<table>
<thead>
<tr>
<th>Title</th>
<th>THE DECOMPOSITION OF THE PERMUTATION CHARACTER $1^{GL(2n,q)}_{GL(n,q^2)}$ (Topics in Young Diagrams and Representation Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
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THE DECOMPOSITION OF THE PERMUTATION CHARACTER

\[ 1^G_{GL(2n,q)} \]

\[ 1^G_{GL(n,q^2)} \]

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INTRODUCTION

Let \( G \) be a finite group acting transitively on a finite set \( X \), and let \( H = G_x \) be the stabilizer of a point \( x \) in \( X \). The permutation character \( \pi \) of \( G \) on \( X \) is equivalent to the induced character \( (1_H)^G \) of the identity character \( 1_H \) of \( H \). We say that the permutation character \( \pi = (1_H)^G \) is multiplicity-free if it is decomposed into a direct sum of inequivalent irreducible characters. In this case, the centralizer algebra (or the Hecke algebra) of the permutation group is commutative, and we also say that \( H \) is a multiplicity-free subgroup of \( G \). A pair \((G, H)\) of a finite group \( G \) and a multiplicity-free subgroup \( H \) is sometimes called a Gelfand pair. A commutative association scheme \( \mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d}) \) is associated with a multiplicity-free transitive action of a finite group \( G \) on a finite set \( X \), by taking the relations \( R_0, R_1, \ldots, R_d \) as the orbits of \( G \) on \( X \times X \). It is an interesting question to know many examples of commutative association schemes and their character tables. (The reader is referred to Bannai-Ito [4], Bannai [1] for the basic concept of commutative association schemes and their character tables.) It should be noted that knowing the character table of a commutative association scheme (associated to a multiplicity-free transitive action of a finite group, i.e., to a Gelfand pair) is equivalent to knowing the zonal spherical functions of the permutation group.

Many examples of Gelfand pairs or commutative association schemes are known (see, e.g. Saxl [16], Inglis [9], Bannai [1], Bannai-Hao-Song [2], Bannai-Hao-Song-Wei [3], Bannai-Kawanaka-Song [5], Lusztig [14], Lawther [13], etc.). In Inglis-Liebeck-Saxl [10], it is stated that the following pairs \((G, H)\) are Gelfand pairs:

\[
\begin{align*}
(i) \quad & (G, H) = (GL(n, q^2), GL(n, q)), \\
(ii) \quad & (G, H) = (GL(n, q^3), GU(n, q)), \\
(iii) \quad & (G, H) = (GL(2n, q), Sp(2n, q)), \\
(iv) \quad & (G, H) = (GL(2n, q), GL(n, q^2)).
\end{align*}
\]

It seems that the structure of the double cosets \( H \backslash G / H \), the decomposition of the permutation character \( \pi = 1^G_H \), and the character table of the associated commutative association scheme are known for the first three cases (Gow [7], Klyachko [12], Bannai-Kawanaka-Song [5], Kawanaka [11], Bannai [1], Lusztig [14]). However, it seems that they are not yet known for the last case (iv) of \( G = GL(2n, q) \) and \( H = GL(n, q^2) \). The decomposition of the permutation character \( 1^G_{GL(2n,q^2)} \) is well-known for \( n = 1 \) (cf. Terras [19, Chapter 21]). When \( n = 2 \), it was determined by the second author [18] by explicitly calculating the inner product \( \langle \chi, 1^G_{GL(2,q^2)} \rangle \) for all irreducible characters \( \chi \) of \( GL(4, q) \). Our purpose in this paper is to determine the decomposition of \( 1^G_{GL(2n,q^2)} \) for general \( n \).
1. Preliminaries on General Linear Groups and Main Results

1.1. First of all, we briefly recall a parametrization of the irreducible characters of the general linear group $G_n = GL(n, q)$, following Macdonald [15, Chapter IV.]. Whenever possible, we use the notation of [15].

A partition is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers $\lambda_i$ containing finitely many non-zero terms. The non-zero $\lambda_i$ are called the parts of $\lambda$. We identify $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots, 0)$. Sometimes we write $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ in place of $\lambda = (\lambda_1, \lambda_2, \ldots)$, where $m_i$ is the number of $j$ such that $\lambda_j = i$. The only partition with no non-zero terms is denoted by $0$. For each partition $\lambda$, the length $l(\lambda)$ of $\lambda$ is the number of parts of $\lambda$, and the weight $|\lambda|$ of $\lambda$ is defined by $|\lambda| = \sum_{i \geq 1} \lambda_i$. We denote the set of all partitions by $\mathcal{P}$. The diagram of $\lambda \in \mathcal{P}$ is the set of points $x = (i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$, and the conjugate $\lambda'$ of $\lambda$ is the partition whose diagram is the transpose of that of $\lambda$. For example, the conjugate of $(2, 2, 1)$ is $(3, 2)$. The hook-length $h(x)$ of $\lambda$ at $x = (i, j) \in \lambda$ (i.e., $1 \leq j \leq \lambda_i$) is defined by $h(x) = \lambda_i + \lambda'_j - i - j + 1$. For $\lambda, \mu \in \mathcal{P}$, we define $\lambda \cup \mu$ to be the partition whose parts are those of $\lambda$ and $\mu$, arranged in descending order. An even (resp. odd) partition is a partition with all parts even (resp. odd). We let $s_\lambda$ denote the Schur function (in countably many independent variables) corresponding to $\lambda \in \mathcal{P}$.

Let $F_q$ be a finite field with $q$ elements, and $\overline{F}_q$ the algebraic closure of $F_q$. For each positive integer $l$ there exists a unique extension $F_{q^l}$ of $F_q$ in $\overline{F}_q$ of degree $l$. We denote the multiplicative group of $F_{q^l}$ by $M_{q^l}$, and the character group of $M_{q^l}$ by $\hat{M}_{q^l}$. If $l$ divides $m$ then $\hat{M}_{q^l}$ is embedded in $\hat{M}_m$ by the transpose of the norm map $N_{m, q^l} : M_m \rightarrow M_{q^l}$. We let $L = \lim\limits_{\longrightarrow} \hat{M}_{q^l}$ be the inductive limit of the $\hat{M}_{q^l}$. The Frobenius map $F : \gamma \mapsto \gamma^q$ acts on $L$, and $\hat{M}_l$ is the set of all $F^l$-fixed elements in $L$. We denote the set of $F$-orbits in $L$ by $\Theta$. Then the irreducible characters of $G_n$ can be parametrized by the partition-valued functions $\mu : \Theta \rightarrow \mathcal{P}$ such that

\[(1) \quad ||\mu|| = \sum_{\varphi \in \Theta} d(\varphi) |\mu(\varphi)| = n\]

where $d(\varphi)$ is the number of elements of $\varphi$. The irreducible character of $G_n$ corresponding to $\mu$ is denoted by $\chi_\mu$. The degree $d_\mu$ of $\chi_\mu$ is given by

\[(2) \quad d_\mu = \psi_n(q) \prod_{\varphi \in \Theta} s_\mu(\varphi)(q^{-1}_\varphi, q^{-2}_\varphi, \ldots)\]

\[= \psi_n(q) \prod_{\varphi \in \Theta} q^{n(\mu(\varphi)'')} \tilde{H}_\mu(q^\varphi)^{-1}\]

where $q^\varphi = q^{d(\varphi)}$,

\[\psi_n(q) = \prod_{i=1}^n (q^i - 1),\]

\[n(\lambda) = \sum_{i \geq 1} (i - 1) \lambda_i,\]

and

\[\tilde{H}_\lambda(q^\varphi) = \prod_{x \in \lambda} (q^h(x) - 1)\]
for $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}$.

Let $\xi_1$ be the identity character of $M_1$, and if $q$ is odd then let $\xi_{-1}$ be the quadratic character of $M_1$. We put $\varphi_1 = \{\xi_1\}$, $\varphi_{-1} = \{\xi_{-1}\} \in \Theta$. For $\varphi = \{\xi, \xi^q, \ldots, \xi^{q^{d-1}}\} \in \Theta$, the reciprocal $F$-orbit $\bar{\varphi}$ of $\varphi$ is defined by

$$\bar{\varphi} = \{\xi^{-1}, \xi^{-q}, \ldots, \xi^{-q^{d-1}}\}.$$ Notice that $\varphi_1$ and $\varphi_{-1}$ are the only elements $\varphi \in \Theta$ such that $d(\varphi) = 1$ and $\bar{\varphi} = \varphi$. Also for each partition-valued function $\mu : \Theta \to \mathcal{P}$, we define $\tilde{\mu} : \Theta \to \mathcal{P}$ by

$$\tilde{\mu}(\varphi) = \mu(\bar{\varphi})$$ for all $\varphi \in \Theta$. Then we can easily verify that the complex conjugate $\overline{\chi_{\mu}}$ of $\chi_{\mu}$ is given by $\chi_{\tilde{\mu}}$ (see for example (4.5) in [15, Chapter IV.]), from which it follows that

1.1.1. An irreducible character $\chi_{\mu}$ of $G_n$ is real-valued if and only if $\tilde{\mu} = \mu$.

1.2. We now present our main results. Let $K_{2n}$ be a subgroup of $G_{2n}$ isomorphic to $GL(n, q^2)$. It is known that

1.2.1. Theorem (Inglis-Liebeck-Saxl [10]). The permutation character $(1_{K_{2n}})^{G_{2n}}$ is multiplicity-free and every irreducible constituent of $(1_{K_{2n}})^{G_{2n}}$ is real-valued.

In this paper, we determine the decomposition of the permutation character $(1_{K_{2n}})^{G_{2n}}$ explicitly. More precisely, we will prove the following:

1.2.2. Theorem. (i) If $q$ is odd, then we have $(1_{K_{2n}})^{G_{2n}} = \sum \chi_{\mu}$, summed over $\mu$ such that $||\mu|| = 2n$, $\tilde{\mu} = \mu$, and both $\mu(\varphi_1)'$ and $\mu(\varphi_{-1})'$ are even.

(ii) If $q$ is even, then we have $(1_{K_{2n}})^{G_{2n}} = \sum \chi_{\mu}$, summed over $\mu$ such that $||\mu|| = 2n$, $\tilde{\mu} = \mu$, and $\mu(\varphi_1)'$ is even.

(iii) In either case, the generating function for the rank (i.e., the number of the irreducible constituents of the permutation character $(1_{K_{2n}})^{G_{2n}}$) is given by

$$\sum_{n \geq 0} \text{rank}(G_{2n}/K_{2n})t^{2n} = \prod_{r \geq 1} (1 - qt^{2r})^{-1}$$ with the understanding that $\text{rank}(G_0/K_0) = 1$. In particular we have

$$\text{rank}(G_{2n}/K_{2n}) = \sum q^{||\lambda||}$$ summed over all partitions $\lambda$ such that $|\lambda| = n$.

1.2.3. Remark. In the notation of Green [8], our character $\chi_{\mu}$ corresponds to the conjugate function $\mu' : \Theta \to \mathcal{P}$ defined by $\mu'(\varphi) = \mu(\varphi)'$ for all $\varphi \in \Theta$. In particular, in our notation the identity character of $G_n$ assigns the partition $(1^n)$ to $\varphi_1$. See Springer-Zelevinsky [17, Remark 1.9].

1.2.4. Remark. Let $\pi(G_n)$ denote the number of the conjugacy classes of $G_n$, then the generating function for the $\pi(G_n)$ is given by

$$\sum_{n \geq 0} \pi(G_n)t^n = \prod_{r \geq 1} (1 - t^r)(1 - qt^r)^{-1}.$$ Hence 1.2.2 (iii) implies that

$$\text{rank}(G_{2n}/K_{2n}) = \sum_{i=0}^{n} p(i) \pi(G_{n-i})$$
where $p(i)$ is the number of partitions $\lambda$ such that $|\lambda| = i$. It is a reasonable guess that there is a natural set of representatives of the double cosets $K_{2n} \backslash G_{2n}/K_{2n}$ which reflects the above equality.

2. Degree Formula

2.1. The starting point of the proof of 1.2.2 is the following proposition:

2.1.1. Proposition. (i) If $q$ is odd, then we have

$$\sum d_\mu = (q^{2n} - q)(q^{2n} - q^3) \cdots (q^{2n} - q^{2n-1})$$

where the sum on the left is over $\mu$ such that $||\mu|| = 2n$, $\tilde{\mu} = \mu$, and both $\mu(\varphi_1)'$ and $\mu(\varphi_{-1})$ are even.

(ii) If $q$ is even, then we have

$$\sum d_\mu = (q^{2n} - q)(q^{2n} - q^3) \cdots (q^{2n} - q^{2n-1})$$

where the sum on the left is over $\mu$ such that $||\mu|| = 2n$, $\tilde{\mu} = \mu$, and $\mu(\varphi_1)'$ is even.

To prove 2.1.1, we need some preparations. In what follows, we assume that $q$ is odd. (The assertion (ii) is proved in exactly the same way as (i).)

Let $\Phi$ denote the set of monic irreducible polynomials $f(t)$ over $\mathbb{F}_q$ with $f(t) \neq t$. We identify $\Phi$ with the set of $F$-orbits in the multiplicative group $M$ of the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$, by assigning to each $f$ the $F$-orbit consisting of its roots in $M$.

Let $f(t) = t^k + a_1 t^{k-1} + \cdots + a_k$ be a monic polynomial in $\mathbb{F}_q[t]$ of degree $k$ with $a_k \neq 0$. The reciprocal polynomial $\tilde{f}$ of $f$ is defined by

$$\tilde{f}(t) = a_k^{-1} t^k f(t^{-1}) = t^k + \frac{a_{k-1}}{a_k} t^{k-1} + \cdots + \frac{1}{a_k}.$$ 

We call the polynomial $f$ self-reciprocal if $f(t) = \tilde{f}(t)$.

Let

$$\Psi = \Phi \cup \{t\} : \text{the set of all monic irreducible polynomials in } \mathbb{F}_q[t],$$

$$S = \{f \in \Phi \backslash \{t \pm 1\} | f : \text{self-reciprocal}\},$$

$$N = \{f \in \Phi \backslash \{t \pm 1\} | f : \text{non-self-reciprocal}\},$$

and let

$$\Psi_k = \{f \in \Psi | \deg f = k\},$$

$$S_k = \{f \in S | \deg f = k\},$$

$$N_k = \{f \in N | \deg f = k\}$$

for $k \geq 1$. Notice that $S_k$ is empty unless $k$ is even.

First we observe the following two one-to-one correspondences due to Carlitz [6]:

2.1.2 ([6, §3.]). We have

$$\Psi_k \overset{1:1}{\leftrightarrow} S_{2k} \cup \{g\bar{g} | g \in N_k\}$$

for $k \geq 2$, and

$$\Psi \backslash \{t \pm 2\} \overset{1:1}{\leftrightarrow} S_2 \cup \{g\bar{g} | g \in N_1\}.$$
THE DECOMPOSITION OF THE PERMUTATION CHARACTER $\chi^{G_{L(2n,q)}}_{G_{L(n,q^2)}}$

Proof. Let $h(t) \in \mathbb{F}_q[t]$ be a monic irreducible polynomial of degree $k$ ($k \geq 1$) such that $h(t) \neq t \pm 2$, then $h(t)$ is decomposed into linear factors in $\mathbb{F}_{q^k}[t]$ as $h(t) = (t - \beta)(t - \beta^q)\ldots(t - \beta^{q^{k-1}})$. Let $\alpha \in \mathbb{F}_{q^{2k}}$ be a root of the polynomial $t^2 - \beta t + 1$, i.e., $\alpha + \alpha^{-1} = \beta$. Since $\beta \neq \pm 2$ it follows that $\alpha \neq \alpha^{-1}$, so that

$\alpha, \alpha^q, \ldots, \alpha^{q^{k-1}}, \alpha^{-1}, \alpha^{-q}, \ldots, \alpha^{-q^{k-1}}$

are distinct. We define

$$f(t) = t^k h(t + t^{-1})$$

$$= (t - \alpha)(t - \alpha^q)\ldots(t - \alpha^{q^{k-1}})(t - \alpha^{-1})(t - \alpha^{-q})\ldots(t - \alpha^{-q^{k-1}}),$$

then $f(t)$ is a monic polynomial of degree $2k$. Now, if $\alpha \in \mathbb{F}_{q^{2k}} \backslash \mathbb{F}_{q^k}$ then we have $f(t) \in S_{2k}$ since $\alpha^{-1} = \alpha^q$, and if $\alpha \in \mathbb{F}_{q^k}$ then we have $f(t) = g(t)\tilde{g}(t)$ where

$g(t) = (t - \alpha)(t - \alpha^q)\ldots(t - \alpha^{q^{k-1}}) \in N_k$,

as desired. \hfill \Box

Let $\sigma_{2k} = |S_{2k}|$ and $\tau_{2k} = |\{g\tilde{g} | g \in N_k\}| = \frac{1}{2}|N_k|$ for $k \geq 1$. Then it follows from 2.1.2 that

$$\sum_{k|N} k(\sigma_{2k} + \tau_{2k}) + 2 = q^N$$

for $N \geq 1$. If $N = 2M$ is even then we also have

$$\sum_{k|M} (2k)\sigma_{2k} + \sum_{k|2M} k(2\tau_{2k}) + 2 = q^N - 1.$$

On the other hand, if $N$ is odd then we have

$$\sum_{k|N} k(2\tau_{2k}) + 2 = q^N - 1.$$

Let $x = (x_1, x_2, \ldots)$ be an infinite sequence of independent variables. We shall need the following four equalities:

2.1.3 (cf. [15, p.63, (4.3)]). $\sum_{\lambda} s_{\lambda}^2 = \prod_{i}(1-x_i^2)^{-1} \prod_{i<j}(1-x_i x_j)^{-2}$, where the sum on the left is over all partitions $\lambda$.

2.1.4 (cf. [15, p.76, Example 4]). $\sum_{\lambda} s_{\lambda} = \prod_{i}(1-x_i)^{-1} \prod_{i<j}(1-x_i x_j)^{-1}$, where the sum on the left is over all partitions $\lambda$.

2.1.5 (cf. [15, p.77, Example 5(a)]). $\sum_{\mu \text{ even}} s_{\mu} = \prod_{i}(1-x_i^2)^{-1} \prod_{i<j}(1-x_i x_j)^{-1}$, where the sum on the left is over all even partitions $\mu$.

2.1.6 (cf. [15, p.77, Example 5(b)]). $\sum_{\nu \text{ even}} s_{\nu} = \prod_{i<j}(1-x_i x_j)^{-1}$, where the sum on the left is over all partitions $\nu$ with $\nu'$ even.
2.2. Proof of 2.1.1. Our proof of 2.1.1 is inspired by [15, p.289, Example 5 of all, notice that the number of elements $\varphi \in \Theta$ such that $d(\varphi) = 2k$ and $\tilde{\varphi}$ equal to $\sigma_{2k}$. We shall compute the following:

\[
D = \sum_{\nu \text{ even}} s_{\nu}(q^{-1}, q^{-2}, \ldots) t^{l_{\nu}} \times \sum_{\mu \text{ even}} s_{\mu}(q^{-1}, q^{-2}, \ldots) t^{l_{\mu}} \\
\times \prod_{k \geq 1} \left\{ \sum_{\lambda} s_{\lambda}(q^{-2k}, q^{-4k}, \ldots) t^{2k|\lambda|} \right\}^{\sigma_{2k}} \\
\times \prod_{k \geq 1} \left\{ \sum_{\lambda} s_{\lambda}^{2}(q^{-k}, q^{-2k}, \ldots) t^{2k|\lambda|} \right\}^{\tau_{2k}}
\]

where $t$ is an indeterminate.

Let

\[
X_1 = \log \prod_{k \geq 1} \left\{ \prod_{i \geq 1} \left( 1 - (tq^{-i})^{2k} \right) \right\}^{\sigma_{2k}}, \\
Y_1 = \log \prod_{k \geq 1} \left\{ \prod_{i \geq 1} \left( 1 - (tq^{-i})^{2k} \right) \right\}^{\tau_{2k}}, \\
Z_1 = \log \prod_{i \geq 1} \left( 1 - (tq^{-i})^{2} \right)^{-1}.
\]

Then we have

\[
X_1 = \sum_{k \geq 1} \sigma_{2k} \sum_{i \geq 1} \sum_{r \geq 1} \frac{(tq^{-i})^{2kr}}{r} = \sum_{k \geq 1} \sigma_{2k} \sum_{r \geq 1} \frac{t^{2kr}}{r} \cdot \frac{1}{q^{2kr}-1} \\
= \sum_{N \geq 1} \frac{t^{2N}}{N(q^{2N}-1)} \sum_{k|N} k \sigma_{2k}.
\]

Similarly, we have

\[
Y_1 = \sum_{N \geq 1} \frac{t^{2N}}{N(q^{2N}-1)} \sum_{k|N} k \tau_{2k}
\]

and

\[
Z_1 = \sum_{N \geq 1} \frac{t^{2N}}{N(q^{2N}-1)}.
\]
Therefore, it follows from (4) that

\[(7) \quad X_1 + Y_1 + Z_1 = \sum_{N \geq 1} \frac{t^{2N}}{N(q^{2N} - 1)}(q^N - 1) = \sum_{N \geq 1} \frac{t^{2N}}{N(q^N + 1)}
= \sum_{N \geq 1} \frac{t^{2N}}{N} \sum_{k \geq 1} (-1)^{k-1} q^{-kN} = \sum_{k \geq 1} (-1)^{k-1} \sum_{N \geq 1} \frac{(t^2q^{-2})^N}{N(q^N - 1)}\]

Let

\[X_2 = \log \prod_{k \geq 1} \{ \prod_{i < j} \left(1 - (t^2q^{-i-j})^{2k} \right)^{-1} \}^{\sigma_{2k}},\]
\[Y_2 = \log \prod_{k \geq 1} \{ \prod_{i < j} \left(1 - (t^2q^{-i-j})^{2k} \right)^{-2} \}^{\tau_{2k}},\]
\[Z_2 = \log \prod_{i < j} \left(1 - t^2q^{-i-j} \right)^{-2}.\]

Then we have

\[X_2 = \sum_{k \geq 1} \sigma_{2k} \sum_{i < j} \sum_{r \geq 1} \frac{(t^2q^{-i-j})^{2kr}}{r} = \sum_{i \geq 1} \sum_{k \geq 1} \frac{t^{4kr}}{r} \sum_{\dot{M} \geq 1} \frac{q^{-4\dot{M}}}{q^{2\dot{M}} - 1}\]

Similarly, we have

\[Y_2 = \sum_{i \geq 1} \sum_{N \geq 1} \frac{t^{2N}}{N(q^N - 1)} \left( \sum_{k|N} k \tau_{2k} \right) q^{-2\dot{M}}\]

and

\[Z_2 = \sum_{i \geq 1} \sum_{N \geq 1} \frac{t^{2N}}{N(q^N - 1)} 2q^{-2\dot{M}}.\]

Therefore, it follows from (5) and (6) that

\[(8) \quad X_2 + Y_2 + Z_2 = \sum_{i \geq 1} \sum_{N \geq 1} \frac{t^{2N}}{N(q^N - 1)}(q^N - 1)q^{-2\dot{M}}\]

\[= \sum_{i \geq 1} \sum_{N \geq 1} \frac{(t^2q^{-2i})^N}{N}.\]

Hence from (7) and (8) we obtain

\[\log D = X_1 + Y_1 + Z_1 + X_2 + Y_2 + Z_2\]

\[= \sum_{i \geq 1} \sum_{N \geq 1} \frac{(t^2q^{-2i+1})^N}{N}\]

\[= \sum_{i \geq 1} \sum_{N \geq 1} \frac{q^{-2i+1}}{\varphi_m(q^2)}\]

so that

\[D = \prod_{i \geq 1} (1 - t^2q^{-2i+1})^{-1} = \sum_{m \geq 0} t^{2m}q^{-m}/\varphi_m(q^{-2})\]
where \( \varphi_m(t) = (1 - t)(1 - t^2) \ldots (1 - t^m) \).

Finally, on picking out the coefficient of \( t^{2n} \), and multiplying by \( \psi_{2n}(q) \), we get the desired result.

3. **Branching Lemmas**

In this section, we prepare two lemmas which enable us to prove 1.2.2 by induction on \( n \). We do not need to assume in this section that \( q \) is odd.

3.1. First, we recall a result of Zelevinsky [21]. Let \( n \geq 2 \) and let \( H_n \) be the subgroup of \( G_n \) consisting of the matrices of the form

\[
g = \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix}
\]

where \( x \in G_{n-1} \). Let \( U_{n-1} \) be the abelian normal subgroup of \( H_n \) defined by

\[
U_{n-1} = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1_{n-1} \end{pmatrix} \right\} \cong \Bbb F_q^{n-1}
\]

where \( 1_{n-1} \) is the identity matrix of degree \( n - 1 \). We identify \( G_{n-1} \) with the following subgroup of \( H_n \):

\[
\left\{ \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \bigmid x \in G_{n-1} \right\}
\]

then we have \( H_n = U_{n-1} \rtimes G_{n-1} \), the semidirect product of \( U_{n-1} \) with \( G_{n-1} \). The irreducible characters of \( H_n \) are determined by applying the method of little groups, and they are parametrized by the partition-valued functions \( \nu : \Theta \to \mathcal P \) such that \( ||\nu|| < n \) (cf. [21, §13.]). The irreducible character of \( H_n \) corresponding to \( \nu \) is denoted by \( \zeta_\nu^{(n)} \). Notice that the irreducible characters \( \zeta_\nu^{(n)} \) of \( H_n \) with \( ||\nu|| = n - 1 \) are exactly those obtained by the irreducible characters \( \chi_\nu \) of \( G_{n-1} \cong H_n/U_{n-1} \), that is, they are constant on \( U_{n-1} \).

If \( \mu : \Theta \to \mathcal P \) and \( \nu : \Theta \to \mathcal P \) are two partition-valued functions, we shall write \( \nu \vdash \mu \) if \( \mu(\varphi)_i - 1 \leq \nu(\varphi)_i \leq \mu(\varphi)_i \) for all \( \varphi \in \Theta \) and \( i \geq 1 \) (i.e., the skew diagram \( \mu(\varphi) - \nu(\varphi) \) is a horizontal strip for any \( \varphi \in \Theta \)).

3.1.1. **Theorem** ([21, §13.5.]). (i) Let \( \mu : \Theta \to \mathcal P \) be a partition-valued function such that \( ||\mu|| = n \). Then we have

\[
\chi_\mu \downarrow_{H_n}^{G_n} = \sum \zeta_\nu^{(n)}
\]

summed over \( \nu \) such that \( ||\nu|| < n \) and \( \nu \vdash \mu \).

(ii) Let \( \nu : \Theta \to \mathcal P \) be a partition-valued function such that \( ||\nu|| < n \). Then we have

\[
\zeta_\nu^{(n)} \downarrow_{G_{n-1}}^{H_n} = \sum \chi_\lambda
\]

summed over \( \lambda \) such that \( ||\lambda|| = n - 1 \) and \( \nu \vdash \lambda \).

The following theorem was first proved by Thoma [20], and is easily derived from 3.1.1.

3.1.2. **Theorem** ([20]). Let \( \mu : \Theta \to \mathcal P \) and \( \lambda : \Theta \to \mathcal P \) be partition-valued functions such that \( ||\mu|| = n \) and \( ||\lambda|| = n - 1 \). Then the multiplicity of \( \chi_\mu \) in the induced character \( \chi_\lambda \downarrow_{G_{n-1}}^{G_n} \) is equal to the number of \( \nu : \Theta \to \mathcal P \) such that \( \nu \vdash \mu \) and \( \nu \vdash \lambda \).
THE DECOMPOSITION OF THE PERMUTATION CHARACTER \(1_{GL(n, q^2)}^{GL(2n, q)}\)

3.2. Let \(V_{2n}\) be the vector space of column \(2n\)-vectors with components in \(F_q\), and let \(\{v_1, v_2, \ldots, v_{2n}\}\) be the standard basis of \(V_{2n}\), that is, \(v_i\) is the vector with 1 in the \(i\)-th component and zeros elsewhere. We fix an element \(\alpha \in F_{q^2}\) such that \(\alpha \in F_q\), and denote by \(f(t) = t^2 + at + b \in F_q[t]\) the minimal polynomial of \(\alpha\) over \(F_q\). Let \(g_0\) be an element in \(G_{2n}\) such that \(g_0^2 + ag_0 + b1_{2n} = 0\). Then \(g_0\) determines a vector space over \(F_{q^2}\) on \(V_{2n}\), of dimension \(n\), such that \(\alpha v = g_0 v\) for \(v \in V_{2n}\).

The centralizer \(K_{2n} = C_{G_{2n}}(g_0)\) of \(g_0\) in \(G_{2n}\) is isomorphic to \(GL(n, q^2)\).

Let \(U\) be the subspace of \(V_{2n}\) over \(F_q\) spanned by \(v_2, v_3, \ldots, v_{2n}\). Clearly, an element \(g \in G_{2n}\) belongs to \(G_{2n-1}\) if and only if \(gU = U\) and \(gv_1 = v_1\). The subspace \(U\) contains a subspace \(W\) of \(V_{2n}\) over \(F_{q^2}\) of dimension \(n-1\) (over \(F_{q^2}\)), defined by

\[ W = \{ u \in U \mid g_0 u \in U \}. \]

It is easily seen that

\[ G_{2n-1} \cap K_{2n} = \{ k \in K_{2n} \mid kW = W, kv_1 = v_1 \}, \]

that is, \(G_{2n-1} \cap K_{2n}\) is isomorphic to \(GL(n-1, q^2)\).

Now for any \(x \in G_{2n}\) we have

\[ |G_{2n-1}xK_{2n}| = \frac{|G_{2n-1}| |K_{2n}|}{|G_{2n-1} \cap xK_{2n}x^{-1}|} = \frac{|G_{2n-1}| |K_{2n}|}{|GL(n-1, q^2)|} = \frac{1}{q} |G_{2n}| \]

since \(xK_{2n}x^{-1} = C_{G_{2n}}(xg_0x^{-1}) \cong GL(n, q^2)\) and \(g_0\) is chosen arbitrarily. Hence it follows from Mackey's theorem that

**3.2.1. Lemma.** \((1_{K_{2n}})^{G_{2n}} \downarrow_{G_{2n-1}^{2n}}^{G_{2n}} = q \cdot (1_{K_{2n-2}})^{G_{2n-2}} \uparrow_{G_{2n-2}}^{G_{2n-1}}\).

3.3. For the sake of simplicity, in what follows we assume that \(g_0\) is of the form

\[ g_0 = \begin{pmatrix} \tilde{g}_0 & 0 & \cdots & 0 \\ 0 & \tilde{g}_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{g}_0 \end{pmatrix} \]

where \(\tilde{g}_0 = \begin{pmatrix} 0 & -b \\ 1 & a \end{pmatrix}\), so that \(v_{2i} = \alpha v_{2i-1}\) (\(1 \leq i \leq n\)). Then it follows that

**3.3.1. For** \(g = (g_{ij}) \in G_{2n}\), \(g\) is contained in \(K_{2n}\) if and only if

\[ g_{2k-1,2l-1} = ag_{2k,2l-1} + g_{2k,2l} \]

\[ g_{2k-1,2l} = -bg_{2k,2l-1} \]

for \(1 \leq k, l \leq n\).

We identify the subgroup \(H_{2n-1}\) of \(G_{2n-1}\) with

\[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & x \end{pmatrix} \mid x \in G_{2n-2} \]
and so on. Clearly, the subgroup \( K_{2n-2} = G_{2n-2} \cap K_{2n} \) of \( G_{2n-2} \) is isomorphic to \( GL(n-1, q^2) \).

**3.3.2. Lemma.** Let \( (1_{K_{2n}})^{G_{2n}} = \sum_{i=1}^{k} \chi_{\mu_{i}} \) and \( (1_{K_{2n-2}})^{G_{2n-2}} = \sum_{j=1}^{l} \chi_{\lambda_{j}} \). Then we have

\[
\sum_{i=1}^{k} \sum_{\nu \dashv \mu} \chi_{\nu} = \sum_{j||\nu||=2n-1=1||\nu}^{l} \lambda_{j} \dashv \nu \sum_{||=2n-1}, \chi_{\nu}.
\]

**3.4. Proof of 3.3.2.** First of all, notice that an element \( g \) in \( G_{2n} \) belongs to \( H_{2n} \) if and only if \( gv_1 = v_1 \). Hence we have

\[
H_{2n} \cap K_{2n} \cong F_{q^{2}}^{n-1} \times GL(n-1, q^{2}),
\]

from which it follows that \( |H_{2n}K_{2n}| = |G_{2n}| \), that is,

\[
(9) \quad G_{2n} = H_{2n}K_{2n} = U_{2n-1}G_{2n-1}K_{2n}.
\]

Let \( \mathbb{C}[G_{2n}] \) be the complex group algebra of \( G_{2n} \). For any subgroup \( K \) of \( G_{2n} \), we define

\[
e_{K} = \frac{1}{|K|} \sum_{k \in K} k,
\]

then \( e_{K}^{2} = e_{K} \) and the left \( \mathbb{C}[G_{2n}] \)-module \( \mathbb{C}[G_{2n}]e_{K} \) affords the induced representation \( (1_{K})^{G_{2n}} \).

By virtue of 3.1.1 (i), in order to prove 3.3.2 it is enough to show that

**3.4.1. The left \( \mathbb{C}[G_{2n-1}] \)-module \( e_{U_{2n-1}} \mathbb{C}[G_{2n}]e_{K_{2n}} \) affords the induced representation \( (1_{U_{2n-1}K_{2n-2}})^{G_{2n-1}} = (1_{U_{2n-1}K_{2n-2}})^{H_{2n-1}} \uparrow_{H_{2n-1}}^{G_{2n-1}} \).**

From (9) it follows that \( e_{U_{2n-1}} \mathbb{C}[G_{2n}]e_{K_{2n}} \) is generated (as vector space) by the elements \( e_{U_{2n-1}} xe_{K_{2n}}, x \in G_{2n-1} \). Moreover, we have

\[
(10) \quad (U_{2n-1}K_{2n}) \cap G_{2n-1} = U_{2n-2}K_{2n-2}.
\]

In fact, if \( x \in G_{2n-1} \) is written as \( x = uk \) for some \( u \in U_{2n-1} \) and \( k \in K_{2n} \), then \( k \) is contained in \( H_{2n} \cap K_{2n} \). Since \( v_1 \) is fixed by \( k \), so is \( v_2 \). That is, \( k \) is of the form

\[
k = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & w \\ 0 & 0 & k_0 \end{pmatrix}
\]

where \( k_0 \in K_{2n-2} \), from which it follows that

\[
x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & w \\ 0 & 0 & k_0 \end{pmatrix} \in U_{2n-2}K_{2n-2}.
\]

Conversely, if \( x \) is written as above, then by 3.3.1 there exists \( z = (z_1, z_2, \ldots, z_{2n-2}) \) such that

\[
\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & w \\ 0 & 0 & k_0 \end{pmatrix} \in K_{2n}
\]

and therefore we have \( x \in U_{2n-1}K_{2n} \), as desired.
THE DECOMPOSITION OF THE PERMUTATION CHARACTER $1^{GL(2n,q)}_{GL(n,q^2)}$

It follows from (10) that for $x, y \in G_{2n-1}$ we have

(11) \[ e_{U_{2n-1}}x e_{K_{2n}} = e_{U_{2n-1}}y e_{K_{2n}} \Leftrightarrow xU_{2n-2}K_{2n-2} = yU_{2n-2}K_{2n-2}. \]

Hence, if $x_1, x_2, \ldots, x_t$ are representatives of the left cosets $xU_{2n-2}K_{2n-2}$ of $U_{2n-2}K_{2n-2}$ in $G_{2n-1}(\mathbb{C}G_{2n})$, then we have

\[ e_{U_{2n-1}}\mathbb{C}[G_{2n}]e_{K_{2n}} = \bigoplus_{j=1}^{t} V_j \]

as vector space over $\mathbb{C}$, where

\[ V_j = \mathbb{C} \cdot e_{U_{2n-1}}x_j e_{K_{2n}}. \]

Clearly, $G_{2n-1}$ acts on $\{V_j\}_{1 \leq j \leq t}$ transitively. Moreover, $U_{2n-2}K_{2n-2}$ is the stabilizer of $V_1$ in $G_{2n-1}$, and $V_1$ affords the trivial representation of $U_{2n-2}K_{2n-2}$. Thus, $e_{U_{2n-1}}\mathbb{C}[G_{2n}]e_{K_{2n}}$ affords the induced representation $(1_{U_{2n-2}K_{2n-2}})^{G_{2n-1}}$, which proves 3.4.1, and hence 3.3.2.

4. PROOF OF THEOREM 1.2.2

In this section, $q$ is assumed to be odd, as in §2. (When $q$ is even, the proof is similar and easier.)

4.1. We prove 1.2.2 (i) by induction on $n$. If $n = 0$, then this is clear. It follows from the induction hypothesis that

4.1.1. If $0 \leq m < n$, then we have $(1_{K_{2m}})^{G_{2m}} = \sum \chi_{\mu}$, summed over $\mu$ such that $||\mu|| = 2m$, $\bar{\mu} = \mu$, and $\mu(\varphi_1)' \cup \mu(\varphi_{-1})$ is even.

Let $(1_{K_{2m}})^{G_{2m}} = \sum_{i=1}^{k} \chi_{\mu_i}$, then from 1.2.1 it follows that $\bar{\mu}_i = \mu_i$ for all $i$. Since as mentioned before $\varphi_1$ and $\varphi_{-1}$ are the only elements $\varphi \in \Theta$ such that $d(\varphi) = 1$ and $\bar{\varphi} = \varphi$, therefore it follows from 3.3.2 that

4.1.2. If $\nu : \Theta \rightarrow \mathcal{P}$ satisfies $||\nu|| = 2n - 1$ and $\nu + \mu_i$ for some $i$, then one of the following holds:

(a) $\nu(\varphi_1)' \cup \nu(\varphi_{-1})$ is even and $\bar{\nu} \neq \nu,$
(b) $\nu(\varphi_1)' \cup \nu(\varphi_{-1})$ has exactly one odd part and $\bar{\nu} = \nu.$

Moreover,

(12) \[ \sum_{i=1}^{k} \sum_{||\nu|| = 2n-1 \text{ and } \nu + \mu_i}^{k} \chi_{\nu} \]

is multiplicity-free.

From 4.1.2 we immediately have

4.1.3. If an irreducible character $\chi_{\mu}$ of $G_{2n}$ with $\bar{\mu} = \mu$ is contained in $(1_{K_{2n}})^{G_{2n}}$, then one of the following holds:

(a) $\mu(\varphi_1)' \cup \mu(\varphi_{-1})$ is even,
(b) $l(\mu(\varphi_1)' \cup \mu(\varphi_{-1})) = 2.$
Let $\mu_\ast : \Theta \longrightarrow \mathcal{P}$ be a partition-valued function such that $||\mu_\ast|| = 2n$, $\bar{\mu}_\ast = \mu_\ast$, $\mu_\ast(\varphi_1) = (1^{2k})$ and $\mu_\ast(\varphi_{-1}) = 0$. For two partitions $\lambda, \rho \in \mathcal{P}$ such that $l(\lambda \cup \rho) \leq 2$ and $|\lambda| + |\rho| = 2k$, we define $\mu_{\lambda,\rho} : \Theta \longrightarrow \mathcal{P}$ by $\mu_{\lambda,\rho}(\varphi_1) = \lambda$, $\mu_{\lambda,\rho}(\varphi_{-1}) = \rho$, and $\mu_{\lambda,\rho}(\varphi) = \mu_\ast(\varphi)$ for all other $\varphi \in \Theta$. Then it follows that

$$d_{\mu_0,(2k)} > d_{\mu_0,(2k-1,1)} > d_{\mu_0,(2k-2,2)} > \cdots$$

In fact, from (2) it follows that

$$d_{\mu_0,(2k)} = \frac{q^{2k-1}}{q^{2k-1} - 1}.\frac{q - 1}{q^{2k-1} - 1}.$$

Then since

$$q^{2k-1}(q - 1) - (q^{2k-1} - 1) = q^{2k-1}(q - 2) + 1 > 0,$$

we have $d_{\mu_0,(2k)} > d_{\mu_0,(2k-1,1)}$. Next, for $1 \leq j \leq k - 1$ it follows that

$$d_{\mu_0,(2k-j,j)} = q^{2k-2j-1} - \frac{(q^{2k-2j+1} - 1)(q^{j+1} - 1)}{(q^{2k-j+1} - 1)(q^{2k-2j-1} - 1)}.$$

Since

$$q^{2k-2j-1}(q^{2k-2j+1} - 1)(q^{j+1} - 1) - (q^{2k-j+1} - 1)(q^{2k-2j-1} - 1) > q^{4k-3j}(q^{2k-2j} - 1) - q^{2k-j} - 1 = 0,$$

we have $d_{\mu_0,(2k-j,j)} > d_{\mu_0,(2k-j-1,j+1)}$, as desired.

4.1.4. Let $\lambda, \rho \in \mathcal{P}$ be as above, and suppose that $\chi_{\mu_\ast}$ is contained in $(1_{K_{2n}})^{G_{2n}}$. Then

(a) if $\lambda \neq 0$ then $\chi_{\mu_{\lambda,\rho}}$ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if $\lambda' \cup \rho$ is even,

(b) if $\lambda = 0$ then exactly one of the following occurs:

(b1) $\chi_{\mu_{0,\rho}}$ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if $\rho$ is even,

(b2) $\chi_{\mu_{0,\rho}}$ is contained in $(1_{K_{2n}})^{G_{2n}}$ if and only if $\rho$ is odd.

Proof. For two partitions $\beta, \gamma \in \mathcal{P}$ such that $l(\beta' \cup \gamma) \leq 2$ and $|\beta| + |\gamma| = 2k - 1$, we also define $\nu_{\beta,\gamma} : \Theta \longrightarrow \mathcal{P}$ such that $||\nu|| = 2n - 1$ by $\nu_{\beta,\gamma}(\varphi_1) = \beta$, $\nu_{\beta,\gamma}(\varphi_{-1}) = \gamma$, and $\nu_{\beta,\gamma}(\varphi) = \mu_\ast(\varphi)$ for all other $\varphi \in \Theta$. First of all, since $\chi_{\nu_{(1^{2k-1})}} \in (1_{K_{2n}})^{G_{2n}}$ appears in (12) and $\nu_{(1^{2k-1}),0} \ni \mu_{(1^{2k})}$, therefore neither $\chi_{\mu_{(1^{2k-2,2})}}$, nor $\chi_{\mu_{(1^{2k-1},1)}}$ is contained in $(1_{K_{2n}})^{G_{2n}}$. Next, since $\chi_{\nu_{(1^{2k-2,2})}}$ appears in (12) by 3.3.2, it follows from 4.1.3 that $\chi_{\mu_{(1^{2k-1})}}$ must be contained in $(1_{K_{2n}})^{G_{2n}}$, and so on. □

4.1.5. Let $1 \leq k \leq n$ and let $\mu_\ast : \Theta \longrightarrow \mathcal{P}$ be a partition-valued function such that $||\mu_\ast|| = 2n$, $\bar{\mu}_\ast = \mu_\ast$, $\mu_\ast(\varphi_1) = (1^{2k})$ and $\mu_\ast(\varphi_{-1}) = 0$. Then $\chi_{\mu_\ast}$ is contained in $(1_{K_{2n}})^{G_{2n}}$.

Proof. We prove 4.1.5 by induction on $k$, starting from $k = n$ and ending with 1. When $k = n$, this is trivial. Let $2 \leq k \leq n$ and assume that the assertion is true for all $l$ such that $k \leq l \leq n$. Let $\nu_\ast : \Theta \longrightarrow \mathcal{P}$ be a partition-valued function such that $||\nu_\ast|| = 2n - 1$, $\nu_\ast(\varphi_1) = (1^{2k-1})$ and $\nu_\ast(\varphi_{-1}) = 0$. If the restriction $\chi_{\mu} \uparrow_{G_{2n}}^{G_{2n-1}}$ of an irreducible constituent $\chi_{\mu}$ of $(1_{K_{2n}})^{G_{2n}}$ to $G_{2n-1}$ contains $\chi_{\nu_\ast}$,
then by 3.1.2, 4.1.3 and 4.1.4 it follows that $\mu(\varphi_1) = (1^{2k})$ or $\mu(\varphi_1) = (1^{2k-2})$, and $\mu(\varphi_{-1}) = (2j)$ for some $j \geq 0$. Hence, we have

\begin{equation}
(14) \quad ((K_{2n})^{G_{2n}} \downarrow_{G_{2n-1}}^{G_{2n}}, \chi_{\nu_{-}})_{G_{2n-1}} = (\sum_{\mu} \chi_{\mu} \downarrow_{G_{2n-1}}^{G_{2n}}, \chi_{\nu_{-}})_{G_{2n-1}}
\end{equation}

where the sum on the right is over $\mu$ such that $||\mu|| = 2n$, $\bar{\mu} = \mu$, $\mu(\varphi_1) = (1^{2k})$ or $\mu(\varphi_1) = (1^{2k-2})$, and $\mu(\varphi_{-1}) = (2j)$ for some $j \geq 0$.

Now, for any $\lambda : \Theta \rightarrow \mathcal{P}$ such that $\lambda(\varphi_1) = (1^m)$ for some $m \geq 2$, we define $\lambda^- : \Theta \rightarrow \mathcal{P}$ by $\lambda^-\varphi_1 = (1^{m-2})$ and $\lambda^-\varphi = \lambda\varphi$ for all $\varphi \in \Theta$. Then it follows from 3.1.2 that the right-hand side of (14) is equal to

\begin{equation}
(\sum_{\mu} \chi_{\mu} \downarrow_{G_{2n-3}}^{G_{2n-3}}, \chi_{\nu_{-}})_{G_{2n-3}}
\end{equation}

summed over $\mu$ as above, which is also equal to

\begin{equation}
((K_{2n})^{G_{2n-2}} \downarrow_{G_{2n-3}}^{G_{2n-3}}, \chi_{\nu_{-}})_{G_{2n-3}} = q \cdot ((K_{2n-4})^{G_{2n-4}} \downarrow_{G_{2n-4}}^{G_{2n-3}}, \chi_{\nu_{-}})_{G_{2n-3}}
\end{equation}

\begin{equation}
= q \cdot ((K_{2n})^{G_{2n-2}} \downarrow_{G_{2n-2}}^{G_{2n-1}}, \chi_{\nu_{-}})_{G_{2n-1}}
\end{equation}

\begin{equation}
= (1(K_{2n})^{G_{2n}} \downarrow_{G_{2n-1}}^{G_{2n}}, \chi_{\nu_{-}})_{G_{2n-1}}
\end{equation}

where the first and the third equalities follow from 3.2.1. Hence, if $\mu : \Theta \rightarrow \mathcal{P}$ satisfies $||\mu|| = 2n$, $\bar{\mu} = \mu$, $\mu(\varphi_1) = (1^{2k-2})$ and $\mu(\varphi_{-1}) = 0$, then since $(\chi_{\mu} \downarrow_{G_{2n-1}}^{G_{2n}}, \chi_{\nu_{-}})_{G_{2n-1}} > 0$ for at least one such $\nu_{-}$ as above, therefore $\chi_{\mu_{-}}$ must be contained in $(1(K_{2n})^{G_{2n}}$.

The proof of 1.2.2 (i) can now be rapidly completed. Let $\mu : \Theta \rightarrow \mathcal{P}$ be a partition-valued function such that $||\mu|| = 2n$ and $\bar{\mu} = \mu$. Then 4.1.5 and 4.1.4 imply that if $\mu(\varphi_1) = 0$ or $l(\mu(\varphi_1) \cup \mu(\varphi_{-1})) \geq 3$ then $\chi_{\mu}$ is contained in $(1(K_{2n})^{G_{2n}}$ if and only if $\mu(\varphi_1) \cup \mu(\varphi_{-1})$ is even. Also, if $\mu(\varphi_1) = 0$ and $l(\mu(\varphi_{-1})) \leq 2$ then there are two possibilities. However, by virtue of 2.1.1 and (13), we can conclude that in this case $\chi_{\mu}$ is contained in $(1(K_{2n})^{G_{2n}}$ if and only if $\mu(\varphi_{-1})$ is even. It also follows from 2.1.1 that $(1(K_{2n})^{G_{2n}}$ contains all irreducible characters $\chi_{\mu}$ of $G_{2n}$ such that $\bar{\mu} = \mu$ and $\mu(\varphi_1) = \mu(\varphi_{-1}) = 0$.

4.2. Finally, we prove 1.2.2 (iii). The left-hand side of (3) is by 2.1.2 equal to

\begin{equation}
\prod_{r \geq 1} (1-t^{2r})^{-1} \prod_{r \geq 1} (1-t^{2r})^{-|\Psi_1|-2} \prod_{k \geq 2} \prod_{r \geq 1} (1-t^{2kr})^{-|\Psi_k|}
\end{equation}

\begin{equation}
\prod_{k \geq 1} \prod_{r \geq 1} (1-t^{2kr})^{-|\Psi_k|} = \prod_{r \geq 1} (1-qt^{2r})^{-1}.
\end{equation}

This completes the proof of 1.2.2.

REFERENCES


