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Fuchsian PDE with applications to normal forms of resonant vector fields †

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Motivations

Let $t \in \mathbf{C}$ or $t \in \mathbf{C}$. We consider Fuchsian ordinary differential equations $P \equiv p(t\frac{d}{dt})$, where $p(\zeta)$ us an polynomial of one variable. We call $p(\zeta)$ an incidencial polynomial of P. We consider the solvability of the equation $Pu = f(t)$, where $f(t)$ is analytic at the origin $t = 0$.

If the "non-log condition"

$$(1) \qquad p(\zeta) \neq 0 \quad \text{for} \quad \zeta = 0, 1, 2, \ldots$$

is fulfilled $Pu = f$ has an analytic solution. Indeed, the solution is constructed by a method of indeterminate coefficients if we expand u in Taylor series.

Now, let us consider the case where a "non-log condition" is not fulfilled. For the sake of simplicity, we consider under the condition

$$(2) \qquad \exists \zeta_0 \in \mathbf{Z}_+ = \{0, 1, 2, \ldots\}, \quad p(\zeta_0) = 0.$$

Remark. If there exists $\zeta_1 \in \mathbf{Z}$ such that $p(\zeta_1) = 0$ Condition (2) is a special case where the difference of characteristic exponents have integral difference.

By Frobenius theorem, the fundamental solutions contain a function of the form $t^\lambda \log t$,where λ is a certain constant. It follows that the solution u of $Pu = f(t)$ is singular, or u has finite differentiability.

Question

What happens in the case of nonlinear partial differential equations of Fuchs type ?

In order to answer to this question, we first introduce a class of so-called

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Fuchsian partial differential equations which appear from geomet
problems.

We also cite related works by Tahara, Mandai, Yamane and Y
mazawa.

Vector fields with an isolated singular poin

We consider

$$\mathcal{X}(x)=\sum_{j=1}^{n}a_j(x)\frac{\partial}{\partial x_j},\quad x=(x_1,\dots,x_n), \tag{3}$$

where $a_j(x)$ is a smooth function of x. We assume

$$\mathcal{X}(0)=0, \tag{4}$$

and the origin $x=0$ is an isolated singular point of $\mathcal{X}$.

We want to linearize $\mathcal{X}(x)$ by a coordinate change

$$x=y+v(y),\quad v=O(|y|^2). \tag{5}$$

We write

$$\mathcal{X}(x)=x\Lambda\frac{\partial}{\partial x}+R(x)\frac{\partial}{\partial x}\equiv X(x)\frac{\partial}{\partial x},\quad \frac{\partial}{\partial x}=\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right) \tag{6}$$

$$X(x)=x\Lambda+R(x), \tag{7}$$

where

$$R(x)=(R_1(x),\dots,R_n(x)),\quad R(x)=O(|x|^2), \tag{8}$$

and Λ is an $n\times n$ constant matrix.

Noting that

$$X(x)\frac{\partial}{\partial x}=X(y+v(y))\frac{\partial y}{\partial x}\frac{\partial}{\partial y}=X(y+v(y))\left(\frac{\partial x}{\partial y}\right)^{-1}\frac{\partial}{\partial y},$$

the linearizability condition implies

$$X(y+v)(1+\partial_y v)^{-1}=y\Lambda.$$

It follows that

$$(y+v)\Lambda+R(y+v)=y\Lambda(1+\partial_y v)=y\Lambda+y\Lambda\partial_y v. \tag{9}$$

Therefore v solves the so-called *homology equation*

$$(*)\qquad \mathcal{L}v \equiv y\Lambda\partial_y v - v\Lambda = R(y+v(y)),\quad v=(v_1,\ldots,v_n).$$

Therefore we have

Eq. (*) has a solution v if and only if $\mathcal{X}$ is linearized by a coordinate change $x = y + v(y)$.

Expression of a homology equation

We calculate the form of $\mathcal{L}$ in case Λ is a diagonal matrix. Namely we assume

$$(10)\qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Because we have

$$y\Lambda\partial_y = \sum_{k=1}^{n} \lambda_k y_k \frac{\partial}{\partial y_k}$$

we have

$$(11)\qquad \mathcal{L}v = \begin{pmatrix} \sum \lambda_k y_k \frac{\partial}{\partial y_k} - \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \sum \lambda_k y_k \frac{\partial}{\partial y_k} - \lambda_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Remark. The homology equation (*) is a special case of totally characteristic Fuchsian PDE. (cf. Tahara [4]). We also cite Shirai [3].

Non-log condition and a non resonant condition

For simplicity, we consider the above example. The indicial polynomial is defined by

$$(12)\qquad \sum_{k=1}^{n} \lambda_k \zeta_k - \lambda_j,\quad (j = 1,\ldots,n).$$

Non resonant condition

$\mathcal{L}$ is said to be *non-resonant* if

$$(13)\qquad \sum_{k=1}^{n} \lambda_k \alpha_k - \lambda_j \neq 0 \quad \text{for } \forall\alpha \in (\alpha_1,\ldots,\alpha_n) \in \mathbf{Z}_+^n, |\alpha| \geq 2.$$

Non resonant condition impies the existence of a formal solution. Indeed, we have

$$\mathcal{L}(\sum_{\alpha} v_\alpha y^\alpha) = \sum_{\alpha}(\sum_{k=1}^{n} \lambda_k \alpha_k - \Lambda) v_\alpha y^\alpha.$$

Hence $\mathcal{L}^{-1}$ exist on a set of formal power series if a non-resonant condition is fulfilled. It should be noted that a non-resonance condition is a non log condition.

Two theorems concerning the solvability of homology equations

As to the solvability of (*), probably the first result was obtained by Poincaré in 19th century. He introduced a so-called *Poincaré condition.* Then the middle of 20th century, Siegel introduced a Siegel condition and he essentially showed the solvability of (*) under a Siegel condition. On the other hand, in the real domain, Sternberg showed the solvability of (*) in a class of smooth functions without any diophantine condition. He essentially assumed the nonresonance condition. As to the resonant case, Hartman showed the solvability of (*) in a class of continuous functions. Our result is closely related to Hartman's theorem.

Sternberg's theorem
Suppose that a hyperbolic condition

(14) $$Re\lambda_k \neq 0, \quad k = 1, \ldots, n$$

is fulfulled. Moreover, assume that a non resonant condition is satisfied. Then, Eq. (*) has a smooth solution.

Sternberg's theorem shows the solvability of (*) under non- log condition.

Grobman- Hartman's theorem
If the hyperbolicity condition is satisfied, Eq.(*) has a continuous solution.

Remark. A continuous solution of (*) is defined by a weak solution.
Hartman's theorem treats the case where a non-log condition is not satisfied.

The Object of the Study

We consider the case where a non-log condition is not satisfied. The typical example is a volumn preserving vector field, $\lambda_1 + \cdots + \lambda_n = 0$. We want to solve (*) in a real domain in a class of finetely differentiable functions, which corresponds to Hartman's theorem in a C^ℓ class. This is closely related to the construction of a singular solution in a complex domain.

Remark. Geometrically, the resonance does not vanish under a formal change of variables. Because the solvability of (*) implies that the change of variables $x = y + v(y)$ linearizes the given vector field, the resonance also vanish under the change of varibles. This implies that *Eq. (*) does not necessarily have a formal solution.*

Heuristic Statement of Results - C^ℓ Hartman theorem -

For the sake of simplicity, we will state the special case of our theorem.

Theorem *Assume that λ_k $(k = 1, \ldots, n)$ are nonzero real number, (a hyperbolicity). Then Eq. (*) has a C^ℓ solution for a certain $\ell \geq 0$ determined by an indicial polynomials.*

Idea of the Proof.

Why Picard's iteration does not work ?

Firstly, we note the loss of derivatives of $\mathcal{L}^{-1}$. In fact, even if there exists $\mathcal{L}^{-1}$, we have a loss of derivatives. In order to see this, let us consider

$$\left(t\frac{d}{dt} - \lambda\right) u = g(t), \quad \mathrm{Re}\lambda < 0.$$

The solution is given by

$$u(t) = \int_0^1 \sigma^{-\lambda-1} g(\sigma t) d\sigma.$$

Clearly, we do not gain the derivatives.

On the other hand, in order to define the right-hand side of (*), $R(y + v)$ one needs derivatives of v. Indeed, Sobolev's embedding theorem implies:

if $0 \leq m < k - n/p < m + 1$ and $0 \leq \alpha < k - m - n/p < 1$, it follows that $W^{k,p}(\Omega) \hookrightarrow C^{m,\alpha}(\overline{\Omega})$.

Here $W^{k,p}(\Omega)$ is the space of distributions whose derivatives up to order

k is in the Lebesgue space $L^p(\Omega)$. $C^{m,\alpha}(\overline{\Omega})$ is a Hölder space, namely the set of functions with derivatives up to m has Hölder exponent α. Therefore the iteration scheme $v = \mathcal{L}^{-1}R(y+v)$ does not seem to converge.

In view of this we need to employ a Nash- Moser scheme, a rapidly convergent iteration scheme.

Rapidly Convergent Iteration Scheme

1. We need a smoothing operator which has not a smoothing effect transversal to the singular locus of the equation, $y_j = 0$, $(j = 1, \ldots, n)$.

2. The crucial step of the Nash-Moser iteration scheme is to solve a linearized equation. The linearized equation of (*) at $v = w$ is given by

$$\mathcal{L}v - \nabla R(y+w)v.$$

We note that w is singular or does not have regularity. The solvability of linear Fuchsian partial differential equations with singular coefficients seems open.

In order to handle these problems we use a Mellin transform, and a Nash-Moser iteration scheme of tangential type.

Statement of the Theorem

Mellin Transform

Let $N \geq 1$ be an integer. Let $f(x) = (f_1(x), \ldots, f_N(x))$ be an integrable function on $\mathbf{R}^n_+$, and let us define a Mellin transform $\hat{f}(\zeta)$ $(\zeta \in \Gamma + i\mathbf{R}^n)$ by

$$\hat{f}(\zeta) = \int_{\mathbf{R}^n_+} f(x)x^{\zeta - e}dx, \quad e = (1, \ldots, 1).$$

The inverse Mellin transform is given by

$$f(x) = M^{-1}(\hat{f})(\zeta) = \frac{1}{(2\pi i)^n}\int_{\mathbf{R}^n} \hat{f}(\eta + i\xi)x^{-\eta - i\xi}d\xi,\ x_j > 0,\ j = 1, \ldots, n,$$

where $\eta \in \Gamma$ is chosen so that the integral converges.

Definition of a function space

Let $\sigma \geq 0$, and let $\Gamma \subset \mathbf{R}^n$ be an open set. We define $H_\sigma \equiv H_{\sigma,\Gamma}$ as the set of holomorphic vector-valued functions

$$v(\zeta) = (v_1(\zeta), \ldots, v_N(\zeta)), \quad \zeta = \eta + i\xi \in \Gamma + i\mathbf{R}^n$$

such that

$$\|v\|_{\sigma,\Gamma} := \sup_{\eta\in\Gamma} \int_{\mathbf{R}^n} \langle\zeta\rangle^{\sigma} |v(\zeta)| d\xi < \infty,$$

where

$$\langle\zeta\rangle = 1 + \sum_{j=1}^{n} |\zeta_j|, \quad |v(\zeta)| = \left(\sum_{j=1}^{N} |v_j(\zeta)|^2\right)^{1/2}.$$

The space $H_{\sigma,\Gamma}$ is a Banach space with the norm $\|\cdot\|_{\sigma,\Gamma}$.

Let $\mathcal{H}_{\sigma,\Gamma}$ be the inverse Mellin transform of $H_{\sigma,\Gamma}$. The norm of $\mathcal{H}_{\sigma,\Gamma}$ is defined by

$$\|u\|_{\mathcal{H}_{\sigma,\Gamma}} \equiv \|u\|_{\sigma,\Gamma} := \|M(u)\|_{H_{\sigma,\Gamma}}.$$

We define an incidencial polynomial by

$$p(\zeta) = -\sum_{j=1}^{n} \zeta_j \lambda_j I - \Lambda,$$

where I is an identity matrix.

We say that $R \in \mathcal{H}_{\nu,\Gamma}$ *at the origin* if $\exists\psi \in C_0^\infty(\mathbf{R}^n)$ being identically equal to 1 in some neighborhood of the origin such that

$$M(\psi R) \in \mathcal{H}_{\nu,\Gamma}.$$

Then we have

Theorem *Suppose that there exist $C > 0$ and an open bounded set Γ, $0 \in \Gamma \subset \mathbf{R}^n$ such that*

$$|p(\eta + i\xi)| > C > 0, \quad \forall\eta \in \Gamma, \forall\xi \in \mathbf{R}^n.$$

Let $\sigma \geq 1$ be an integer. Then there exists $\nu \geq 0$ such that, if

$$R \in \mathcal{H}_{\nu,\Gamma} \quad \text{and} \quad \nabla R_j \in \mathcal{H}_{\nu,\Gamma}, \quad j = 1, \ldots, n$$

at the origin, Eq. (*) has a solution $v \in \mathcal{H}_{\sigma,\Gamma'}$ for every $\Gamma' \subset\subset \Gamma$.

Remark The set Γ determine the vanishing order of $v \in \mathcal{H}_{\sigma,\Gamma'}$. Hence Γ expresses the smoothness up to the set $y_j = 0$ $(j = 1, \ldots, n)$, because we have the interior regularity, $x_j > 0$ $j = 1, \ldots, n$.

In order to construct a solution in some neighborhood of the origin, we construct solutions in the domain $\pm y_j \geq 0$ $(j = 1, \ldots, n)$. Then we patch up these solutions.

Further extensions

We will briefly mention how the above theorem is extended to more general systems. We consider N ($N \geq 1$) system of equations for the unknown vector $v = (v_1, \ldots, v_N)$

$$p_j(\delta)u_j + a_j(x, \delta^\alpha u; |\alpha| \leq s) = 0, \quad j = 1, \ldots, N,$$

where $1 \leq s \leq m$ are integers and $\delta_j = \partial/\partial x_j$,

$$\delta^\alpha = \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n},$$

and $p_j(\zeta)$ is a polynomial of ζ. The nonlinear term $a_j(x, z)$, $z = (z_\alpha^j)$ is supposed to be real-valued and smooth in $\mathbf{R}^n \times \Omega$, where Ω is a neighborhood of the origin $z = 0$.

Then we have the same assertion as to the above theorem.

Example. We consider Monge-Ampère operator

$$M(u) = u_{xx}u_{yy} - u_{xy}^2 + kxyu_{xy} + cu$$

in some neighborhood of the origin $(x, y) \in \mathbf{R}^2$. Here k is a real constant and c is a complex constant.

Let $u_0 = x^2y^2$ and $f_0 = M(u_0)$. We want to solve

$$M(u_0 + v) = f_0(x, y) + g(x, y), \quad \text{in } \mathbf{R}^2,$$

where $g(x, y)$ is a given function. This equation is related to find a surface with a prescribed Gaussian curvature. The general theory does not apply this equation because of the degeneracy of u_0.

The incidencial polynomial is given by

$$p(\zeta) = -2\zeta_1(\zeta_1 + 1) - 2\zeta_2(\zeta_2 + 1) - (k - 8)\zeta_1\zeta_2 - c.$$

Our theorem shows that *If* $4 < k < 12$ *and* $c = iK$, $K \gg 1$ *there exists a solution* v *of the above equation.*

Proof of the Theorem

Definition of a smoothing operator in $\mathcal{H}_{s,\Gamma}$

Let $\phi \in C_0^\infty(\mathbf{R}^n)$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ near the origin $x = 0$. Let $N \geq 1$ and let $\ell \geq 1$ be positive integers. We set

$$\psi_N(\zeta) = \exp\left(\frac{1}{N^{2\tau}} \sum_{j=1}^n \zeta_j^{2\tau}\right),$$

and define

$$\chi_N^\ell = \int_{\mathbf{R}^n} \left\{ \psi_N(\zeta) \left(e^{-\sigma\zeta/N} - \sum_{\nu=1}^{\ell} \left(-\frac{\sigma\zeta}{N} \right)^\nu \frac{1}{\nu!} \right) + (1 - \psi_N(\zeta)) e^{-\sigma\zeta/N} \right\} d\zeta$$

where τ is an odd integer such that $2\tau \geq \ell$. We can easily see that $\chi_N^\ell(\zeta)$ is an entire function of ζ and real,

$$\overline{\chi_N^\ell(\zeta)} = \chi_N^\ell(\bar{\zeta}).$$

We define a *smoothing operator* S_N by

$$S_N v := M^{-1}(\chi_{N+1}^\ell(\zeta)\hat{v}(\zeta)), \quad v \in \mathcal{H}_{s,\Gamma},$$

where $\hat{v}(\zeta)$ denotes the Mellin transform of v, and M^{-1} denotes the inverse Mellin transform.

Proof of the Theorem

Let $1 < \tau < 2$ and $d > 1$ be the constants chosen later. Let S_k $(k = 0, 1, 2, \ldots)$ be a smoothing operator defined above with $N + 1 = \mu_k := d^{\tau^k}$. We define

$$G(v) = \mathcal{L}v - R(y + v).$$

Let L_w be the linearized operator of G at $v = w$. We define $g_0 = G(0)$.

Iterative scheme
We construct an approximate sequence $\{w_k\}$ by

$$w_0 = 0,\ w_{k+1} = w_k + S_k\rho_k,\ L_{w_k}\rho_k = g_k,\ g_k = -G(w_k),\ k = 0, 1, 2, \ldots$$

Estimates
There exist $\exists\nu$, $\exists\kappa$ and $\exists c > 0$, $\nu > \kappa > 1$ such that

$$\|g_k\|_{0,\Gamma} \leq c\mu_k^{-\kappa} d^\kappa \|g_0\|_{\nu+1,\Gamma}.$$

If we can show this estimate we see that $g_k \to 0$ as $k \to \infty$ and that $\{w_k\}$ is a Cauchy sequence. It follows that $w := \lim_k w_k$ satisfies $G(w) = 0$.

Step 1 *A priori estimate of w_k.*
There exists $C > 0$ independent of k such that, for $j = 1, \ldots, k + 1$

$$\|w_j\|_{\ell,\Gamma} \leq C d^\kappa \|g_0\|_{\nu+1,\Gamma}, \quad \text{if } \ell < \kappa + s,$$

$$\|w_j\|_{\ell,\Gamma} \le C\mu_{j-1}^{\ell+1-\kappa-s} d^{\kappa} \|g_0\|_{\nu+1,\Gamma}, \quad \text{if } \ell \ge \kappa + s.$$

Step 2 *A priori estimate of* g_k
There exists $C > 0$ independent of k such that

$$\|g_k\|_{\nu,\Gamma} \le Cd^{\kappa}\|g_0\|_{\nu+1,\Gamma}(1 + \mu_k^{(\nu+m+n+2-\kappa-s)/\tau}).$$

Using these estimates we can show the desired estimate. The constants τ and d are determined by the equation.

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