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THE DECOMPOSITION OF THE PERMUTATION CHARACTER $1_{GL(n,q^2)}^{GL(2n,q)}$

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INTRODUCTION

Let G be a finite group acting transitively on a finite set X, and let $H = G_x$ be the stabilizer of a point x in X. The permutation character π of G on X is equivalent to the induced character $(1_H)^G$ of the identity character 1_H of H. We say that the permutation character $\pi = (1_H)^G$ is multiplicity-free if it is decomposed into a direct sum of inequivalent irreducible characters. In this case, the centralizer algebra (or the Hecke algebra) of the permutation group is commutative, and we also say that H is a multiplicity-free subgroup of G. A pair (G, H) of a finite group G and a multiplicity-free subgroup H is sometimes called a Gelfand pair. A commutative association scheme $\mathfrak{X} = (X, \{R_i\}_{0\leq i\leq d})$ is associated with a multiplicity-free transitive action of a finite group G on a finite set X, by taking the relations $R_0, R_1, \ldots, R_d$ as the orbits of G on $X \times X$. It is an interesting question to know many examples of commutative association schemes and their character tables. (The reader is referred to Bannai-Ito [4], Bannai [1] for the basic concept of commutative association schemes and their character tables.) It should be noted that knowing the character table of a commutative association scheme (associated to a multiplicity-free transitive action of a finite group, i.e., to a Gelfand pair) is equivalent to knowing the zonal spherical functions of the permutation group.

Many examples of Gelfand pairs or commutative association schemes are known (see, e.g. Saxl [16], Inglis [9], Bannai [1], Bannai-Hao-Song [2], Bannai-Hao-Song-Wei [3], Bannai-Kawanaka-Song [5], Lusztig [14], Lawther [13], etc.). In Inglis-Liebeck-Saxl [10], it is stated that the following pairs (G, H) are Gelfand pairs:

(i) $(G, H) = (GL(n, q^2), GL(n, q))$,
(ii) $(G, H) = (GL(n, q^2), GU(n, q))$,
(iii) $(G, H) = (GL(2n, q), Sp(2n, q))$,
(iv) $(G, H) = (GL(2n, q), GL(n, q^2))$.

It seems that the structure of the double cosets $H\backslash G/H$, the decomposition of the permutation character $\pi = 1_H^G$, and the character table of the associated commutative association scheme are known for the first three cases (Gow [7], Klyachko [12], Bannai-Kawanaka-Song [5], Kawanaka [11], Bannai [1], Lusztig [14]). However, it seems that they are not yet known for the last case (iv) of $G = GL(2n, q)$ and $H = GL(n, q^2)$. The decomposition of the permutation character $1_{GL(n,q^2)}^{GL(2n,q)}$ is well-known for $n = 1$ (cf. Terras [19, Chapter 21]). When $n = 2$, it was determined by the second author [18] by explicitly calculating the inner product $(\chi, 1_{GL(2,q^2)}^{GL(4,q)})$ for all irreducible characters χ of $GL(4, q)$. Our purpose in this paper is to determine the decomposition of $1_{GL(n,q^2)}^{GL(2n,q)}$ for general n.

1. Preliminaries on General Linear Groups and Main Results

1.1. First of all, we briefly recall a parametrization of the irreducible characters of the general linear group $G_n = GL(n,q)$, following Macdonald [15, Chapter IV.]. Whenever possible, we use the notation of [15].

A partition is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers λ_i containing finitely many non-zero terms. The non-zero λ_i are called the parts of λ. We identify $(\lambda_1, \lambda_2, \dots, \lambda_r)$ with $(\lambda_1, \lambda_2, \dots, \lambda_r, 0, \dots, 0)$. Sometimes we write $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ in place of $\lambda = (\lambda_1, \lambda_2, \dots)$, where m_i is the number of j such that $\lambda_j = i$. The only partition with no non-zero terms is denoted by 0. For each partition λ, the length $l(\lambda)$ of λ is the number of parts of λ, and the weight $|\lambda|$ of λ is defined by $|\lambda| = \sum_{i\geq 1} \lambda_i$. We denote the set of all partitions by $\mathscr{P}$. The diagram of $\lambda \in \mathscr{P}$ is the set of points $x = (i,j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$, and the conjugate λ' of λ is the partition whose diagram is the transpose of that of λ. For example, the conjugate of $(2,2,1)$ is $(3,2)$. The hook-length $h(x)$ of λ at $x = (i,j) \in \lambda$ (i.e., $1 \leq j \leq \lambda_i$) is defined by $h(x) = \lambda_i + \lambda'_j - i - j + 1$. For $\lambda, \mu \in \mathscr{P}$, we define $\lambda \cup \mu$ to be the partition whose parts are those of λ and μ, arranged in descending order. An even (resp. odd) partition is a partition with all parts even (resp. odd). We let s_λ denote the Schur function (in countably many independent variables) corresponding to $\lambda \in \mathscr{P}$.

Let $\mathbb{F}_q$ be a finite field with q elements, and $\overline{\mathbb{F}}_q$ the algebraic closure of $\mathbb{F}_q$. For each positive integer l there exists a unique extension $\mathbb{F}_{q^l}$ of $\mathbb{F}_q$ in $\overline{\mathbb{F}}_q$ of degree l. We denote the multiplicative group of $\mathbb{F}_{q^l}$ by M_l, and the character group of M_l by $\hat{M}_l$. If l divides m then $\hat{M}_l$ is embedded in $\hat{M}_m$ by the transpose of the norm map $N_{m,l} : M_m \longrightarrow M_l$. We let $L = \varinjlim \hat{M}_l$ be the inductive limit of the $\hat{M}_l$. The Frobenius map $F : \gamma \longrightarrow \gamma^q$ acts on L, and $\hat{M}_l$ is the set of all F^l-fixed elements in L. We denote the set of F-orbits in L by Θ. Then the irreducible characters of G_n can be parametrized by the partition-valued functions $\boldsymbol{\mu} : \Theta \longrightarrow \mathscr{P}$ such that

$$||\boldsymbol{\mu}|| = \sum_{\varphi\in\Theta} d(\varphi)\, |\boldsymbol{\mu}(\varphi)| = n \tag{1}$$

where $d(\varphi)$ is the number of elements of φ. The irreducible character of G_n corresponding to $\boldsymbol{\mu}$ is denoted by $\chi_{\boldsymbol{\mu}}$. The degree $d_{\boldsymbol{\mu}}$ of $\chi_{\boldsymbol{\mu}}$ is given by

$$\begin{aligned} d_{\boldsymbol{\mu}} &= \psi_n(q) \prod_{\varphi\in\Theta} s_{\boldsymbol{\mu}(\varphi)}(q_\varphi^{-1}, q_\varphi^{-2}, \dots) \\ &= \psi_n(q) \prod_{\varphi\in\Theta} q_\varphi^{n(\boldsymbol{\mu}(\varphi)')} \tilde{H}_{\boldsymbol{\mu}(\varphi)}(q_\varphi)^{-1} \end{aligned} \tag{2}$$

where $q_\varphi = q^{d(\varphi)}$,

$$\psi_n(q) = \prod_{i=1}^{n} (q^i - 1),$$

$$n(\lambda) = \sum_{i\geq 1} (i-1)\lambda_i,$$

and

$$\tilde{H}_\lambda(q_\varphi) = \prod_{x\in\lambda} (q_\varphi^{h(x)} - 1)$$

for $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}$.

Let ξ_1 be the identity character of M_1, and if q is odd then let ξ_{-1} be the quadratic character of M_1. We put $\varphi_1 = \{\xi_1\}$, $\varphi_{-1} = \{\xi_{-1}\} \in \Theta$. For $\varphi = \{\xi, \xi^q, \dots, \xi^{q^{d-1}}\} \in \Theta$, the reciprocal F-orbit $\tilde{\varphi}$ of φ is defined by

$$\tilde{\varphi} = \{\xi^{-1}, \xi^{-q}, \dots, \xi^{-q^{d-1}}\}.$$

Notice that φ_1 and φ_{-1} are the only elements $\varphi \in \Theta$ such that $d(\varphi) = 1$ and $\tilde{\varphi} = \varphi$. Also for each partition-valued function $\boldsymbol{\mu} : \Theta \longrightarrow \mathscr{P}$, we define $\tilde{\boldsymbol{\mu}} : \Theta \longrightarrow \mathscr{P}$ by

$$\tilde{\boldsymbol{\mu}}(\varphi) = \boldsymbol{\mu}(\tilde{\varphi})$$

for all $\varphi \in \Theta$. Then we can easily verify that the complex conjugate $\overline{\chi_{\boldsymbol{\mu}}}$ of $\chi_{\boldsymbol{\mu}}$ is given by $\chi_{\tilde{\boldsymbol{\mu}}}$ (see for example (4.5) in [15, Chapter IV.]), from which it follows that

1.1.1. *An irreducible character $\chi_{\boldsymbol{\mu}}$ of G_n is real-valued if and only if $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu}$.*

1.2. We now present our main results. Let K_{2n} be a subgroup of G_{2n} isomorphic to $GL(n, q^2)$. It is known that

1.2.1. Theorem (Inglis-Liebeck-Saxl [10]). *The permutation character $(1_{K_{2n}})^{G_{2n}}$ is multiplicity-free and every irreducible constituent of $(1_{K_{2n}})^{G_{2n}}$ is real-valued.*

In this paper, we determine the decomposition of the permutation character $(1_{K_{2n}})^{G_{2n}}$ explicitly. More precisely, we will prove the following:

1.2.2. Theorem. (i) *If q is odd, then we have $(1_{K_{2n}})^{G_{2n}} = \sum \chi_{\boldsymbol{\mu}}$, summed over $\boldsymbol{\mu}$ such that $||\boldsymbol{\mu}|| = 2n$, $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu}$, and both $\boldsymbol{\mu}(\varphi_1)'$ and $\boldsymbol{\mu}(\varphi_{-1})$ are even.*

(ii) *If q is even, then we have $(1_{K_{2n}})^{G_{2n}} = \sum \chi_{\boldsymbol{\mu}}$, summed over $\boldsymbol{\mu}$ such that $||\boldsymbol{\mu}|| = 2n$, $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu}$, and $\boldsymbol{\mu}(\varphi_1)'$ is even.*

(iii) *In either case, the generating function for the rank (i.e., the number of the irreducible constituents of the permutation character $(1_{K_{2n}})^{G_{2n}}$) is given by*

$$\sum_{n \geq 0} \operatorname{rank}(G_{2n}/K_{2n}) t^{2n} = \prod_{r \geq 1} (1 - qt^{2r})^{-1} \tag{3}$$

with the understanding that $\operatorname{rank}(G_0/K_0) = 1$. *In particular we have*

$$\operatorname{rank}(G_{2n}/K_{2n}) = \sum q^{l(\lambda)}$$

summed over all partitions λ such that $|\lambda| = n$.

1.2.3. *Remark.* In the notation of Green [8], our character $\chi_{\boldsymbol{\mu}}$ correponds to the conjugate function $\boldsymbol{\mu}' : \Theta \longrightarrow \mathscr{P}$ defined by $\boldsymbol{\mu}'(\varphi) = \boldsymbol{\mu}(\varphi)'$ for all $\varphi \in \Theta$. In particular, in our notation the identity character of G_n assigns the partition (1^n) to φ_1. See Springer-Zelevinsky [17, Remark 1.9.].

1.2.4. *Remark.* Let $\pi(G_n)$ denote the number of the conjugacy classes of G_n, then the generating function for the $\pi(G_n)$ is given by

$$\sum_{n \geq 0} \pi(G_n) t^n = \prod_{r \geq 1} (1 - t^r)(1 - qt^r)^{-1}.$$

Hence 1.2.2 (iii) implies that

$$\operatorname{rank}(G_{2n}/K_{2n}) = \sum_{i=0}^{n} p(i) \pi(G_{n-i})$$

where $p(i)$ is the number of partitions λ such that $|\lambda| = i$. It is a reasonable guess that there is a natural set of representatives of the double cosets $K_{2n}\backslash G_{2n}/K_{2n}$ which reflects the above equality.

2. Degree Formula

2.1. The starting point of the proof of 1.2.2 is the following proposition:

2.1.1. Proposition. (i) *If q is odd, then we have*

$$\sum d_\mu = (q^{2n} - q)(q^{2n} - q^3)\dots(q^{2n} - q^{2n-1})$$

where the sum on the left is over μ such that $||\mu|| = 2n$, $\tilde{\mu} = \mu$, and both $\mu(\varphi_1)'$ and $\mu(\varphi_{-1})$ are even.

(ii) *If q is even, then we have*

$$\sum d_\mu = (q^{2n} - q)(q^{2n} - q^3)\dots(q^{2n} - q^{2n-1})$$

where the sum on the left is over μ such that $||\mu|| = 2n$, $\tilde{\mu} = \mu$, and $\mu(\varphi_1)'$ is even.

To prove 2.1.1, we need some preparations. In what follows, we assume that q is odd. (The assertion (ii) is proved in exactly the same way as (i).)

Let Φ denote the set of monic irreducible polynomials $f(t)$ over $\mathbb{F}_q$ with $f(t) \neq t$. We identify Φ with the set of F-orbits in the multiplicative group M of the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$, by assigning to each f the F-orbit consisting of its roots in M.

Let $f(t) = t^k + a_1 t^{k-1} + \dots + a_k$ be a monic polynomial in $\mathbb{F}_q[t]$ of degree k with $a_k \neq 0$. The reciprocal polynomial $\tilde{f}$ of f is defined by

$$\tilde{f}(t) = a_k^{-1} t^k f(t^{-1}) = t^k + \frac{a_{k-1}}{a_k} t^{k-1} + \dots + \frac{1}{a_k}.$$

We call the polynomial f self-reciprocal if $f(t) = \tilde{f}(t)$.

Let

$$\begin{aligned} \Psi &= \Phi \cup \{t\} : \text{the set of all monic irreducible polynomials in } \mathbb{F}_q[t], \\ S &= \{f \in \Phi\backslash\{t \pm 1\} \mid f : \text{self-reciprocal}\}, \\ N &= \{f \in \Phi\backslash\{t \pm 1\} \mid f : \text{non-self-reciprocal}\}, \end{aligned}$$

and let

$$\begin{aligned} \Psi_k &= \{f \in \Psi \mid \deg f = k\}, \\ S_k &= \{f \in S \mid \deg f = k\}, \\ N_k &= \{f \in N \mid \deg f = k\} \end{aligned}$$

for $k \geq 1$. Notice that S_k is empty unless k is even.

First we observe the following two one-to-one correspondences due to Carlitz [6]:

2.1.2 ([6, §3.]). *We have*

$$\Psi_k \stackrel{1:1}{\longleftrightarrow} S_{2k} \cup \{g\tilde{g} \mid g \in N_k\}$$

for $k \geq 2$, and

$$\Psi_1\backslash\{t \pm 2\} \stackrel{1:1}{\longleftrightarrow} S_2 \cup \{g\tilde{g} \mid g \in N_1\}.$$

Proof. Let $h(t) \in \mathbb{F}_q[t]$ be a monic irreducible polynomial of degree k ($k \geq 1$) such that $h(t) \neq t \pm 2$, then $h(t)$ is decomposed into linear factors in $\mathbb{F}_{q^k}[t]$ as $h(t) = (t-\beta)(t-\beta^q)\ldots(t-\beta^{q^{k-1}})$. Let $\alpha \in \mathbb{F}_{q^{2k}}$ be a root of the polynomial $t^2 - \beta t + 1$, i.e., $\alpha + \alpha^{-1} = \beta$. Since $\beta \neq \pm 2$ it follows that $\alpha \neq \alpha^{-1}$, so that

$$\alpha, \alpha^q, \ldots, \alpha^{q^{k-1}}, \alpha^{-1}, \alpha^{-q}, \ldots, \alpha^{-q^{k-1}}$$

are distinct. We define

$$\begin{aligned} f(t) &= t^k h(t+t^{-1}) \\ &= (t-\alpha)(t-\alpha^q)\ldots(t-\alpha^{q^{k-1}})(t-\alpha^{-1})(t-\alpha^{-q})\ldots(t-\alpha^{-q^{k-1}}), \end{aligned}$$

then $f(t)$ is a monic polynomial of degree $2k$. Now, if $\alpha \in \mathbb{F}_{q^{2k}} \backslash \mathbb{F}_{q^k}$ then we have $f(t) \in S_{2k}$ since $\alpha^{-1} = \alpha^{q^k}$, and if $\alpha \in \mathbb{F}_{q^k}$ then we have $f(t) = g(t)\tilde{g}(t)$ where

$$g(t) = (t-\alpha)(t-\alpha^q)\ldots(t-\alpha^{q^{k-1}}) \in N_k,$$

as desired. □

Let $\sigma_{2k} = |S_{2k}|$ and $\tau_{2k} = |\{g\tilde{g} \mid g \in N_k\}| = \frac{1}{2}|N_k|$ for $k \geq 1$. Then it follows from 2.1.2 that

$$\sum_{k|N} k(\sigma_{2k} + \tau_{2k}) + 2 = q^N \tag{4}$$

for $N \geq 1$. If $N = 2M$ is even then we also have

$$\sum_{k|M} (2k)\sigma_{2k} + \sum_{k|2M} k(2\tau_{2k}) + 2 = q^N - 1. \tag{5}$$

On the other hand, if N is odd then we have

$$\sum_{k|N} k(2\tau_{2k}) + 2 = q^N - 1. \tag{6}$$

Let $x = (x_1, x_2, \ldots)$ be an infinite sequence of independent variables. We shall need the following four equalities:

2.1.3 (cf. [15, p.63, (4.3)]). $\sum_\lambda s_\lambda^2 = \prod_i (1-x_i^2)^{-1} \prod_{i<j} (1-x_ix_j)^{-2}$, *where the sum on the left is over all partitions* λ.

2.1.4 (cf. [15, p.76, Example 4]). $\sum_\lambda s_\lambda = \prod_i (1-x_i)^{-1} \prod_{i<j} (1-x_ix_j)^{-1}$, *where the sum on the left is over all partitions* λ.

2.1.5 (cf. [15, p.77, Example 5(a)]). $\sum_{\mu \text{ even}} s_\mu = \prod_i (1-x_i^2)^{-1} \prod_{i<j} (1-x_ix_j)^{-1}$, *where the sum on the left is over all even partitions* μ.

2.1.6 (cf. [15, p.77, Example 5(b)]). $\sum_{\nu' \text{ even}} s_\nu = \prod_{i<j} (1-x_ix_j)^{-1}$, *where the sum on the left is over all partitions* ν *with* ν' *even.*

2.2. *Proof of 2.1.1.* Our proof of 2.1.1 is inspired by [15, p.289, Example 5 of all, notice that the number of elements $\varphi \in \Theta$ such that $d(\varphi) = 2k$ and $\tilde{\varphi}$ equal to σ_{2k}. We shall compute the following:

$$
\begin{aligned}
D = & \sum_{\nu' \text{ even}} s_\nu(q^{-1}, q^{-2}, \ldots) t^{|\nu|} \times \sum_{\mu \text{ even}} s_\mu(q^{-1}, q^{-2}, \ldots) t^{|\mu|} \\
& \times \prod_{k \geq 1} \Big\{ \sum_{\lambda} s_\lambda(q^{-2k}, q^{-4k}, \ldots) t^{2k|\lambda|} \Big\}^{\sigma_{2k}} \\
& \times \prod_{k \geq 1} \Big\{ \sum_{\lambda} s_\lambda^2(q^{-k}, q^{-2k}, \ldots) t^{2k|\lambda|} \Big\}^{\tau_{2k}} \\
= & \prod_{i<j} (1 - (t^2 q^{-i-j}))^{-1} \times \prod_i (1 - (tq^{-i})^2)^{-1} \prod_{i<j} (1 - (t^2 q^{-i-j}))^{-1} \\
& \times \prod_{k \geq 1} \Big\{ \prod_i (1 - (tq^{-i})^{2k})^{-1} \prod_{i<j} (1 - (t^2 q^{-i-j})^{2k})^{-1} \Big\}^{\sigma_{2k}} \\
& \times \prod_{k \geq 1} \Big\{ \prod_i (1 - (tq^{-i})^{2k})^{-1} \prod_{i<j} (1 - (t^2 q^{-i-j})^{k})^{-2} \Big\}^{\tau_{2k}}
\end{aligned}
$$

where t is an indeterminate.

Let

$$
\begin{aligned}
X_1 &= \log \prod_{k \geq 1} \Big\{ \prod_{i \geq 1} (1 - (tq^{-i})^{2k})^{-1} \Big\}^{\sigma_{2k}}, \\
Y_1 &= \log \prod_{k \geq 1} \Big\{ \prod_{i \geq 1} (1 - (tq^{-i})^{2k})^{-1} \Big\}^{\tau_{2k}}, \\
Z_1 &= \log \prod_{i \geq 1} (1 - (tq^{-i})^2)^{-1}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
X_1 &= \sum_{k \geq 1} \sigma_{2k} \sum_{i \geq 1} \sum_{r \geq 1} \frac{(tq^{-i})^{2kr}}{r} = \sum_{k \geq 1} \sigma_{2k} \sum_{r \geq 1} \frac{t^{2kr}}{r} \cdot \frac{1}{q^{2kr} - 1} \\
&= \sum_{N \geq 1} \frac{t^{2N}}{N(q^{2N} - 1)} \sum_{k|N} k\sigma_{2k}.
\end{aligned}
$$

Similarly, we have

$$
Y_1 = \sum_{N \geq 1} \frac{t^{2N}}{N(q^{2N} - 1)} \sum_{k|N} k\tau_{2k}
$$

and

$$
Z_1 = \sum_{N \geq 1} \frac{t^{2N}}{N(q^{2N} - 1)}.
$$

Therefore, it follows from (4) that

$$(7)\qquad X_1+Y_1+Z_1=\sum_{N\geq 1}\frac{t^{2N}}{N(q^{2N}-1)}(q^N-1)=\sum_{N\geq 1}\frac{t^{2N}}{N(q^N+1)}$$
$$=\sum_{N\geq 1}\frac{t^{2N}}{N}\sum_{k\geq 1}(-1)^{k-1}q^{-kN}=\sum_{k\geq 1}(-1)^{k-1}\sum_{N\geq 1}\frac{(t^2q^-}{N}$$

Let

$$X_2=\log\prod_{k\geq 1}\Big\{\prod_{i<j}(1-(t^2q^{-i-j})^{2k})^{-1}\Big\}^{\sigma_{2k}},$$
$$Y_2=\log\prod_{k\geq 1}\Big\{\prod_{i<j}(1-(t^2q^{-i-j})^{k})^{-2}\Big\}^{\tau_{2k}},$$
$$Z_2=\log\prod_{i<j}(1-t^2q^{-i-j})^{-2}.$$

Then we have

$$X_2=\sum_{k\geq 1}\sigma_{2k}\sum_{i<j}\sum_{r\geq 1}\frac{(t^2q^{-i-j})^{2kr}}{r}=\sum_{k\geq 1}\sigma_{2k}\sum_{r\geq 1}\frac{t^{4kr}}{r}\sum_{i\geq 1}\frac{q^{-4ikr}}{q^{2kr}-1}$$
$$=\sum_{i\geq 1}\sum_{M\geq 1}\frac{t^{4M}}{(2M)(q^{2M}-1)}\Big(\sum_{k|M}(2k)\sigma_{2k}\Big)q^{-4iM}.$$

Similarly, we have

$$Y_2=\sum_{i\geq 1}\sum_{N\geq 1}\frac{t^{2N}}{N(q^N-1)}\Big(\sum_{k|N}k(2\tau_{2k})\Big)q^{-2iN}$$

and

$$Z_2=\sum_{i\geq 1}\sum_{N\geq 1}\frac{t^{2N}}{N(q^N-1)}2q^{-2iN}.$$

Therefore, it follows from (5) and (6) that

$$(8)\qquad X_2+Y_2+Z_2=\sum_{i\geq 1}\sum_{N\geq 1}\frac{t^{2N}}{N(q^N-1)}(q^N-1)q^{-2iN}$$
$$=\sum_{i\geq 1}\sum_{N\geq 1}\frac{(t^2q^{-2i})^N}{N}.$$

Hence from (7) and (8) we obtain

$$\log D=X_1+Y_1+Z_1+X_2+Y_2+Z_2$$
$$=\sum_{l\geq 1}\sum_{N\geq 1}\frac{(t^2q^{-2l+1})^N}{N}$$
$$=\log\prod_{l\geq 1}(1-t^2q^{-2l+1})^{-1}$$

so that

$$D=\prod_{l\geq 1}(1-t^2q^{-2l+1})^{-1}=\sum_{m\geq 0}t^{2m}q^{-m}/\varphi_m(q^{-2})$$

where $\varphi_m(t) = (1-t)(1-t^2)\dots(1-t^m)$.

Finally, on picking out the coefficient of t^{2n}, and multiplying by $\psi_{2n}(q)$, we get the desired result.

3. Branching Lemmas

In this section, we prepare two lemmas which enable us to prove 1.2.2 by induction on n. We do not need to assume in this section that q is odd.

3.1. First, we recall a result of Zelevinsky [21]. Let $n \geq 2$ and let H_n be the subgroup of G_n consisting of the matrices of the form

$$g = \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix}$$

where $x \in G_{n-1}$. Let U_{n-1} be the abelian normal subgroup of H_n defined by

$$U_{n-1} = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1_{n-1} \end{pmatrix} \right\} \cong \mathbb{F}_q^{n-1}$$

where 1_{n-1} is the identity matrix of degree $n-1$. We identify G_{n-1} with the following subgroup of H_n:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \middle| \; x \in G_{n-1} \right\}$$

then we have $H_n = U_{n-1} \rtimes G_{n-1}$, the semidirect product of U_{n-1} with G_{n-1}. The irreducible characters of H_n are determined by applying the method of little groups, and they are parametrized by the partition-valued functions $\boldsymbol{\nu} : \Theta \longrightarrow \mathscr{P}$ such that $||\boldsymbol{\nu}|| < n$ (cf. [21, §13.]). The irreducible character of H_n corresponding to $\boldsymbol{\nu}$ is denoted by $\zeta_{\boldsymbol{\nu}}^{(n)}$. Notice that the irreducible characters $\zeta_{\boldsymbol{\nu}}^{(n)}$ of H_n with $||\boldsymbol{\nu}|| = n-1$ are exactly those obtained by the irreducible characters $\chi_{\boldsymbol{\nu}}$ of $G_{n-1} \cong H_n/U_{n-1}$, that is, they are constant on U_{n-1}.

If $\boldsymbol{\mu} : \Theta \longrightarrow \mathscr{P}$ and $\boldsymbol{\nu} : \Theta \longrightarrow \mathscr{P}$ are two partition-valued functions, we shall write $\boldsymbol{\nu} \dashv \boldsymbol{\mu}$ if $\boldsymbol{\mu}(\varphi)'_i - 1 \leq \boldsymbol{\nu}(\varphi)'_i \leq \boldsymbol{\mu}(\varphi)'_i$ for all $\varphi \in \Theta$ and $i \geq 1$ (i.e., the skew diagram $\boldsymbol{\mu}(\varphi) - \boldsymbol{\nu}(\varphi)$ is a horizontal strip for any $\varphi \in \Theta$).

3.1.1. Theorem ([21, §13.5.]). (i) *Let $\boldsymbol{\mu} : \Theta \longrightarrow \mathscr{P}$ be a partition-valued function such that $||\boldsymbol{\mu}|| = n$. Then we have*

$$\chi_{\boldsymbol{\mu}} \downarrow_{H_n}^{G_n} = \sum \zeta_{\boldsymbol{\nu}}^{(n)}$$

summed over $\boldsymbol{\nu}$ such that $||\boldsymbol{\nu}|| < n$ and $\boldsymbol{\nu} \dashv \boldsymbol{\mu}$.

(ii) *Let $\boldsymbol{\nu} : \Theta \longrightarrow \mathscr{P}$ be a partition-valued function such that $||\boldsymbol{\nu}|| < n$. Then we have*

$$\zeta_{\boldsymbol{\nu}}^{(n)} \downarrow_{G_{n-1}}^{H_n} = \sum \chi_{\boldsymbol{\lambda}}$$

summed over $\boldsymbol{\lambda}$ such that $||\boldsymbol{\lambda}|| = n-1$ and $\boldsymbol{\nu} \dashv \boldsymbol{\lambda}$.

The following theorem was first proved by Thoma [20], and is easily derived from 3.1.1.

3.1.2. Theorem ([20]). *Let $\boldsymbol{\mu} : \Theta \longrightarrow \mathscr{P}$ and $\boldsymbol{\lambda} : \Theta \longrightarrow \mathscr{P}$ be partition-valued functions such that $||\boldsymbol{\mu}|| = n$ and $||\boldsymbol{\lambda}|| = n-1$. Then the multiplicity of $\chi_{\boldsymbol{\mu}}$ in the induced character $\chi_{\boldsymbol{\lambda}} \uparrow_{G_{n-1}}^{G_n}$ is equal to the number of $\boldsymbol{\nu} : \Theta \longrightarrow \mathscr{P}$ such that $\boldsymbol{\nu} \dashv \boldsymbol{\mu}$ and $\boldsymbol{\nu} \dashv \boldsymbol{\lambda}$.*

3.2. Let $\mathbb{V}_{2n}$ be the vector space of column $2n$-vectors with components in $\mathbb{F}_q$, and let $\{v_1, v_2, \ldots, v_{2n}\}$ be the standard basis of $\mathbb{V}_{2n}$, that is, v_i is the vector with 1 in the i-th component and zeros elsewhere. We fix an element $\alpha \in \mathbb{F}_{q^2}$ such that $\alpha \notin \mathbb{F}_q$, and denote by $f(t) = t^2 + at + b \in \mathbb{F}_q[t]$ the minimal polynomial of α over $\mathbb{F}_q$. Let g_0 be an element in G_{2n} such that $g_0^2 + ag_0 + b1_{2n} = 0$. Then g_0 determines a vector space over $\mathbb{F}_{q^2}$ on $\mathbb{V}_{2n}$, of dimension n, such that $\alpha v = g_0 v$ for $v \in \mathbb{V}_{2n}$. The centralizer $K_{2n} = C_{G_{2n}}(g_0)$ of g_0 in G_{2n} is isomorphic to $GL(n, q^2)$.

Let U be the subspace of $\mathbb{V}_{2n}$ over $\mathbb{F}_q$ spanned by $v_2, v_3, \ldots, v_{2n}$. Clearly, an element $g \in G_{2n}$ belongs to G_{2n-1} if and only if $gU = U$ and $gv_1 = v_1$. The subspace U contains a subspace W of $\mathbb{V}_{2n}$ over $\mathbb{F}_{q^2}$ of dimension $n-1$ (over $\mathbb{F}_{q^2}$), defined by

$$W = \{u \in U \mid g_0 u \in U\}.$$

It is easily seen that

$$G_{2n-1} \cap K_{2n} = \{k \in K_{2n} \mid kW = W,\ kv_1 = v_1\},$$

that is, $G_{2n-1} \cap K_{2n}$ is isomorphic to $GL(n-1, q^2)$.

Now for any $x \in G_{2n}$ we have

$$\begin{aligned} |G_{2n-1} x K_{2n}| &= \frac{|G_{2n-1}|\,|K_{2n}|}{|G_{2n-1} \cap xK_{2n}x^{-1}|} \\ &= \frac{|G_{2n-1}|\,|K_{2n}|}{|GL(n-1, q^2)|} \\ &= \frac{1}{q}|G_{2n}| \end{aligned}$$

since $xK_{2n}x^{-1} = C_{G_{2n}}(xg_0x^{-1}) \cong GL(n, q^2)$ and g_0 is chosen arbitrarily. Hence it follows from Mackey's theorem that

3.2.1. Lemma. $(1_{K_{2n}})^{G_{2n}} \downarrow^{G_{2n}}_{G_{2n-1}} = q \cdot (1_{K_{2n-2}})^{G_{2n-2}} \uparrow^{G_{2n-1}}_{G_{2n-2}}$.

3.3. For the sake of simplicity, in what follows we assume that g_0 is of the form

$$g_0 = \begin{pmatrix} \tilde{g_0} & 0 & \cdots & 0 \\ 0 & \tilde{g_0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{g_0} \end{pmatrix}$$

where $\tilde{g_0} = \left(\begin{smallmatrix} 0 & -b \\ 1 & -a \end{smallmatrix}\right)$, so that $v_{2i} = \alpha v_{2i-1}$ $(1 \le i \le n)$. Then it follows that

3.3.1. *For $g = (g_{ij}) \in G_{2n}$, g is contained in K_{2n} if and only if*

$$g_{2k-1,2l-1} = ag_{2k,2l-1} + g_{2k,2l}$$

and

$$g_{2k-1,2l} = -bg_{2k,2l-1}$$

for $1 \le k, l \le n$.

We identify the subgroup H_{2n-1} of G_{2n-1} with

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & x \end{pmatrix} \;\middle|\; x \in G_{2n-2} \right\},$$

and so on. Clearly, the subgroup $K_{2n-2} = G_{2n-2} \cap K_{2n}$ of G_{2n-2} is isomorphic to $GL(n-1, q^2)$.

3.3.2. Lemma. *Let* $(1_{K_{2n}})^{G_{2n}} = \sum_{i=1}^{k} \chi_{\mu_i}$ *and* $(1_{K_{2n-2}})^{G_{2n-2}} = \sum_{j=1}^{l} \chi_{\lambda_j}$. *Then we have*

$$\sum_{i=1}^{k} \sum_{\substack{||\nu||=2n-1 \\ \nu \dashv \mu_i}} \chi_\nu = \sum_{j=1}^{l} \sum_{\substack{||\nu||=2n-1 \\ \lambda_j \dashv \nu}} \chi_\nu.$$

3.4. *Proof of 3.3.2.* First of all, notice that an element g in G_{2n} belongs to H_{2n} if and only if $gv_1 = v_1$. Hence we have

$$H_{2n} \cap K_{2n} \cong \mathbb{F}_{q^2}^{n-1} \rtimes GL(n-1, q^2),$$

from which it follows that $|H_{2n}K_{2n}| = |G_{2n}|$, that is,

$$G_{2n} = H_{2n}K_{2n} = U_{2n-1}G_{2n-1}K_{2n}. \tag{9}$$

Let $\mathbb{C}[G_{2n}]$ be the complex group algebra of G_{2n}. For any subgroup K of G_{2n}, we define

$$e_K = \frac{1}{|K|} \sum_{k \in K} k,$$

then $e_K^2 = e_K$ and the left $\mathbb{C}[G_{2n}]$-module $\mathbb{C}[G_{2n}]e_K$ affords the induced representation $(1_K)^{G_{2n}}$.

By virtue of 3.1.1 (i), in order to prove 3.3.2 it is enough to show that

3.4.1. *The left* $\mathbb{C}[G_{2n-1}]$*-module* $e_{U_{2n-1}}\mathbb{C}[G_{2n}]e_{K_{2n}}$ *affords the induced representation* $(1_{U_{2n-2}K_{2n-2}})^{G_{2n-1}} = (1_{U_{2n-2}K_{2n-2}})^{H_{2n-1}} \uparrow_{H_{2n-1}}^{G_{2n-1}}$.

From (9) it follows that $e_{U_{2n-1}}\mathbb{C}[G_{2n}]e_{K_{2n}}$ is generated (as vector space) by the elements $e_{U_{2n-1}}xe_{K_{2n}}$, $x \in G_{2n-1}$. Moreover, we have

$$(U_{2n-1}K_{2n}) \cap G_{2n-1} = U_{2n-2}K_{2n-2}. \tag{10}$$

In fact, if $x \in G_{2n-1}$ is written as $x = uk$ for some $u \in U_{2n-1}$ and $k \in K_{2n}$, then k is contained in $H_{2n} \cap K_{2n}$. Since v_1 is fixed by k, so is v_2. That is, k is of the form

$$k = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & w \\ 0 & 0 & k_0 \end{pmatrix}$$

where $k_0 \in K_{2n-2}$, from which it follows that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & w \\ 0 & 0 & k_0 \end{pmatrix} \in U_{2n-2}K_{2n-2}.$$

Conversely, if x is written as above, then by 3.3.1 there exists $z = (z_1, z_2, \ldots, z_{2n-2})$ such that

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & w \\ 0 & 0 & k_0 \end{pmatrix} \in K_{2n}$$

and therefore we have $x \in U_{2n-1}K_{2n}$, as desired.

It follows from (10) that for $x, y \in G_{2n-1}$ we have

$$e_{U_{2n-1}} x e_{K_{2n}} = e_{U_{2n-1}} y e_{K_{2n}} \Leftrightarrow xU_{2n-2}K_{2n-2} = yU_{2n-2}K_{2n-2}. \tag{11}$$

Hence, if $x_1 = 1_{2n}, x_2, \ldots, x_t$ are representatives of the left cosets $xU_{2n-2}K_{2n-2}$ of $U_{2n-2}K_{2n-2}$ in $G_{2n-1}(\subset G_{2n})$, then we have

$$e_{U_{2n-1}}\mathbb{C}[G_{2n}]e_{K_{2n}} = \bigoplus_{j=1}^{t} V_j$$

as vector space over $\mathbb{C}$, where

$$V_j = \mathbb{C} \cdot e_{U_{2n-1}} x_j e_{K_{2n}}.$$

Clearly, G_{2n-1} acts on $\{V_j\}_{1\leq j\leq t}$ transitively. Moreover, $U_{2n-2}K_{2n-2}$ is the stabilizer of V_1 in G_{2n-1}, and V_1 affords the trivial representation of $U_{2n-2}K_{2n-2}$. Thus, $e_{U_{2n-1}}\mathbb{C}[G_{2n}]e_{K_{2n}}$ affords the induced representation $(1_{U_{2n-2}K_{2n-2}})^{G_{2n-1}}$, which proves 3.4.1, and hence 3.3.2.

4. Proof of Theorem 1.2.2

In this section, q is assumed to be odd, as in §2. (When q is even, the proof is similar and easier.)

4.1. We prove 1.2.2 (i) by induction on n. If $n = 0$, then this is clear. It follows from the induction hypothesis that

4.1.1. *If $0 \leq m < n$, then we have $(1_{K_{2m}})^{G_{2m}} = \sum \chi_{\mu}$, summed over μ such that $||\mu|| = 2m$, $\tilde{\mu} = \mu$, and $\mu(\varphi_1)' \cup \mu(\varphi_{-1})$ is even.*

Let $(1_{K_{2n}})^{G_{2n}} = \sum_{i=1}^{k} \chi_{\mu_i}$, then from 1.2.1 it follows that $\tilde{\mu}_i = \mu_i$ for all i. Since as mentioned before φ_1 and φ_{-1} are the only elements $\varphi \in \Theta$ such that $d(\varphi) = 1$ and $\tilde{\varphi} = \varphi$, therefore it follows from 3.3.2 that

4.1.2. *If $\nu : \Theta \longrightarrow \mathcal{P}$ satisfies $||\nu|| = 2n - 1$ and $\nu \dashv \mu_i$ for some i, then one of the following holds:*

(a) *$\nu(\varphi_1)' \cup \nu(\varphi_{-1})$ is even and $\tilde{\nu} \neq \nu$,*
(b) *$\nu(\varphi_1)' \cup \nu(\varphi_{-1})$ has exactly one odd part and $\tilde{\nu} = \nu$.*

Moreover,

$$\sum_{i=1}^{k} \sum_{\substack{||\nu||=2n-1 \\ \nu \dashv \mu_i}} \chi_{\nu} \tag{12}$$

is multiplicity-free.

From 4.1.2 we immediately have

4.1.3. *If an irreducible character χ_{μ} of G_{2n} with $\tilde{\mu} = \mu$ is contained in $(1_{K_{2n}})^{G_{2n}}$, then one of the following holds:*

(a) *$\mu(\varphi_1)' \cup \mu(\varphi_{-1})$ is even,*
(b) *$l(\mu(\varphi_1)' \cup \mu(\varphi_{-1})) = 2$.*

Let $\mu_* : \Theta \longrightarrow \mathscr{P}$ be a partition-valued function such that $||\mu_*|| = 2n$, $\tilde{\mu}_* = \mu_*$, $\mu_*(\varphi_1) = (1^{2k})$ and $\mu_*(\varphi_{-1}) = 0$. For two partitions $\lambda, \rho \in \mathscr{P}$ such that $l(\lambda' \cup \rho) \leq 2$ and $|\lambda| + |\rho| = 2k$, we define $\mu_{\lambda,\rho} : \Theta \longrightarrow \mathscr{P}$ by $\mu_{\lambda,\rho}(\varphi_1) = \lambda$, $\mu_{\lambda,\rho}(\varphi_{-1}) = \rho$, and $\mu_{\lambda,\rho}(\varphi) = \mu_*(\varphi)$ for all other $\varphi \in \Theta$. Then it follows that

$$d_{\mu_{0,(2k)}} > d_{\mu_{0,(2k-1,1)}} > d_{\mu_{0,(2k-2,2)}} > \cdots \tag{13}$$

In fact, from (2) it follows that

$$\frac{d_{\mu_{0,(2k)}}}{d_{\mu_{0,(2k-1,1)}}} = q^{2k-1} \cdot \frac{q-1}{q^{2k-1}-1}.$$

Then since

$$q^{2k-1}(q-1) - (q^{2k-1}-1) = q^{2k-1}(q-2) + 1 > 0,$$

we have $d_{\mu_{0,(2k)}} > d_{\mu_{0,(2k-1,1)}}$. Next, for $1 \leq j \leq k-1$ it follows that

$$\frac{d_{\mu_{0,(2k-j,j)}}}{d_{\mu_{0,(2k-j-1,j+1)}}} = q^{2k-2j-1} \cdot \frac{(q^{2k-2j+1}-1)(q^{j+1}-1)}{(q^{2k-j+1}-1)(q^{2k-2j-1}-1)}.$$

Since

$$\begin{aligned} &q^{2k-2j-1}(q^{2k-2j+1}-1)(q^{j+1}-1) - (q^{2k-j+1}-1)(q^{2k-2j-1}-1) \\ &> q^{4k-3j}(q - q^{-j} - 1) - q^{2k-j} - 1 \geq q^{4k-3j} - q^{2k-j} - 1 \\ &= q^{2k-j}(q^{2k-2j}-1) - 1 > 0, \end{aligned}$$

we have $d_{\mu_{0,(2k-j,j)}} > d_{\mu_{0,(2k-j-1,j+1)}}$, as desired.

4.1.4. *Let $\lambda, \rho \in \mathscr{P}$ be as above, and suppose that χ_{μ_*} is contained in $(1_{K_{2n}})^{G_{2n}}$. Then*

(a) *if $\lambda \neq 0$ then $\chi_{\mu_{\lambda,\rho}}$ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if $\lambda' \cup \rho$ is even,*
(b) *if $\lambda = 0$ then exactly one of the following occurs:*
  (b1) *$\chi_{\mu_{0,\rho}}$ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if ρ is even,*
  (b2) *$\chi_{\mu_{0,\rho}}$ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if ρ is odd.*

Proof. For two partitions $\beta, \gamma \in \mathscr{P}$ such that $l(\beta' \cup \gamma) \leq 2$ and $|\beta| + |\gamma| = 2k-1$, we also define $\nu_{\beta,\gamma} : \Theta \longrightarrow \mathscr{P}$ such that $||\nu|| = 2n-1$ by $\nu_{\beta,\gamma}(\varphi_1) = \beta$, $\nu_{\beta,\gamma}(\varphi_{-1}) = \gamma$, and $\nu_{\beta,\gamma}(\varphi) = \mu_*(\varphi)$ for all other $\varphi \in \Theta$. First of all, since $\chi_{\nu_{(1^{2k-1}),0}}$ appears in (12) and $\nu_{(1^{2k-1}),0} \dashv \mu_{(1^{2k}),0}$, therefore neither $\chi_{\mu_{(1^{2k-2},2),0}}$ nor $\chi_{\mu_{(1^{2k-1}),(1)}}$ is contained in $(1_{K_{2n}})^{G_{2n}}$. Next, since $\chi_{\nu_{(1^{2k-3},2),0}}$ appears in (12) by 3.3.2, it follows from 4.1.3 that $\chi_{\mu_{(1^{2k-4},2^2),0}}$ must be contained in $(1_{K_{2n}})^{G_{2n}}$, and so on. □

4.1.5. *Let $1 \leq k \leq n$ and let $\mu_* : \Theta \longrightarrow \mathscr{P}$ be a partition-valued function such that $||\mu_*|| = 2n$, $\tilde{\mu}_* = \mu_*$, $\mu_*(\varphi_1) = (1^{2k})$ and $\mu_*(\varphi_{-1}) = 0$. Then χ_{μ_*} is contained in $(1_{K_{2n}})^{G_{2n}}$.*

Proof. We prove 4.1.5 by induction on k, starting from $k = n$ and ending with 1. When $k = n$, this is trivial. Let $2 \leq k \leq n$ and assume that the assertion is true for all l such that $k \leq l \leq n$. Let $\nu_* : \Theta \longrightarrow \mathscr{P}$ be a partition-valued function such that $||\nu_*|| = 2n-1$, $\nu_*(\varphi_1) = (1^{2k-1})$ and $\nu_*(\varphi_{-1}) = 0$. If the restriction $\chi_\mu \downarrow^{G_{2n}}_{G_{2n-1}}$ of an irreducible constituent χ_μ of $(1_{K_{2n}})^{G_{2n}}$ to G_{2n-1} contains χ_{ν_*}, then by 3.1.2, 4.1.3 and 4.1.4 it follows that $\mu(\varphi_1) = (1^{2k})$ or $\mu(\varphi_1) = (1^{2k-2})$, and $\mu(\varphi_{-1}) = (2j)$ for some $j \geq 0$. Hence, we have

$$(14) \qquad ((1_{K_{2n}})^{G_{2n}} \downarrow_{G_{2n-1}}^{G_{2n}}, \chi_{\nu_*})_{G_{2n-1}} \leq (\sum \chi_\mu \downarrow_{G_{2n-1}}^{G_{2n}}, \chi_{\nu_*})_{G_{2n-1}}$$

where the sum on the right is over μ such that $||\mu|| = 2n$, $\tilde{\mu} = \mu$, $\mu(\varphi_1) = (1^{2k})$ or $\mu(\varphi_1) = (1^{2k-2})$, and $\mu(\varphi_{-1}) = (2j)$ for some $j \geq 0$.

Now, for any $\lambda : \Theta \longrightarrow \mathcal{P}$ such that $\lambda(\varphi_1) = (1^m)$ for some $m \geq 2$, we define $\lambda^- : \Theta \longrightarrow \mathcal{P}$ by $\lambda^-(\varphi_1) = (1^{m-2})$ and $\lambda^-(\varphi) = \lambda(\varphi)$ for all other $\varphi \in \Theta$. Then it follows from 3.1.2 that the right-hand side of (14) is equal to

$$(\sum \chi_{\mu^-} \downarrow_{G_{2n-3}}^{G_{2n-2}}, \chi_{\nu_*^-})_{G_{2n-3}}$$

summed over μ as above, which is also equal to

$$\begin{aligned}((1_{K_{2n-2}})^{G_{2n-2}} \downarrow_{G_{2n-3}}^{G_{2n-2}}, \chi_{\nu_*^-})_{G_{2n-3}} &= q \cdot ((1_{K_{2n-4}})^{G_{2n-4}} \uparrow_{G_{2n-4}}^{G_{2n-3}}, \chi_{\nu_*^-})_{G_{2n-3}} \\ &= q \cdot ((1_{K_{2n-2}})^{G_{2n-2}} \uparrow_{G_{2n-2}}^{G_{2n-1}}, \chi_{\nu_*})_{G_{2n-1}} \\ &= ((1_{K_{2n}})^{G_{2n}} \downarrow_{G_{2n-1}}^{G_{2n}}, \chi_{\nu_*})_{G_{2n-1}}\end{aligned}$$

where the first and the third equalities follow from 3.2.1. Hence, if $\mu_* : \Theta \longrightarrow \mathcal{P}$ satisfies $||\mu_*|| = 2n$, $\tilde{\mu}_* = \mu_*$, $\mu_*(\varphi_1) = (1^{2k-2})$ and $\mu_*(\varphi_{-1}) = 0$, then since $(\chi_{\mu_*} \downarrow_{G_{2n-1}}^{G_{2n}}, \chi_{\nu_*})_{G_{2n-1}} > 0$ for at least one such ν_* as above, therefore χ_{μ_*} must be contained in $(1_{K_{2n}})^{G_{2n}}$. □

The proof of 1.2.2 (i) can now be rapidly completed. Let $\mu : \Theta \longrightarrow \mathcal{P}$ be a partition-valued function such that $||\mu|| = 2n$ and $\tilde{\mu} = \mu$. Then 4.1.5 and 4.1.4 imply that if $\mu(\varphi_1) \neq 0$ or $l(\mu(\varphi_1)' \cup \mu(\varphi_{-1})) \geq 3$ then χ_μ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if $\mu(\varphi_1)' \cup \mu(\varphi_{-1})$ is even. Also, if $\mu(\varphi_1) = 0$ and $l(\mu(\varphi_{-1})) \leq 2$ then there are two posibilities. However, by virtue of 2.1.1 and (13), we can conclude that in this case χ_μ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if $\mu(\varphi_{-1})$ is even. It also follows from 2.1.1 that $(1_{K_{2n}})^{G_{2n}}$ contains all irreducible characters χ_μ of G_{2n} such that $\tilde{\mu} = \mu$ and $\mu(\varphi_1) = \mu(\varphi_{-1}) = 0$.

4.2. Finally, we prove 1.2.2 (iii). The left-hand side of (3) is by 2.1.2 equal to

$$\begin{aligned}&\prod_{r\geq 1}(1-t^{2r})^{-2} \cdot \prod_{r\geq 1}(1-t^{2r})^{-(|\Psi_1|-2)} \cdot \prod_{k\geq 2}\prod_{r\geq 1}(1-t^{2kr})^{-|\Psi_k|} \\ &= \prod_{k\geq 1}\prod_{r\geq 1}(1-t^{2kr})^{-|\Psi_k|} = \prod_{r\geq 1}(1-qt^{2r})^{-1}.\end{aligned}$$

This completes the proof of 1.2.2.

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