G(w_k), \quad k = 0, 1, 2, \ldots \]

**Estimates**

There exist \( \exists \nu, \exists \kappa \) and \( \exists c > 0, \nu > \kappa > 1 \) such that
\[ \|g_k\|_0,\Gamma \leq c \mu_k^{-\kappa} d^{\kappa} \|g_0\|_{\nu+1,\Gamma}. \]

If we can show this estimate we see that \( g_k \to 0 \) as \( k \to \infty \) and that \( \{w_k\} \) is a Cauchy sequence. It follows that \( w := \lim_k w_k \) satisfies \( G(w) = 0 \).

**Step 1 A priori estimate of \( w_k \).**

There exists \( C > 0 \) independent of \( k \) such that, for \( j = 1, \ldots, k + 1 \)
\[ \|w_j\|_{\ell,\Gamma} \leq Cd^\kappa \|g_0\|_{\nu+1,\Gamma}, \quad \text{if} \ \ell < \kappa + s, \]

$$\|w_j\|_{\ell,\Gamma} \leq C\mu_j^{\ell+1-\kappa-s}d^\kappa\|g_0\|_{\nu+1,\Gamma}, \text{ if } \ell \geq \kappa + s.$$  

**Step 2** *A priori estimate of $g_k$* 

There exists $C > 0$ independent of $k$ such that 

$$\|g_k\|_{\nu,\Gamma} \leq Cd^\kappa\|g_0\|_{\nu+1,\Gamma}(1 + \mu_k^{(\nu+m+n+2-\kappa-s)/\tau}).$$ 

Using these estimates we can show the desired estimate. The constants $\tau$ and $d$ are determined by the equation.

**References**


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