THE DECOMPOSITION OF THE PERMUTATION CHARACTER
\[ \chi_{GGL(2n,q)}^{GL(n,q^2)} \]

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INTRODUCTION

Let \( G \) be a finite group acting transitively on a finite set \( X \), and let \( H = G_x \) be the stabilizer of a point \( x \) in \( X \). The permutation character \( \pi \) of \( G \) on \( X \) is equivalent to the induced character \((1_H)^G\) of the identity character \( 1_H \) of \( H \). We say that the permutation character \( \pi = (1_H)^G \) is multiplicity-free if it is decomposed into a direct sum of inequivalent irreducible characters. In this case, the centralizer algebra (or the Hecke algebra) of the permutation group is commutative, and we also say that \( H \) is a multiplicity-free subgroup of \( G \). A pair \((G, H)\) of a finite group \( G \) and a multiplicity-free subgroup \( H \) is sometimes called a Gelfand pair. A commutative association scheme \( \mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d}) \) is associated with a multiplicity-free transitive action of a finite group \( G \) on a finite set \( X \), by taking the relations \( R_0, R_1, \ldots, R_d \) as the orbits of \( G \) on \( X \times X \). It is an interesting question to know many examples of commutative association schemes and their character tables. (The reader is referred to Bannai-Ito [4], Bannai [1] for the basic concept of commutative association schemes and their character tables.) It should be noted that knowing the character table of a commutative association scheme (associated to a multiplicity-free transitive action of a finite group, i.e., to a Gelfand pair) is equivalent to knowing the zonal spherical functions of the permutation group.

Many examples of Gelfand pairs or commutative association schemes are known (see, e.g. Saxl [16], Inglis [9], Bannai [1], Bannai-Hao-Song [2], Bannai-Hao-Song-Wei [3], Bannai-Kawanaka-Song [5], Lusztig [14], Lawther [13], etc.). In Inglis-Liebeck-Saxl [10], it is stated that the following pairs \((G, H)\) are Gelfand pairs:

(i) \((G, H) = (GL(n, q^2), GL(n, q))\),
(ii) \((G, H) = (GL(n, q^2), GU(n, q))\),
(iii) \((G, H) = (GL(2n, q), Sp(2n, q))\),
(iv) \((G, H) = (GL(2n, q), GL(n, q^2))\).

It seems that the structure of the double cosets \( H \backslash G / H \), the decomposition of the permutation character \( \chi = 1^G_H \), and the character table of the associated commutative association scheme are known for the first three cases (Gow [7], Klyachko [12], Bannai-Kawanaka-Song [5], Kawanaka [11], Bannai [1], Lusztig [14]). However, it seems that they are not yet known for the last case (iv) of \( G = GL(2n, q) \) and \( H = GL(n, q^2) \). The decomposition of the permutation character \( 1^{GL(2n,q)}_{GL(n,q^2)} \) is well-known for \( n = 1 \) (cf. Terras [19, Chapter 21]). When \( n = 2 \), it was determined by the second author [18] by explicitly calculating the inner product \( \langle \chi, 1^{GL(4,q^2)}_{GL(2,q^2)} \rangle \) for all irreducible characters \( \chi \) of \( GL(4, q) \). Our purpose in this paper is to determine the decomposition of \( 1^{GL(2n,q)}_{GL(n,q^2)} \) for general \( n \).
1. Preliminaries on General Linear Groups and Main Results

1.1. First of all, we briefly recall a parametrization of the irreducible characters of the general linear group $G_n = GL(n, q)$, following Macdonald [15, Chapter IV.]. Whenever possible, we use the notation of [15].

A partition is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers $\lambda_i$ containing finitely many non-zero terms. The non-zero $\lambda_i$ are called the parts of $\lambda$. We identify $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots, 0)$. Sometimes we write $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ in place of $\lambda = (\lambda_1, \lambda_2, \ldots)$, where $m_i$ is the number of $j$ such that $\lambda_j = i$. The only partition with no non-zero terms is denoted by $0$. For each partition $\lambda$, the length $l(\lambda)$ of $\lambda$ is the number of parts of $\lambda$, and the weight $|\lambda|$ of $\lambda$ is defined by $|\lambda| = \sum_{i \geq 1} \lambda_i$. We denote the set of all partitions by $\mathcal{P}$. The diagram of $\lambda \in \mathcal{P}$ is the set of points $x = (i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$, and the conjugate $\lambda'$ of $\lambda$ is the partition whose diagram is the transpose of that of $\lambda$. For example, the conjugate of (2, 2, 1) is (3, 2). The hook-length $h(x)$ of $\lambda$ at $x = (i, j) \in \lambda$ (i.e., $1 \leq j \leq \lambda_i$) is defined by $h(x) = \lambda_i + \lambda_j' - i - j + 1$. For $\lambda, \mu \in \mathcal{P}$, we define $\lambda \cup \mu$ to be the partition whose parts are those of $\lambda$ and $\mu$, arranged in descending order. An even (resp. odd) partition is a partition with all parts even (resp. odd). We let $s_\lambda$ denote the Schur function (in countably many independent variables) corresponding to $\lambda \in \mathcal{P}$.

Let $F_q$ be a finite field with $q$ elements, and $\overline{F}_q$ the algebraic closure of $F_q$. For each positive integer $l$ there exists a unique extension $F_{q^l}$ of $F_q$ in $\overline{F}_q$ of degree $l$. We denote the multiplicative group of $F_{q^l}$ by $M_{q^l}$, and the character group of $M_{q^l}$ by $\hat{M}_{q^l}$. If $l$ divides $m$ then $\hat{M}_{q^l}$ is embedded in $\hat{M}_{q^m}$ by the transpose of the norm map $N_{m, q^l} : M_{q^m} \rightarrow M_{q^l}$. We let $L = \lim_{\rightarrow} \hat{M}_{q^l}$ be the inductive limit of the $\hat{M}_{q^l}$. The Frobenius map $F : \gamma \mapsto \gamma^q$ acts on $L$, and $\hat{M}_{q^l}$ is the set of all $F^l$-fixed elements in $L$. We denote the set of $F$-orbits in $L$ by $\Theta$. Then the irreducible characters of $G_n$ can be parametrized by the partition-valued functions $\mu : \Theta \rightarrow \mathcal{P}$ such that

\begin{equation}
||\mu|| = \sum_{\varphi \in \Theta} d(\varphi)|\mu(\varphi)| = n
\end{equation}

where $d(\varphi)$ is the number of elements of $\varphi$. The irreducible character of $G_n$ corresponding to $\mu$ is denoted by $\chi_{\mu}$. The degree $d_\mu$ of $\chi_{\mu}$ is given by

\begin{equation}
d_\mu = \psi_n(q) \prod_{\varphi \in \Theta} s_{\mu(\varphi)}(q_{\varphi}^{-1}, q_{\varphi}^{-2}, \ldots)
= \psi_n(q) \prod_{\varphi \in \Theta} q_{\varphi}^{n(\mu(\varphi)')} \tilde{H}_{\mu(\varphi)}(q_{\varphi})^{-1}
\end{equation}

where $q_{\varphi} = q^{d(\varphi)}$,

\begin{align*}
\psi_n(q) &= \prod_{i=1}^{n} (q^i - 1), \\
n(\lambda) &= \sum_{i \geq 1} (i - 1) \lambda_i,
\end{align*}

and

\begin{align*}
\tilde{H}_\lambda(q_{\varphi}) &= \prod_{x \in \lambda} (q_{\varphi}^{h(x)} - 1) \quad (x \in \lambda)
\end{align*}
for \( \lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P} \).

Let \( \xi_1 \) be the identity character of \( M_1 \), and if \( q \) is odd then let \( \xi_{-1} \) be the quadratic character of \( M_1 \). We put \( \varphi_1 = \{ \xi_1 \}, \varphi_{-1} = \{ \xi_{-1} \} \in \Theta \). For \( \varphi = \{ \xi, \xi^q, \ldots, \xi^{q^{d-1}} \} \in \Theta \), the reciprocal \( F \)-orbit \( \tilde{\varphi} \) of \( \varphi \) is defined by

\[
\tilde{\varphi} = \{ \xi^{-1}, \xi^{-q}, \ldots, \xi^{-q^{d-1}} \}.
\]

Notice that \( \varphi_1 \) and \( \varphi_{-1} \) are the only elements \( \varphi \in \Theta \) such that \( d(\varphi) = 1 \) and \( \tilde{\varphi} = \varphi \). Also for each partition-valued function \( \mu : \Theta \rightarrow \mathcal{P} \), we define \( \tilde{\mu} : \Theta \rightarrow \mathcal{P} \) by

\[
\tilde{\mu}(\varphi) = \mu(\tilde{\varphi})
\]
for all \( \varphi \in \Theta \). Then we can easily verify that the complex conjugate \( \overline{\chi_{\mu}} \) of \( \chi_{\mu} \) is given by \( \chi_{\tilde{\mu}} \) (see for example (4.5) in [15, Chapter IV.]), from which it follows that

1.1.1. An irreducible character \( \chi_{\mu} \) of \( G_n \) is real-valued if and only if \( \tilde{\mu} = \mu \).

1.2. We now present our main results. Let \( K_{2n} \) be a subgroup of \( G_{2n} \) isomorphic to \( GL(n, q^2) \). It is known that

1.2.1. Theorem (Inglis-Liebeck-Saxl [10]). The permutation character \( (1_{K_{2n}})^{G_{2n}} \) is multiplicity-free and every irreducible constituent of \( (1_{K_{2n}})^{G_{2n}} \) is real-valued.

In this paper, we determine the decomposition of the permutation character \((1_{K_{2n}})^{G_{2n}}\) explicitly. More precisely, we will prove the following:

1.2.2. Theorem. (i) If \( q \) is odd, then we have \((1_{K_{2n}})^{G_{2n}} = \sum \chi_{\mu}, \) summed over \( \mu \) such that \( ||\mu|| = 2n, \mu = \mu, \) and both \( \mu(\varphi_1)' \) and \( \mu(\varphi_{-1})' \) are even.

(ii) If \( q \) is even, then we have \((1_{K_{2n}})^{G_{2n}} = \sum \chi_{\mu}, \) summed over \( \mu \) such that \( ||\mu|| = 2n, \mu = \mu, \) and \( \mu(\varphi_1)' \) is even.

(iii) In either case, the generating function for the rank (i.e., the number of the irreducible constituents of the permutation character \((1_{K_{2n}})^{G_{2n}}\)) is given by

\[
\sum_{n \geq 0} \mathrm{rank}(G_{2n}/K_{2n}) t^{2n} = \prod_{r \geq 1} (1 - qt^{2r})^{-1}
\]

with the understanding that \( \mathrm{rank}(G_0/K_0) = 1 \). In particular we have

\[
\mathrm{rank}(G_{2n}/K_{2n}) = \sum q^{(\lambda)}
\]

summed over all partitions \( \lambda \) such that \( |\lambda| = n \).

1.2.3. Remark. In the notation of Green [8], our character \( \chi_{\mu} \) corresponds to the conjugate function \( \mu' : \Theta \rightarrow \mathcal{P} \) defined by \( \mu'(\varphi) = \mu(\varphi)' \) for all \( \varphi \in \Theta \). In particular, in our notation the identity character of \( G_n \) assigns the partition \((1^n)\) to \( \varphi_1 \). See Springer-Zelevinsky [17, Remark 1.9.].

1.2.4. Remark. Let \( \pi(G_n) \) denote the number of the conjugacy classes of \( G_n \), then the generating function for the \( \pi(G_n) \) is given by

\[
\sum_{n \geq 0} \pi(G_n) t^n = \prod_{r \geq 1} (1 - t^r)(1 - qt^r)^{-1}.
\]

Hence 1.2.2 (iii) implies that

\[
\mathrm{rank}(G_{2n}/K_{2n}) = \sum_{i=0}^{n} p(i) \pi(G_{n-i})
\]
where $p(i)$ is the number of partitions $\lambda$ such that $|\lambda| = i$. It is a reasonable guess that there is a natural set of representatives of the double cosets $K_{2n} \backslash G_{2n} / K_{2n}$ which reflects the above equality.

2. DEGREE FORMULA

2.1. The starting point of the proof of 1.2.2 is the following proposition:

2.1.1. Proposition. (i) If $q$ is odd, then we have

$$\sum d_\mu = (q^{2n} - q)(q^{2n} - q^3) \cdots (q^{2n} - q^{2n-1})$$

where the sum on the left is over $\mu$ such that $||\mu|| = 2n$, $\bar{\mu} = \mu$, and both $\mu(\varphi_1)'$ and $\mu(\varphi_{-1})$ are even.

(ii) If $q$ is even, then we have

$$\sum d_\mu = (q^{2n} - q)(q^{2n} - q^3) \cdots (q^{2n} - q^{2n-1})$$

where the sum on the left is over $\mu$ such that $||\mu|| = 2n$, $\bar{\mu} = \mu$, and $\mu(\varphi_1)'$ is even.

To prove 2.1.1, we need some preparations. In what follows, we assume that $q$ is odd. (The assertion (ii) is proved in exactly the same way as (i).)

Let $\Phi$ denote the set of monic irreducible polynomials $f(t)$ over $\mathbb{F}_q$ with $f(t) \neq t$. We identify $\Phi$ with the set of $F$-orbits in the multiplicative group $M$ of the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$, by assigning to each $f$ the $F$-orbit consisting of its roots in $M$.

Let $f(t) = t^k + a_1 t^{k-1} + \cdots + a_k$ be a monic polynomial in $\mathbb{F}_q[t]$ of degree $k$ with $a_k \neq 0$. The reciprocal polynomial $\tilde{f}$ of $f$ is defined by

$$\tilde{f}(t) = a_k^{-1} t^k f(t^{-1}) = t^k + \frac{a_{k-1}}{a_k} t^{k-1} + \cdots + \frac{1}{a_k}.$$

We call the polynomial $f$ self-reciprocal if $f(t) = \tilde{f}(t)$.

Let $\Psi = \Phi \cup \{t\}$: the set of all monic irreducible polynomials in $\mathbb{F}_q[t]$, $S = \{f \in \Phi \backslash \{t \pm 1\} \mid f : \text{self-reciprocal}\}$, $N = \{f \in \Phi \backslash \{t \pm 1\} \mid f : \text{non-self-reciprocal}\}$, and let

$$\Psi_k = \{f \in \Psi \mid \deg f = k\},$$

$$S_k = \{f \in S \mid \deg f = k\},$$

$$N_k = \{f \in N \mid \deg f = k\}$$

for $k \geq 1$. Notice that $S_k$ is empty unless $k$ is even.

First we observe the following two one-to-one correspondences due to Carlitz [6]:

2.1.2 ([6, §3.]). We have

$$\Psi_k \leftrightarrow_{1:1} S_{2k} \cup \{g\bar{g} \mid g \in N_k\}$$

for $k \geq 2$, and

$$\Psi_1 \backslash \{t \pm 2\} \leftrightarrow_{1:1} S_2 \cup \{g\bar{g} \mid g \in N_1\}.$$
Proof. Let \( h(t) \in \mathbb{F}_q[t] \) be a monic irreducible polynomial of degree \( k \) \((k \geq 1)\) such that \( h(t) \neq t \pm 2 \), then \( h(t) \) is decomposed into linear factors in \( \mathbb{F}_q[t] \) as \( h(t) = (t - \beta)(t - \beta^q) \ldots (t - \beta^{q^{k-1}}) \). Let \( \alpha \in \mathbb{F}_{q^2k} \) be a root of the polynomial \( t^2 - \beta t + 1 \), i.e., \( \alpha + \alpha^{-1} = \beta \). Since \( \beta \neq \pm 2 \) it follows that \( \alpha \neq \alpha^{-1} \), so that

\[
\alpha, \alpha^q, \ldots, \alpha^{q^{k-1}}, \alpha^{-1}, \alpha^{-q}, \ldots, \alpha^{-q^{k-1}}
\]

are distinct. We define

\[
f(t) = t^k h(t + t^{-1})
\]

\[
= (t - \alpha)(t - \alpha^q) \ldots (t - \alpha^{q^{k-1}})(t - \alpha^{-1})(t - \alpha^{-q}) \ldots (t - \alpha^{-q^{k-1}}),
\]

then \( f(t) \) is a monic polynomial of degree \( 2k \). Now, if \( \alpha \in \mathbb{F}_{q^2k} \setminus \mathbb{F}_q \) then we have \( f(t) \in S_{2k} \) since \( \alpha^{-1} = \alpha^q \), and if \( \alpha \in \mathbb{F}_q \) then we have \( f(t) = g(t)\tilde{g}(t) \) where

\[
g(t) = (t - \alpha)(t - \alpha^q) \ldots (t - \alpha^{q^{k-1}}) \in N_k,
\]

as desired. \( \square \)

Let \( \sigma_{2k} = |S_{2k}| \) and \( \tau_{2k} = |\{ g\tilde{g} \mid g \in N_k \}| = \frac{1}{2}|N_k| \) for \( k \geq 1 \). Then it follows from 2.1.2 that

\[
\sum_{k|N} k(\sigma_{2k} + \tau_{2k}) + 2 = q^N
\]

for \( N \geq 1 \). If \( N = 2M \) is even then we also have

\[
\sum_{k|M}(2k)\sigma_{2k} + \sum_{k|2M} k(2\tau_{2k}) + 2 = q^N - 1.
\]

On the other hand, if \( N \) is odd then we have

\[
\sum_{k|N} k(2\tau_{2k}) + 2 = q^N - 1.
\]

Let \( x = (x_1, x_2, \ldots) \) be an infinite sequence of independent variables. We shall need the following four equalities:

2.1.3 (cf. [15, p.63, (4.3)]). \[
\sum_{\lambda} s_\lambda^2 = \prod_i (1-x_i^2)^{-1} \prod_{i<j} (1-x_ix_j)^{-2}, \text{ where the sum on the left is over all partitions } \lambda.
\]

2.1.4 (cf. [15, p.76, Example 4]). \[
\sum_{\lambda} s_\lambda = \prod_i (1-x_i)^{-1} \prod_{i<j} (1-x_ix_j)^{-1}, \text{ where the sum on the left is over all partitions } \lambda.
\]

2.1.5 (cf. [15, p.77, Example 5(a)]). \[
\sum_{\mu \text{ even}} s_\mu = \prod_i (1-x_i^2)^{-1} \prod_{i<j} (1-x_ix_j)^{-1}, \text{ where the sum on the left is over all even partitions } \mu.
\]

2.1.6 (cf. [15, p.77, Example 5(b)]). \[
\sum_{\nu' \text{ even}} s_\nu = \prod_i (1-x_ix_j)^{-1}, \text{ where the sum on the left is over all partitions } \nu \text{ with } \nu' \text{ even}.
\]
2.2. Proof of 2.1.1. Our proof of 2.1.1 is inspired by [15, p.289, Example 5 of all, notice that the number of elements $\varphi \in \Theta$ such that $d(\varphi) = 2k$ and $\tilde{\varphi}$ equal to $\sigma_{2k}$. We shall compute the following:

\[
D = \sum_{\nu' \text{ even}} s_{\nu}(q^{-1}, q^{-2}, \ldots) t^{\nu} \times \sum_{\mu \text{ even}} s_{\mu}(q^{-1}, q^{-2}, \ldots) t^{\mu} \times
\prod_{k \geq 1} \left\{ \sum_{\lambda} s_{\lambda}(q^{-2k}, q^{-4k}, \ldots) t^{2k|\lambda|} \right\}^{\sigma_{2k}}
\times \prod_{k \geq 1} \left\{ \sum_{\lambda} s_{\lambda}^{2}(q^{-k}, q^{-2k}, \ldots) t^{2k|\lambda|} \right\}^{\tau_{2k}}
\]

\[
= \prod_{i<j} (1 - (t^{2}q^{-i-j})^{-1}) \times \prod_{i} (1 - (tq^{-i})^{-2}) \times \prod_{i<j} (1 - (t^{2}q^{-i-j})^{-1})^{-1}
\times \prod_{k \geq 1} \left\{ \prod_{i} (1 - (tq^{-i})^{-2k}) \prod_{i<j} (1 - (t^{2}q^{-i-j})^{-2k}) \right\}^{\sigma_{2k}}
\times \prod_{k \geq 1} \left\{ \prod_{i} (1 - (tq^{-i})^{-2k}) \prod_{i<j} (1 - (t^{2}q^{-i-j})^{-k}) \right\}^{\tau_{2k}}
\]

where $t$ is an indeterminate.

Let

\[
X_{1} = \log \prod_{k \geq 1} \left\{ \prod_{i \geq 1} (1 - (tq^{-i})^{-2k}) \right\}^{\sigma_{2k}}
\]

\[
Y_{1} = \log \prod_{k \geq 1} \left\{ \prod_{i \geq 1} (1 - (tq^{-i})^{-2k}) \right\}^{\tau_{2k}}
\]

\[
Z_{1} = \log \prod_{i \geq 1} (1 - (tq^{-i})^{-2})^{-1}
\]

Then we have

\[
X_{1} = \sum_{k \geq 1} \sigma_{2k} \sum_{i \geq 1} \sum_{r \geq 1} \frac{(tq^{-i})^{2kr}}{r} = \sum_{k \geq 1} \sigma_{2k} \sum_{r \geq 1} \frac{t^{2kr}}{r} \cdot \frac{1}{q^{2kr} - 1}
\]

\[
= \sum_{N \geq 1} \frac{t^{2N}}{N(q^{2N} - 1)} \sum_{k|N} k \sigma_{2k}.
\]

Similarly, we have

\[
Y_{1} = \sum_{N \geq 1} \frac{t^{2N}}{N(q^{2N} - 1)} \sum_{k|N} k \tau_{2k}
\]

and

\[
Z_{1} = \sum_{N \geq 1} \frac{t^{2N}}{N(q^{2N} - 1)}
\]
Therefore, it follows from (4) that

$$X_1 + Y_1 + Z_1 = \sum_{N \geq 1} \frac{t^{2N}}{N(q^{2N}-1)}(q^N - 1) = \sum_{N \geq 1} \frac{t^{2N}}{N(q^N+1)}$$

$$= \sum_{N \geq 1} \frac{t^{2N}}{N} \sum_{k \geq 1} (-1)^{k-1} q^{-kN} = \sum_{k \geq 1} (-1)^{k-1} \sum_{N \geq 1} \frac{(t^2q^{-k})^N}{N}$$

Let

$$X_2 = \log \prod_{k \geq 1} \left\{ \prod_{i < j} (1 - (t^2q^{-i-j})^{2k})^{-1} \right\}^{\sigma_{2k}},$$

$$Y_2 = \log \prod_{k \geq 1} \left\{ \prod_{i < j} (1 - (t^2q^{-i-j})^{k})^{-2} \right\}^{\tau_{2k}},$$

$$Z_2 = \log \prod_{i < j} (1 - t^{2}q^{-i-j})^{-2}.$$}

Then we have

$$X_2 = \sum_{k \geq 1} \sigma_{2k} \sum_{i < j} \sum_{r \geq 1} \frac{(t^2q^{-i-j})^{2kr}}{r} = \sum_{k \geq 1} \sigma_{2k} \sum_{r \geq 1} \frac{t^{4kr}}{r} \sum_{i \geq 1} \frac{q^{-4kr}}{q^{2kr}-1}.$$

Similarly, we have

$$Y_2 = \sum_{i \geq 1} \sum_{N \geq 1} \frac{t^{2N}}{N(q^N-1)} \left( \sum_{k|N} k(2\tau_{2k}) \right) q^{-2iN}$$

and

$$Z_2 = \sum_{i \geq 1} \sum_{N \geq 1} \frac{t^{2N}}{N(q^N-1)} 2q^{-2iN}.$$

Therefore, it follows from (5) and (6) that

$$X_2 + Y_2 + Z_2 = \sum_{i \geq 1} \sum_{N \geq 1} \frac{t^{2N}}{N(q^N-1)} (q^N - 1)q^{-2iN}$$

$$= \sum_{i \geq 1} \sum_{N \geq 1} \frac{(t^2q^{-2i})^N}{N}.$$

Hence from (7) and (8) we obtain

$$\log D = X_1 + Y_1 + Z_1 + X_2 + Y_2 + Z_2$$

$$= \sum_{i \geq 1} \sum_{N \geq 1} \frac{(t^2q^{-2i+1})^N}{N}$$

$$= \log \prod_{i \geq 1} (1 - t^2q^{-2i+1})^{-1}$$

so that

$$D = \prod_{i \geq 1} (1 - t^2q^{-2i+1})^{-1} = \sum_{m \geq 0} t^{2m}q^{-m}/\varphi_m(q^{-2})$$
where $\varphi_m(t) = (1 - t)(1 - t^2) \ldots (1 - t^m)$.

Finally, on picking out the coefficient of $t^{2n}$, and multiplying by $\psi_{2n}(q)$, we get the desired result.

3. Branching Lemmas

In this section, we prepare two lemmas which enable us to prove 1.2.2 by induction on $n$. We do not need to assume in this section that $q$ is odd.

3.1. First, we recall a result of Zelevinsky [21]. Let $n \geq 2$ and let $H_n$ be the subgroup of $G_n$ consisting of the matrices of the form

$$g = \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix}$$

where $x \in G_{n-1}$. Let $U_{n-1}$ be the abelian normal subgroup of $H_n$ defined by

$$U_{n-1} = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1_{n-1} \end{pmatrix} \right\} \cong \mathbb{F}_q^{n-1}$$

where $1_{n-1}$ is the identity matrix of degree $n - 1$. We identify $G_{n-1}$ with the following subgroup of $H_n$:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \mid x \in G_{n-1} \right\}$$

then we have $H_n = U_{n-1} \rtimes G_{n-1}$, the semidirect product of $U_{n-1}$ with $G_{n-1}$. The irreducible characters of $H_n$ are determined by applying the method of little groups, and they are parametrized by the partition-valued functions $\nu : \Theta \rightarrow \mathcal{P}$ such that $||\nu|| < n$ (cf. [21, §13.]). The irreducible character of $H_n$ corresponding to $\nu$ is denoted by $\zeta^{(n)}_{\nu}$ . Notice that the irreducible characters $\zeta^{(n)}_{\nu}$ of $H_n$ with $||\nu|| = n - 1$ are exactly those obtained by the irreducible characters $\chi_{\nu}$ of $G_{n-1} \cong H_n/U_{n-1}$, that is, they are constant on $U_{n-1}$.

If $\mu : \Theta \rightarrow \mathcal{P}$ and $\nu : \Theta \rightarrow \mathcal{P}$ are two partition-valued functions, we shall write $\nu \ominus \mu$ if $\mu(\varphi)'_i - 1 \leq \nu(\varphi)'_i \leq \mu(\varphi)'_i$ for all $\varphi \in \Theta$ and $i \geq 1$ (i.e., the skew diagram $\mu(\varphi) - \nu(\varphi)$ is a horizontal strip for any $\varphi \in \Theta$).

3.1.1. Theorem ([21, §13.5.]). (i) Let $\mu : \Theta \rightarrow \mathcal{P}$ be a partition-valued function such that $||\mu|| = n$. Then we have

$$\chi_{\mu} \downarrow^{G_n}_{H_n} = \sum \zeta^{(n)}_{\nu}$$

summed over $\nu$ such that $||\nu|| < n$ and $\nu \ominus \mu$.

(ii) Let $\nu : \Theta \rightarrow \mathcal{P}$ be a partition-valued function such that $||\nu|| < n$. Then we have

$$\zeta^{(n)}_{\nu} \downarrow^{G_n}_{G_{n-1}} = \sum \chi_{\lambda}$$

summed over $\lambda$ such that $||\lambda|| = n - 1$ and $\nu \ominus \lambda$.

The following theorem was first proved by Thoma [20], and is easily derived from 3.1.1.

3.1.2. Theorem ([20]). Let $\mu : \Theta \rightarrow \mathcal{P}$ and $\lambda : \Theta \rightarrow \mathcal{P}$ be partition-valued functions such that $||\mu|| = n$ and $||\lambda|| = n - 1$. Then the multiplicity of $\chi_{\mu}$ in the induced character $\chi_{\lambda} \uparrow_{G_{n-1}}^{G_n}$ is equal to the number of $\nu : \Theta \rightarrow \mathcal{P}$ such that $\nu \ominus \mu$ and $\nu \ominus \lambda$. 
3.2. Let $V_{2n}$ be the vector space of column $2n$-vectors with components in $\mathbb{F}_q$, and let $\{v_1, v_2, \ldots, v_{2n}\}$ be the standard basis of $V_{2n}$, that is, $v_i$ is the vector with 1 in the $i$-th component and zeros elsewhere. We fix an element $\alpha \in \mathbb{F}_q^2$ such that $\alpha \in \mathbb{F}_q$, and denote by $f(t) = t^2 + at + b \in \mathbb{F}_q[t]$ the minimal polynomial of $\alpha$ over $\mathbb{F}_q$. Let $g_0$ be an element in $G_{2n}$ such that $g_0^2 + ag_0 + b1_{2n} = 0$. Then $g_0$ determines a vector space over $\mathbb{F}_q^2$ on $V_{2n}$, of dimension $n$, such that $\alpha v = g_0 v$ for $v \in V_{2n}$. The centralizer $K_{2n} = C_{G_{2n}}(g_0)$ of $g_0$ in $G_{2n}$ is isomorphic to $GL(n, q^2)$.

Let $U$ be the subspace of $V_{2n}$ over $\mathbb{F}_q$ spanned by $v_2, v_3, \ldots, v_{2n}$. Clearly, an element $g \in G_{2n}$ belongs to $G_{2n-1}$ if and only if $gU = U$ and $gv_1 = v_1$. The subspace $U$ contains a subspace $W$ of $V_{2n}$ over $\mathbb{F}_q^2$ of dimension $n - 1$ (over $\mathbb{F}_q^2$), defined by

$$W = \{u \in U \mid g_0 u \in U\}.$$ 

It is easily seen that

$$G_{2n-1} \cap K_{2n} = \{k \in K_{2n} \mid kW = W, kv_1 = v_1\},$$

that is, $G_{2n-1} \cap K_{2n}$ is isomorphic to $GL(n-1, q^2)$.

Now for any $x \in G_{2n}$ we have

$$|G_{2n-1}xK_{2n}| = \frac{|G_{2n-1}||K_{2n}|}{|G_{2n-1} \cap xK_{2n}x^{-1}|} = \frac{|G_{2n-1}||K_{2n}|}{|GL(n-1, q^2)|} = \frac{1}{q}|G_{2n}|$$

since $xK_{2n}x^{-1} = C_{G_{2n}}(xg_0x^{-1}) \cong GL(n, q^2)$ and $g_0$ is chosen arbitrarily. Hence it follows from Mackey's theorem that

3.2.1. **Lemma.** $(1_{K_{2n}})^{G_{2n}} \downarrow_{G_{2n-1}}^{G_{2n}} = q \cdot (1_{K_{2n-2}})^{G_{2n-2}} \uparrow_{G_{2n-2}}^{G_{2n-1}}$.

3.3. For the sake of simplicity, in what follows we assume that $g_0$ is of the form

$$g_0 = \begin{pmatrix} \tilde{g}_0 & 0 & \ldots & 0 \\ 0 & \tilde{g}_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \tilde{g}_0 \end{pmatrix}$$

where $\tilde{g}_0 = (\begin{smallmatrix} 0 & b \\ 1 & -a \end{smallmatrix})$, so that $v_{2i} = \alpha v_{2i-1}$ ($1 \leq i \leq n$). Then it follows that

3.3.1. For $g = (g_{ij}) \in G_{2n}$, $g$ is contained in $K_{2n}$ if and only if

$$g_{2k-1,2l-1} = ag_{2k,2l-1} + g_{2k,2l}$$

and

$$g_{2k-1,2l} = -bg_{2k,2l-1}$$

for $1 \leq k, l \leq n$.

We identify the subgroup $H_{2n-1}$ of $G_{2n-1}$ with

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & x \end{pmatrix} \mid x \in G_{2n-2} \right\},$$
and so on. Clearly, the subgroup $K_{2n-2} = G_{2n-2} \cap K_{2n}$ of $G_{2n-2}$ is isomorphic to $GL(n-1, q^2)$.

3.3.2. Lemma. Let $(1_{K_{2n}})^{G_{2n}} = \sum_{i=1}^{k} \chi_{\mu_{i}}$ and $(1_{K_{2n-2}})^{G_{2n-2}} = \sum_{j=1}^{l} \chi_{\lambda_{j}}$. Then we have

$$\sum_{i=1}^{k} \sum_{\nu \vdash \mu_{i}} \chi_{\nu} = \sum_{j=1}^{l} \sum_{\lambda_{j} \vdash \nu} \chi_{\nu}.$$ 

3.4. Proof of 3.3.2. First of all, notice that an element $g$ in $G_{2n}$ belongs to $H_{2n}$ if and only if $gv_1 = v_1$. Hence we have

$$H_{2n} \cap K_{2n} \cong \mathbb{F}_{q^{2}}^{n-1} \times GL(n-1, q^{2})$$ 

from which it follows that $|H_{2n}K_{2n}| = |G_{2n}|$, that is,

$$G_{2n} = H_{2n}K_{2n} = U_{2n-1}G_{2n-1}K_{2n}.$$ 

Let $\mathbb{C}[G_{2n}]$ be the complex group algebra of $G_{2n}$. For any subgroup $K$ of $G_{2n}$, we define

$$e_{K} = \frac{1}{|K|} \sum_{k \in K} k,$$

then $e_{K}^{2} = e_{K}$ and the left $\mathbb{C}[G_{2n}]$-module $\mathbb{C}[G_{2n}]e_{K}$ affords the induced representation $(1_{K})^{G_{2n}}$.

By virtue of 3.1.1 (i), in order to prove 3.3.2 it is enough to show that

3.4.1. The left $\mathbb{C}[G_{2n-1}]$-module $e_{U_{2n-1}} \mathbb{C}[G_{2n}]e_{K_{2n}}$ affords the induced representation $(1_{U_{2n-2}K_{2n-2}})^{G_{2n-1}} = (1_{U_{2n-2}K_{2n-2}})^{H_{2n-1}} \uparrow_{H_{2n-1}}^{G_{2n-1}}$.

From (9) it follows that $e_{U_{2n-1}} \mathbb{C}[G_{2n}]e_{K_{2n}}$ is generated (as vector space) by the elements $e_{U_{2n-1}}xe_{K_{2n}}, x \in G_{2n-1}$. Moreover, we have

$$e_{U_{2n-1}} \mathbb{C}[G_{2n}]e_{K_{2n}} \subset (U_{2n-1}K_{2n}) \cap G_{2n-1} = U_{2n-2}K_{2n-2}.$$ 

In fact, if $x \in G_{2n-1}$ is written as $x = uk$ for some $u \in U_{2n-1}$ and $k \in K_{2n}$, then $k$ is contained in $H_{2n} \cap K_{2n}$. Since $v_1$ is fixed by $k$, so is $v_2$. That is, $k$ is of the form

$$k = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & w \\ 0 & 0 & k_0 \end{pmatrix}$$

where $k_0 \in K_{2n-2}$, from which it follows that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & w \\ 0 & 0 & k_0 \end{pmatrix} \in U_{2n-2}K_{2n-2}.$$ 

Conversely, if $x$ is written as above, then by 3.3.1 there exists $z = (z_1, z_2, \ldots, z_{2n-2})$ such that

$$x = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & w \\ 0 & 0 & k_0 \end{pmatrix} \in K_{2n}$$

and therefore we have $x \in U_{2n-1}K_{2n}$, as desired.
THE DECOMPOSITION OF THE PERMUTATION CHARACTER $1^{	ext{GL}(2n,q)}_{\text{GL}(n,q^2)}$

It follows from (10) that for $x, y \in G_{2n-1}$ we have

$$e_{U_{2n-1}}xe_{K_{2n}} = e_{U_{2n-1}}ye_{K_{2n}} \iff xU_{2n-2}K_{2n-2} = yU_{2n-2}K_{2n-2}.$$  

Hence, if $x_1 = 1_{2n}, x_2, \ldots, x_t$ are representatives of the left cosets $xU_{2n-2}K_{2n-2}$ of $U_{2n-2}K_{2n-2}$ in $G_{2n-1}(\mathbb{C} G_{2n})$, then we have

$$e_{U_{2n-1}} \mathbb{C}[G_{2n}]e_{K_{2n}} = \bigoplus_{j=1}^t V_j$$

as vector space over $\mathbb{C}$, where

$$V_j = \mathbb{C} \cdot e_{U_{2n-1}} x_j e_{K_{2n}}.$$  

Clearly, $G_{2n-1}$ acts on $\{V_j\}_{1 \leq j \leq t}$ transitively. Moreover, $U_{2n-2}K_{2n-2}$ is the stabilizer of $V_1$ in $G_{2n-1}$, and $V_1$ affords the trivial representation of $U_{2n-2}K_{2n-2}$. Thus, $e_{U_{2n-1}} \mathbb{C}[G_{2n}]e_{K_{2n}}$ affords the induced representation $(1_{U_{2n-2}K_{2n-2}})^{G_{2n-1}}$, which proves 3.4.1, and hence 3.3.2.

4. PROOF OF THEOREM 1.2.2

In this section, $q$ is assumed to be odd, as in §2. (When $q$ is even, the proof is similar and easier.)

4.1. We prove 1.2.2 (i) by induction on $n$. If $n = 0$, then this is clear. It follows from the induction hypothesis that

4.1.1. If $0 \leq m < n$, then we have $(1_{K_{2m}})^{G_{2m}} = \sum \chi_\mu$, summed over $\mu$ such that $||\mu|| = 2m$, $\hat{\mu} = \mu$, and $\mu(\varphi_1)' \cup \mu(\varphi_{-1})$ is even.

Let $(1_{K_{2m}})^{G_{2m}} = \sum_{i=1}^k \chi_{\mu_i}$, then from 1.2.1 it follows that $\hat{\mu}_i = \mu_i$ for all $i$. Since as mentioned before $\varphi_1$ and $\varphi_{-1}$ are the only elements $\varphi \in \Theta$ such that $d(\varphi) = 1$ and $\hat{\varphi} = \varphi$, therefore it follows from 3.3.2 that

4.1.2. If $\nu : \Theta \rightarrow \mathcal{P}$ satisfies $||\nu|| = 2n - 1$ and $\nu \vdash \mu_i$ for some $i$, then one of the following holds:

(a) $\nu(\varphi_1)' \cup \nu(\varphi_{-1})$ is even and $\hat{\nu} \neq \nu$,

(b) $\nu(\varphi_1)' \cup \nu(\varphi_{-1})$ has exactly one odd part and $\hat{\nu} = \nu$.

Moreover,

$$\sum_{i=1}^k \sum_{||\nu|| = 2n-1 \nu \vdash \mu_i} \chi_\nu$$

is multiplicity-free.

From 4.1.2 we immediately have

4.1.3. If an irreducible character $\chi_\mu$ of $G_{2n}$ with $\hat{\mu} = \mu$ is contained in $(1_{K_{2n}})^{G_{2n}}$, then one of the following holds:

(a) $\mu(\varphi_1)' \cup \mu(\varphi_{-1})$ is even,

(b) $l(\mu(\varphi_1)' \cup \mu(\varphi_{-1})) = 2$. 
Let $\mu_* : \Theta \rightarrow \mathcal{P}$ be a partition-valued function such that $||\mu_*|| = 2n$, $\mu_* = \mu*$, $\mu_*(\varphi_1) = (1^{2k})$ and $\mu_*(\varphi_{-1}) = 0$. For two partitions $\lambda, \rho \in \mathcal{P}$ such that $l(\lambda \cup \rho) \leq 2$ and $|\lambda| + |\rho| = 2k$, we define $\mu_{\lambda,\rho} : \Theta \rightarrow \mathcal{P}$ by $\mu_{\lambda,\rho}(\varphi_1) = \lambda$, $\mu_{\lambda,\rho}(\varphi_{-1}) = \rho$, and $\mu_{\lambda,\rho}(\varphi) = \mu_*(\varphi)$ for all other $\varphi \in \Theta$. Then it follows that

$$d_{\mu_0,(2k)} > d_{\mu_0,(2k-1,1)} > d_{\mu_0,(2k-2,2)} > \cdots$$

In fact, from (2) it follows that

$$\frac{d_{\mu_0,(2k)}}{d_{\mu_0,(2k-1,1)}} = q^{2k-1} \cdot \frac{q-1}{q^{2k-1} - 1}.$$ 

Then since

$$q^{2k-1}(q - 1) - (q^{2k-1} - 1) = q^{2k-1}(q - 2) + 1 > 0,$$

we have $d_{\mu_0,(2k)} > d_{\mu_0,(2k-1,1)}$. Next, for $1 \leq j \leq k - 1$ it follows that

$$\frac{d_{\mu_0,(2k-j,j)}}{d_{\mu_0,(2k-j-1,j+1)}} = q^{2k-2j-1} \cdot \frac{(q^{2k-2j+1} - 1)(q^{j+1} - 1)}{(q^{2k-j+1} - 1)(q^{2k-2j-1} - 1)}.$$ 

Since

$$q^{2k-2j-1}(q^{2k-2j+1} - 1)(q^{j+1} - 1) - (q^{2k-j+1} - 1)(q^{2k-2j-1} - 1) > q^{4k-3j}(q - q^{-j} - 1) - q^{2k-j} - 1 \geq q^{4k-3j} - q^{2k-j} - 1$$

$$= q^{2k-j}(q^{2k-2j} - 1) - 1 > 0,$$

we have $d_{\mu_0,(2k-j,j)} > d_{\mu_0,(2k-j-1,j+1)}$, as desired.

4.1.4. Let $\lambda, \rho \in \mathcal{P}$ be as above, and suppose that $\chi_{\mu_*}$ is contained in $(1_{K_{2n}})^{G_{2n}}$. Then

(a) if $\lambda \neq 0$ then $\chi_{\mu_{\lambda,\rho}}$ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if $\lambda \cup \rho$ is even,

(b) if $\lambda = 0$ then exactly one of the following occurs:

(b1) $\chi_{\mu_0,\rho}$ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if $\rho$ is even,

(b2) $\chi_{\mu_0,\rho}$ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if $\rho$ is odd.

Proof. For two partitions $\beta, \gamma \in \mathcal{P}$ such that $l(\beta \cup \gamma) \leq 2$ and $|\beta| + |\gamma| = 2k - 1$, we also define $\nu_{\beta,\gamma} : \Theta \rightarrow \mathcal{P}$ such that $||\nu|| = 2n - 1$ by $\nu_{\beta,\gamma}(\varphi_1) = \beta$, $\nu_{\beta,\gamma}(\varphi_{-1}) = \gamma$, and $\nu_{\beta,\gamma}(\varphi) = \mu_*(\varphi)$ for all other $\varphi \in \Theta$. First of all, since $\chi_{\nu_{(1^{2k-1})},0}$ appears in (12) and $\nu_{(1^{2k-1}),0} \vdash \mu_{(1^{2k})},0$, therefore neither $\chi_{\mu_{(1^{2k-2},2)},0}$ nor $\chi_{\mu_{(1^{2k-1})},(1)}$ is contained in $(1_{K_{2n}})^{G_{2n}}$. Next, since $\nu_{(1^{2k-3}),0}$ appears in (12) by 3.3.2, it follows from 4.1.3 that $\chi_{\mu_{(1^{2k-4},2)},0}$ must be contained in $(1_{K_{2n}})^{G_{2n}}$, and so on.

4.1.5. Let $1 \leq k \leq n$ and let $\mu_* : \Theta \rightarrow \mathcal{P}$ be a partition-valued function such that $||\mu_*|| = 2n$, $\mu_* = \mu_*$, $\mu_*(\varphi_1) = (1^{2k})$ and $\mu_*(\varphi_{-1}) = 0$. Then $\chi_{\mu_*}$ is contained in $(1_{K_{2n}})^{G_{2n}}$.

Proof. We prove 4.1.5 by induction on $k$, starting from $k = n$ and ending with 1. When $k = n$, this is trivial. Let $2 \leq k \leq n$ and assume that the assertion is true for all $l$ such that $k \leq l \leq n$. Let $\nu_* : \Theta \rightarrow \mathcal{P}$ be a partition-valued function such that $||\nu_*|| = 2n - 1$, $\nu_*(\varphi_1) = (1^{2k-1})$ and $\nu_*(\varphi_{-1}) = 0$. If the restriction $\chi_{\mu_*} \downarrow_{G_{2n-1}}^{G_{2n}}$ of an irreducible constituent $\chi_{\mu}$ of $(1_{K_{2n}})^{G_{2n}}$ to $G_{2n-1}$ contains $\chi_{\nu_*}$,
then by 3.1.2, 4.1.3 and 4.1.4 it follows that $\mu(\varphi_{1}) = (1^{2k})$ or $\mu(\varphi_{1}) = (1^{2k-2})$, and $\mu(\varphi_{-1}) = (2j)$ for some $j \geq 0$. Hence, we have

$$(14) \quad ((1_{K_{2n-2}})^{G_{2n-2}} \downarrow_{G_{2n-3}}^{G_{2n-2}}, \chi_{\nu_{*}})_{G_{2n-3}} = q \cdot ((1_{K_{2n-4}})^{G_{2n-4}} \uparrow_{G_{2n-4}}^{G_{2n-3}}, \chi_{\nu_{*}})_{G_{2n-3}}$$

where the sum on the right is over $\mu$ such that $||\mu|| = 2n$, $\bar{\mu} = \mu$, $\mu(\varphi_{1}) = (1^{2k})$ or $\mu(\varphi_{1}) = (1^{2k-2})$, and $\mu(\varphi_{-1}) = (2j)$ for some $j \geq 0$.

Now, for any $\lambda : \Theta \rightarrow \mathscr{A}$ such that $\lambda(\varphi_{1}) = (1^{m})$ for some $m \geq 2$, we define $\lambda^{-} : \Theta \rightarrow \mathcal{A}$ by $\lambda^{-}(\varphi_{1}) = (1^{m-2})$ and $\lambda^{-}(\varphi) = \lambda(\varphi)$ for all $\varphi \in \Theta$. Then it follows from 3.1.2 that the right-hand side of (14) is equal to

$$\left( \sum \chi_{\mu^{-}} \downarrow_{G_{2n-3}}^{G_{2n-2}}, \chi_{\nu_{*}} \right)_{G_{2n-3}}$$

summed over $\mu$ as above, which is also equal to

$$(1_{K_{2n-2}})^{G_{2n-2}} \downarrow_{G_{2n-3}}^{G_{2n-2}}, \chi_{\nu_{*}})_{G_{2n-3}} = q \cdot \left((1_{K_{2n-4}})^{G_{2n-4}} \downarrow_{G_{2n-4}}^{G_{2n-3}}, \chi_{\nu_{*}}\right)_{G_{2n-3}}$$

where the first and the third equalities follow from 3.2.1. Hence, if $\mu_{*} : \Theta \rightarrow \mathcal{A}$ satisfies $||\mu_{*}|| = 2n$, $\bar{\mu}_{*} = \mu_{*}$, $\mu_{*}(\varphi_{1}) = (1^{2k-2})$ and $\mu_{*}(\varphi_{-1}) = 0$, then since $(\chi_{\mu_{*}} \downarrow_{G_{2n-1}}^{G_{2n}}, \chi_{\nu_{*}})_{G_{2n-1}} > 0$ for at least one such $\nu_{*}$ as above, therefore $\chi_{\mu_{*}}$ must be contained in $(1_{K_{2n}})^{G_{2n}}$.

The proof of 1.2.2 (i) can now be rapidly completed. Let $\mu : \Theta \rightarrow \mathcal{A}$ be a partition-valued function such that $||\mu|| = 2n$ and $\bar{\mu} = \mu$. Then 4.1.5 and 4.1.4 imply that if $\mu(\varphi_{1}) \neq 0$ or $l(\mu(\varphi_{1}) \cup \mu(\varphi_{-1})) \geq 3$ then $\chi_{\mu}$ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if $\mu(\varphi_{1}) \cup \mu(\varphi_{-1})$ is even. Also, if $\mu(\varphi_{1}) = 0$ and $l(\mu(\varphi_{-1})) \leq 2$ then there are two possibilities. However, by virtue of 2.1.1 and (13), we can conclude that in this case $\chi_{\mu}$ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if $\mu(\varphi_{-1})$ is even. It also follows from 2.1.1 that $(1_{K_{2n}})^{G_{2n}}$ contains all irreducible characters $\chi_{\mu}$ of $G_{2n}$ such that $\bar{\mu} = \mu$ and $\mu(\varphi_{1}) = \mu(\varphi_{-1}) = 0$.

4.2. Finally, we prove 1.2.2 (iii). The left-hand side of (3) is by 2.1.2 equal to

$$\prod_{r \geq 1} (1 - t^{2r})^{-2} \cdot \prod_{r \geq 1} (1 - t^{2r})^{-|\Psi_{1}|} \cdot \prod_{k \geq 2} \prod_{r \geq 1} (1 - t^{2kr})^{-|\Psi_{k}|}$$

This completes the proof of 1.2.2.

REFERENCES


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