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<tr>
<td>Author(s)</td>
<td>Mazalov, Vladimir V.; Zabelin, Anatoliy A.</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2002), 1263: 117-130</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42040">http://hdl.handle.net/2433/42040</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Discrete final-offer arbitration model

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Abstract

A bargaining problem with two players Labor (player L) and Management (player M) is considered. The players must decide the monthly wage paid to L by M. At the beginning players L and M submit their offers $s_1$ and $s_2$. If $s_1 \leq s_2$ there is an agreement at $(s_1 + s_2)/2$. If not, the arbitrator is called in and he chooses the offer which is nearest for his solution $\alpha$. We suppose that a solution $\alpha$ is concentrated in two points $a, 1 - a$ at the interval $[0,1]$ with probabilities $p, q = 1 - p$. The equilibrium in the arbitration game among pure and mixed strategies is derived.

Key words: bargaining problem, arbitration, equilibrium strategy.

AMS Subject Classification: 91A05, 91A80, 91B26.

1 Introduction

We consider a zero-sum game related with a model of the labor-management negotiations using an arbitration procedure. Imagine that two players: Labor (player $L$) and Management (player $M$) bargain on a wage bill which has to be in the range $[0,1]$ where the current wage bill is normalised at zero, and the known maximum management ability to pay is at 1. Player $L$ is interested to maximize a wage bill as much as possible and the player $M$ has the opposite goal.

At the beginning the players $L$ and $M$ submit their offers $s_1$ and $s_2$ respectively, $s_1, s_2 \in [0,1]$. If $s_1 \leq s_2$ there is an agreement at $(s_1 + s_2)/2$. If not, the arbitrator $A$ is called in and he has to choose one of the decisions.

There are different approaches in analyzing the arbitration models [1-6]. We consider here the final-offer arbitration procedure [3] which allows the arbitrator only to choose one of the two final offers made by the players. We suppose here that the arbitrator imposes a solution $\alpha$ which is random variable being concentrated in two points $a$ and $b = 1 - a$ with different probabilities $p$ and $q = 1 - p$, $0 \leq a, p \leq 1$. The arbitrator chooses the offer which is nearest for his solution $\alpha$. The solution of this game with equal $p = q = 1/2$ was obtained in [6]. In this paper we obtain the solution of this game where $p$ and $q$ can be non-equal.
So, we have a zero-sum game determined in the unit square where the strategies of players $L$ and $M$ are the real numbers $s_1, s_2 \in [0, 1]$ and payoff function in this game has form $H(s_1, s_2) = E H_\alpha(s_1, s_2)$, where

$$H_\alpha(s_1, s_2) = \begin{cases} 
\frac{(s_1 + s_2)}{2}, & \text{if } s_1 \leq s_2 \\
 s_1, & \text{if } s_1 > s_2, |s_1 - \alpha| < |s_2 - \alpha| \\
 s_2, & \text{if } s_1 > s_2, |s_1 - \alpha| > |s_2 - \alpha| \\
 \alpha, & \text{if } s_1 > s_2, |s_1 - \alpha| = |s_2 - \alpha| 
\end{cases}$$

(1)

Below we show that the equilibrium in this game in dependence on value $\alpha$ can be among pure (section 2) and mixed (sections 3-4) strategies.

## 2 Solution of the game. Pure strategies

**Theorem 1.** Let $p \in (0, 0.5]$ and $a \in [0, p/2]$. Equilibrium consists of pure strategies and has form $s_1^* = 1, s_2^* = 0$. The value of the game $v = q$.

**Proof.** Let player II uses $s_2 = 0$. The payoff of player I is equal to:

for $s_1 \in [0, 2a) H(s_1, 0) = ps_1 + qs_1 = s_1 < 2a \leq p \leq q$,

for $s_1 = 2a H(2a, 0) = pa + (1-p)2a = (2-p)a < 2a \leq p \leq q$,

for $s_1 \in (2a, 1] H(s_1, 0) = p0 + q s_1 = qs_1$.

The maximum of the function is reached for $s_1 = 1$ and equals to $q$. Now, suppose that player I uses $s_1 = 1$. For $s_2 \in [0, 1-2a) H(1, s_2) = ps_2 + q s_2 = s_2 < 2a \leq p \leq q$.

According to condition $p \geq 2a$ we have $s_2 \geq 1 - 2a > 1 - p = q$. So, for all $s_2 H(1, s_2) \geq q$ and $H(s_1, 0) \leq q$ for all $s_1$. Hence, $\{s_1 = 1, s_2 = 0\}$ is an equilibrium in the game and $v = q$.

Analogous arguments leads to

**Theorem 2.** Let $p \in (0.5, 1)$ and $a \in [0, q/2]$. Equilibrium consists of pure strategies and has form $s_1^* = 1, s_2^* = 0$, and value of the game $v = q$.

## 3 Method for obtaining the equilibrium among mixed strategies

In case $a > \min \{p/2, q/2\}$ equilibrium consists of mixed strategies, i.e. randomised strategies of players L and M. Denote $F_1(s_1)$ and $F_2(s_2)$ distribution functions of the strategies for L and M, respectively. Suppose, that $F_1(s_1) \left[ F_2(s_2) \right]$ is continuous and its support consists of two intervals $(\alpha_1; \alpha_2]$ and $(\alpha_3; \alpha_4]$ at the $[0; 1]$ with $\alpha_2 \leq \alpha_3 \left[ \beta_2 \leq \beta_3 \right]$.
In extreme points of the interval \([0; 1]\) functions \(F_1(s_1)\) and \(F_2(s_2)\) can have a gap. Let also \(\beta_4 \leq \alpha_1, F_1(\alpha_1) = 0\) and \(F_2(\beta_4) = 1\).

Let \(F_{1,12}(s_1)\) and \(F_{1,34}(s_1)\) denote the form of \(F_1(s_1)\) at the intervals \((\alpha_1; \alpha_2]\) and \((\alpha_3; \alpha_4]\); and, respectivcly, \(F_{2,12}(s_2)\) and \(F_{2,34}(s_2)\) - for the function \(F_2(s_2)\) at \((\beta_1; \beta_2]\) and \((\beta_3; \beta_4]\).

Firstly, consider the case \(p \leq 0.5\). Admit, that the intervals \((\alpha_1; \alpha_2]\) and \((\beta_1; \beta_2]\) are symmetric in respect on the point \(a\) and the intervals \((\alpha_3; \alpha_4]\) and \((\beta_3; \beta_4]\) are symmetric in respect on \(b\). Otherwords,

\[
\alpha_1 = 2a - \beta_2, \quad \beta_1 = 2a - \alpha_2, \quad \alpha_4 = 2b - \beta_3, \quad \beta_4 = 2b - \alpha_3. \tag{2}
\]

Suppose, that player \(L\) (M) uses a mixed strategy \(F_1(s_1)(F_2(s_2))\) and consider the payoffs of the players.

For \(s_1 \in (\alpha_1; \alpha_2]\),

\[
H(s_1, F_2(s_2)) = p \left\{ s_1 F_{2,12}(2a - s_1) + \int_{2a-s_1}^{\beta_2} s_2 dF_{2,12}(s_2) + \int_{\beta_3}^{2b-\alpha_3} s_2 dF_{2,34}(s_2) \right\} + qs_1. \tag{3}
\]

For \(s_1 \in (\alpha_3; \alpha_4]\),

\[
H(s_1, F_2(s_2)) = p \left\{ 0 \cdot F_2(0) + \int_{2a-\alpha_2}^{\beta_2} s_2 dF_{2,12}(s_2) + \int_{\beta_3}^{2b-\alpha_3} s_2 dF_{2,34}(s_2) \right\} + q \left\{ s_1 F_{2,34}(2b - s_1) + \int_{2b-s_1}^{2b-\alpha_3} s_2 dF_{2,34}(s_2) \right\}. \tag{4}
\]

For \(s_2 \in (\beta_1; \beta_2]\),

\[
H(F_1(s_1), s_2) = p \left\{ \int_{2a-s_2}^{\alpha_2} s_1 dF_{1,12}(s_1) + s_2 (1 - F_{1,12}(2a - s_2)) \right\} + q \left\{ \int_{2a-\beta_2}^{\alpha_2} s_1 dF_{1,12}(s_1) + \int_{\alpha_3}^{2b-\beta_3} s_1 dF_{1,34}(s_1) + 1 \cdot (1 - F_1(1)) \right\}. \tag{5}
\]

For \(s_2 \in (\beta_3; \beta_4]\),

\[
H(F_1(s_1), s_2) = ps_2 + q \left\{ \int_{2a-\beta_2}^{\alpha_2} s_1 dF_{1,12}(s_1) + \int_{\alpha_3}^{2b-s_2} s_1 dF_{1,34}(s_1) + s_2 (1 - F_{1,34}(2b - s_2)) \right\}. \tag{6}
\]

If \(F_1^*(s_1), F_2^*(s_2)\) are optimal then the equations \(H(s_1, F_2^*(s_2)) = v\) and \(H(F_1^*(s_1), s_2) = v\), must be satisfied in the support-intervals where \(v\)-value of the game. Hence,

\[H(s_1, F_2^*(s_2)) = v, \quad s_1 \in (\alpha_1; \alpha_2] \cup (\alpha_3; \alpha_4],\]
\[ H(F_1^*(s_1), s_2) = v, \quad s_2 \in (\beta_1; \beta_2] \cup (\beta_3; \beta_4]. \]

From here,
\[
\frac{\partial H(s_1, F_2^*(s_2))}{\partial s_1} = 0, \quad s_1 \in (\alpha_1; \alpha_2] \cup (\alpha_3; \alpha_4], \\
\frac{\partial H(F_1^*(s_1), s_2)}{\partial s_2} = 0, \quad s_2 \in (\beta_1; \beta_2] \cup (\beta_3; \beta_4].
\]

Finding the derivative of (3-4) in \( s_1 \) and putting it equal to 0, and using the admission that \( F_2^*(\beta_4) = 1 \) and \( F_2^*(s_2) \) is continuous at \([\beta_2; \beta_3]\), consequently, \( F_2^*(\beta_2) = F_2^*(\beta_3) \), we obtain the system of differential equations with boundary conditions:
\[
\begin{align*}
p \left\{ 2(s_1 - a)F_{2,12}^*(2a - s_1) - F_{2,12}^*(2a - s_1) \right\} - q = 0, \quad s_1 \in (\alpha_1; \alpha_2], \\
q \left\{ 2(b - s_1)F_{2,34}^*(2b - s_1) + F_{2,34}^*(2b - s_1) \right\} = 0, \quad s_1 \in (\alpha_3; \alpha_4], \\
F_{2,34}^*(\beta_4) = 1, \quad F_{2,12}^*(\beta_2) = F_{2,12}^*(\beta_3).
\end{align*}
\]

Changing the arguments \( t_1 = 2a - s_1, t_1 \in (\beta_1; \beta_2] \) in the first equation and \( t_2 = 2b - s_1, t_2 \in (\beta_3; \beta_4] \) in the second one we obtain the system:
\[
\frac{dt_1}{2(a-t_1)} = \frac{dF_{2,12}^*}{F_{2,12}^* + p/q}, \quad \frac{dt_2}{2(b-t_2)} = \frac{dF_{2,34}^*}{F_{2,34}^*}.
\]

The solution which satisfies the boundary conditions has the following form
\[
F_2^*(s_2) = \begin{cases} 
0, & \text{if } s_2 \leq 2a - \alpha_2, \\
\left( \frac{\sqrt{a - \beta_2}}{\sqrt{a - s_2}} + \frac{2}{p} \right) \sqrt{\frac{a - \beta_2}{a - s_2}} - \frac{2}{p}, & \text{if } 2a - \alpha_2 < s_2 \leq \beta_2, \\
\left( \frac{\sqrt{a - \beta_3}}{\sqrt{a - s_2}} \right), & \text{if } \beta_2 < s_2 \leq \beta_3, \\
\left( \frac{\sqrt{a - s_2}}{\sqrt{a - \beta_3}} \right), & \text{if } \beta_3 < s_2 \leq 2b - \alpha_3, \\
1, & \text{if } 2b - \alpha_3 < s_2.
\end{cases}
\]

Finding the derivative of (5-6) in \( s_2 \) and putting it equal to 0, and using the admission \( F_1^*(\alpha_1) = 0 \) and \( F_1^*(\alpha_2) = F_1^*(\alpha_3) \), we obtain the system:
\[
\begin{align*}
p \left\{ 1 - F_{1,12}^*(2a - s_2) - 2(a - s_2)F_{1,12}^*(2a - s_2) \right\} = 0, \quad s_2 \in (\beta_1; \beta_2], \\
p + q \left\{ 1 - F_{1,34}^*(2b - s_2) - 2(b - s_2)F_{1,34}^*(2b - s_2) \right\} = 0, \quad s_2 \in (\beta_3; \beta_4], \\
F_{1,12}^*(\alpha_1) = 0, \quad F_{1,12}^*(\alpha_2) = F_{1,34}^*(\alpha_3).
\end{align*}
\]

Let change the arguments \( t_1 = 2a - s_2, t_1 \in (\alpha_1; \alpha_2] \) in the first equation, and \( t_2 = 2b - s_2, t_2 \in (\alpha_3; \alpha_4] \) in the second equation:
\[
\frac{dt_1}{2(t_1 - a)} = \frac{dF_{1,12}^*}{1 - F_{1,12}^*}, \quad \frac{dt_2}{2(t_2 - b)} = \frac{dF_{1,34}^*}{1 + p/q + F_{2,34}^*}.
\]

The solution of the system:
\[
F_1^*(s_1) = \begin{cases} 
0, & \text{if } s_1 \leq 2a - \beta_2, \\
1 - \frac{\sqrt{a - \beta_2}}{\sqrt{a - s_1}}, & \text{if } 2a - \beta_2 < s_1 \leq \alpha_2, \\
1 - \frac{\sqrt{a - \beta_3}}{\sqrt{a - s_1}}, & \text{if } \alpha_2 < s_1 \leq \alpha_3, \\
1 + \frac{p}{q} - \left( \frac{\sqrt{a - \beta_2}}{\sqrt{a - \beta_3}} + \frac{2}{p} \right), \frac{\sqrt{a - \beta_2}}{\sqrt{a - s_1}} - \frac{2}{p}, & \text{if } \alpha_3 < s_1 \leq 2b - \beta_3, \\
1, & \text{if } 2b - \beta_3 < s_1.
\end{cases}
\]
Now let us substitute the functions (7)–(8) to (3) – (6). For $s_1 \in (\alpha_1; \alpha_2]$, 
\[ H_1 = H(s_1, F_2^*(s_2)) = p\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}((2a - \beta_2) - (2b - \beta_3)) + p\alpha_3 + q(2a - \beta_2). \]

For $s_1 \in (\alpha_3; \alpha_4]$, 
\[ H_2 = H(s_1, F_2^*(s_2)) = p\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}((2a - \beta_2) - (2b - \beta_3)) + p\alpha_3 + q(2a - \beta_2) - 
- p\alpha_2\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}\cdot\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} - q\alpha_2\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + q\alpha_3. \]

For $s_2 \in (\beta_1; \beta_2]$, 
\[ H_3 = H(F_1^*(s_1), s_2) = q\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}}((\alpha_2 - 2a) - (\alpha_3 - 2b)) + q\beta_2 - p(\alpha_3 - 2b) - 
- q\beta_3\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}\cdot\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} - p\beta_3\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} + p\beta_2 + q\theta \]

For $s_2 \in (\beta_3; \beta_4]$, 
\[ H_4 = H(F_1^*(s_1), s_2) = q\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}}((\alpha_2 - 2a) - (\alpha_3 - 2b)) + q\beta_2 - p(\alpha_3 - 2b), \]

where 
\[ \theta = \begin{cases} 
0, & \text{if } F_1^*(1) = 1, \\
-\frac{p}{q} + \left(\frac{\sqrt{\alpha_3 - b}}{\sqrt{\alpha_3 - a}} + \frac{\epsilon}{q}\right)\frac{\sqrt{\alpha_3 - b}}{\sqrt{a}}, & \text{if } F_1^*(1) < 1. 
\end{cases} \]

So, take place 
\[ H_2 = H_1 + \chi_1, \]
\[ H_3 = H_4 + \chi_2, \]

where 
\[ \chi_1 = -p\alpha_2\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}\cdot\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} - q\alpha_2\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + q\alpha_3, \]
\[ \chi_2 = -q\beta_3\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}\cdot\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} - p\beta_3\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} + p\beta_2 + q\theta. \]

But must be $H_1 = H_2 = v$ and $H_3 = H_4 = v$, hence 
\[ \chi_1 = 0, \]
\[ \chi_2 = 0, \]
\[ H_1 = H_4. \]

Below we will find a solution of the system (9) in different cases. The value of the equal 
\[ v = p\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} \cdot \left((2a - \beta_2) - (2b - \beta_3)\right) + p\alpha_3 + q(2a - \beta_2). \]
Denote $\frac{\sqrt{a-\beta_2}}{\sqrt{a_2-a}} = \frac{1}{x}$, $\alpha_{-3} = y$. After symplifications (9) can be rewritten:

\[-\frac{\alpha_2}{x}(py + q) + q_3 = 0,\]  
\[-\beta_3 y(q/x + p) + p_2 + q_3 = 0,\]  

\[p(y(2a - 2b - \beta_2 + \beta_3) + 2\alpha_3 - 2b) = q \left( \frac{1}{x}(\alpha_2 - \alpha_3 - 2a + 2b) + 2\beta_2 - 2a \right).\]  

If $F_1^*(1) = 1$ (or, $F_{1,34}^*(2b - \beta_3) = 1$), then $y = \frac{q}{x + p}$. Substituting it to (11) we receive $\beta_2 = \beta_3$. If $F_1^*(1) < 1$ (2b - \beta_3 = 1), then $\beta_3 = 2b - 1$, $y = \frac{\sqrt{a_3 - b}}{\sqrt{a}}$ and $q_3 = -p + (q/x + p)y$.

Analogously, if $F_2^*(0+) = 0$ (2a - \alpha_2 = 0), then $1/x(py + q) = q$. Substituting to (11), we receive $\alpha_2 = \alpha_3$. If $F_2^*(0+) > 0$ (2a - \alpha_2 = 0), then $\alpha_2 = 2a$ and $1/x = \frac{\sqrt{a_2 - \beta_2}}{\sqrt{a}}$.

Thus, taking place $F_1^*(1) = 1 \Rightarrow \beta_2 = \beta_3$ and $F_2^*(0+) = 0 \Rightarrow \alpha_2 = \alpha_3$.

Varying different collections of the values $F_1^*(1)$ and $F_2^*(0+)$ and demanding that the support of optimal strategies belongs to [0;1], we will obtain the form of optimal strategies depending on values of $a$ and $p$ (see Fig. 1).

### 4 Solution of the game. Mixed Strategies

#### 4.1 Equilibrium for $(p, a) \in D_1$

Suppose that $F_1^*(1) = 1$ and $F_2^*(0+) = 0$ (i.e. $\alpha_2 = \alpha_3 = A$, $\beta_2 = \beta_3 = B$). From the equations $\frac{1}{x} = \frac{\sqrt{a-B}}{\sqrt{A}}$, $y = \frac{\sqrt{A-B}}{\sqrt{a}}$ it follows

\[\alpha_2 = \alpha_3 = A = \frac{bx^2(1 + y^2) - ay^2(1 + x^2)}{x^2 - y^2}, \quad \beta_2 = \beta_3 = B = \frac{a(1 + x^2) - b(1 + y^2)}{x^2 - y^2}. \]

The first two equations in (11) give

\[
\begin{align*}
qx &= py + q, \\
y \left(\frac{q}{x} + p\right) &= p,
\end{align*}
\]

which positive solution is

\[x = \frac{p^2 + pq - q^2 + \sqrt{p^4 + 2p^3q - p^2q^2 + 2pq^3 + q^4}}{2pq}, \quad (13)\]

\[y = \frac{p^2 - pq - q^2 + \sqrt{p^4 + 2p^3q - p^2q^2 + 2pq^3 + q^4}}{2p^2}. \quad (14)\]

It is not difficult to check that it satisfies to the third equation in (11).

The values $x, y$ and (12) give the solution of the game iff the following system of inequalities be satisfied

\[\beta_1 \geq 0, \quad \alpha_4 \leq 1, \]

\[\beta_1 \geq 0, \quad \alpha_4 \leq 1, \]

\[\beta_1 \geq 0, \quad \alpha_4 \leq 1, \]
Theorem 3. For \((p, a) \in D_1\) the equilibrium is \((F_1^*, F_2^*)\) of the form (7–8) with parameters determined by (12–14). The value of the game: \(v = q(2a - \beta_2) + p\alpha_3 - 2p(2b - 1)f^\alpha f^{-} - i_2\).

Notice some properties of the solution:

\[
\lim_{p \to 0} a_1(p) = 0.4, \quad \lim_{p \to 0.5} a_1(p) = z^2,
\]

where \(z\) is the "golden section" of the interval \([0, 1]\). It follows from

\[
\lim_{p \to 0^+} x = 1, \quad \lim_{p \to 0} y = 0, \quad \lim_{p \to 0^+} x = \frac{\sqrt{5} + 1}{2}, \quad \lim_{p \to 0^+} y = z = \frac{\sqrt{5} - 1}{2}.
\]

Notice also, that for fixed \(p\) if \(a\) decreases then \(\alpha_4\) increases to 1 and reaches it for \(a = a_1(p)\) (to obtain it we can substitute \(a_1(p)\) instead of \(a\) to \(\alpha_4 = 2 - 2a - \beta_3\)). For values \(a \leq a_1(p)\), the solution of the game is different.

### 4.2 Equilibrium for \((p, a) \in D_2\)

If \(F_1^*(1) < 1\) and \(F_2^*(0+) = 0\) (or, equivalently, \(\alpha_2 = \alpha_3 = A, \beta_2 = B, \beta_3 = 2b - 1\)), then from the equations \(\frac{1}{x} = \frac{A-B}{\sqrt{a}}\) and \(y = \frac{\sqrt{a} - \beta}{\sqrt{a}}\) we obtain

\[
\alpha_2 = \alpha_3 = A = ay^2 + b, \quad \beta_2 = B = \frac{a(1 + x^2) - (ay^2 + b)}{x^2}.
\]
The first two equations of (11) take form

\[
\begin{align*}
qx &= py + q, \\
2ay \left( \frac{q}{x} + p \right) &= p(1 - B).
\end{align*}
\]

From the first equation it follows \( x = \frac{py + q}{q} \). Substituting it to the second equation we receive after simplification

\[
(2y^3a p^3 + (-p^3 + 4a p^2 q + paq^2 + ap^3)y^2 + (2aq^3 - 2p^2 q + 2paq^2 + 2ap^2 q)y + 3paq^2 - 2pq^2)/(py + q)^2 = 0.
\]

Substituting it to the third equation in (11) we obtain

\[
y(2y^3a p^3 + (-p^3 + 4a p^2 q + paq^2 + ap^3)y^2 + (2aq^3 - 2p^2 q + 2paq^2 + 2ap^2 q)y + 3paq^2 - 2pq^2)/(py + q)^2 = 0.
\]

It is sufficient to find only positive roots of (17).

Denoting \( \lambda = p/q \) we have

\[
2a\lambda^3y^3 + \lambda(a + 4a\lambda - \lambda^2 + a\lambda^2)y^2 + (2a + a\lambda - \lambda^2 + a\lambda^2)y + \lambda(3a - 2) = 0.
\]

Denote the cubic polynomial at the left side of (18) as \( \nu(y) \), \( \nu(0) = \lambda(3a - 2) < 0 \), \( a \in [0; 0.5) \). The coefficient in higher degree of \( y \) in (18) is positive, hence, at least one positive root exists. From here also follows that the maximum lies before minimum. The function \( \nu = \nu(y) \) has two extreme points \( y_1 = \frac{1}{3} \left( \frac{1}{a} - \frac{1 + \lambda + \lambda^2}{\lambda^2} \right) \) and \( y_2 = -\frac{1}{\lambda} < 0 \). With \( \nu(0) < 0 \) it gives the uniqueness of the positive root of (18).

The solution takes place in case of \( \beta_1 \geq 0 \), or \( a(3 - y^2) \geq 1 \). It determines the lower border \( a_2(p) \) of the region \( D_2 \) on the plane \( (p, a) \).

**Theorem 4.** For \( (p, a) \in D_2 \) the equilibrium is \( (F_1^*, F_2^*) \) of the form (7–8) with parameters determined by (15–17). The value of the game: \( v = q(2a - \beta_2) + p\alpha_3 - p(2b - 1 + \beta_2)\sqrt{\frac{a_3 - \beta}{a}} \).

In case \( a < a_2(p) \) the following solution will take place.

**4.3 Equilibrium for \( (p, a) \in D_3 \)**

If \( F_1^*(1) < 1 \) and \( F_2^*(0+)) > 0 \) (or, equivalently, \( \alpha_2 = 2a, \beta_2 = 2b - 1, \alpha_3 = A, \beta_2 = B \), the first two equations in (11) with \( 1/x = \sqrt{a - B} \) and \( y = \sqrt{A - B} \) (or, \( \beta_2 = B = a - a/x^2 \) and \( \alpha_3 = A = ay^2 + b \)) take the form

\[
\begin{align*}
2a(py + q) &= q(ay^2 + b)x, \\
2ay \left( \frac{q}{x} + p \right) &= p \left( b + \frac{a}{x^2} \right).
\end{align*}
\]

From the first equation in (19) it follows \( x = \frac{2a(py + q)}{q(ay^2 + b)} \). Substituting it to the second equation in (19) and the third equation in (11) we obtain

\[
(3a^2y^4q^2p + (8a^2p^3 + 4a^2q^3)y^3 + (-2p a^2 q^2 - 4p^3a + 16a^2 p^2 q + 4a^2 p^3 + 2paq^2)y^2 +
\]
\[ + (8a^2p^2q + 8pa^2q^2 - 4a^2q^3 - 8p^2aq + 4q^3a) y + 3pa^2q^2 - pq^2 - 2paq^2)/(4a(py + q)^2) = 0. \]  \hspace{1cm} (20)

and

\[
y(3a^2y^4q^2p + (8a^2p^3 + 4a^2q^3)y^3 + (-2pa^2q^2 - 4p^3a + 16a^2p^2q + 4a^2p^3 + 2paq^2)y^2 + \\
(8a^2p^2q + 8pa^2q^2 - 4a^2q^3 - 8p^2aq + 4q^3a) y + 3pa^2q^2 - pq^2 - 2paq^2)/(4a(py + q)^2) = 0.
\]

It is sufficient to find only positive solutions of (20).

Denoting \( \lambda = p/q \) we rewrite (20) in the form

\[ 3a^2\lambda y^4 + 4a^2(1 + 2\lambda^3)y^3 + 2a\lambda((1-a)(1-2\lambda^2) + 8a\lambda)y^2 + \\
+ 4a(1-a + 2\lambda - 2\lambda^2 + 2a\lambda^2)y - (1-a)(1-2\lambda^2 + 8a\lambda) = 0. \]

Denote \( \nu(y) \) polynomials at the left side of the equation. Then \( \nu(0) = -(1-a)(1+3a)\lambda < 0 \), and because the coefficient in higher degree of \( y \) is positive then there exists at least one positive root of the equation. Let us show that it is unique. It follows from the fact that the points where \( \nu'(y) = 0 \) are negative.

\[ \nu'(y) = 36a^2\lambda y^2 + 24a^2(1+2\lambda^3)y + 4a\lambda((1-a)(1-2\lambda^2) + 8a\lambda). \]

If this parabola has no roots then \( \nu(y) \) is concave and the positive root is unique. Let there are two roots

\[ y_{1,2} = -a(1 + 2\lambda^3) \pm \sqrt{a(4a\lambda^6 - 2a\lambda^4 + 2\lambda^4 - 4a\lambda^3 + a\lambda^2 - \lambda^2 + a)}/3a\lambda. \]

The root \( y_1 \) is negative. Coefficient in higher degree of \( y \) of \( \nu''(y) \) is positive, hence, the largest root \( y_2 \) is negative, iff the coefficient in lower degree of \( \nu(y) \) is positive. It is equal to \( \xi(a,\lambda) = (1-a)(1-2\lambda^2) + 8a\lambda \). We have: \( \xi(a,0) = 1 - a > 0 \), the function \( \xi(a,\lambda) \) is convex in \( \lambda \), \( \xi(a,1) = 9a - 1 \). If \( a > \frac{1}{9} \), then \( \xi(a,\lambda) > 0 \), coefficient in lower degree in \( \nu''(y) \) is positive, \( y_2 \) is negative, hence, the positive root of the equation is unique.

The solution takes place, iff \( \beta_2 \geq 0 \) or \( \frac{a\sqrt{2} + b}{2a(1+\lambda b)} \leq 1 \). This inequality determines the lower border \( a_3(p) \) of the region \( D_3 \) on the plane \( (p,a) \). Notice, that in \( D_3 \) the inequality \( a < \frac{1}{9} \) is satisfied automatically.

**Theorem 5.** For \( (p,a) \in D_3 \) the equilibrium is \( (F_1^*, F_2^*) \) of the form (7–8) with parameters determined by (19–20). The value of the game: \( v = q(2a - \beta_2) + p\alpha_3 - p(2b - 1 + \beta_2)\sqrt{\frac{\alpha_3}{\sqrt{2}}} \).

For fixed \( p \), if \( a \) decreases from \( a_2(p) \) to \( a_3(p) \), then \( \beta \) decreases to zero. Finally, consider the case \( a < a_3(p) \).

### 4.4 Equilibrium for \( (p,a) \in D_4 \)

For \( \alpha_1 = \alpha_2 = 2a \), \( \alpha_4 = 1 \), \( \beta_1 = \beta_2 = 0 \), \( \beta_3 = 2b - 1 \) the optimal strategies are

\[
F_1^*(s_1) = \begin{cases} 
0, & \text{if } s_1 \leq \alpha_3, \\
\frac{q}{q}(1 - \frac{\sqrt{\alpha_3 - b}}{s_1 - b}), & \text{if } \alpha_3 < s_1 \leq 1, \\
1, & \text{if } 1 < s_1, 
\end{cases}
\]  \hspace{1cm} (21)
\[ F_2^*(s_2) = \begin{cases} 
0, & \text{if } s_2 \leq 0, \\
\frac{\sqrt{a_3}}{\sqrt{a}}, & \text{if } 0 < s_2 \leq 2b - 1, \\
\frac{\sqrt{a_3} - \delta}{\sqrt{a} - s_2}, & \text{if } 2b - 1 < s_2 \leq 2b - a_3, \\
1, & \text{if } 2b - a_3 < s_2. 
\end{cases} \]

Then, for \( s_1 \in (a_3; 1) \)

\[
H_2 = H(s_1, F_2^*(s_2)) = p \left\{ 0 \cdot F_2^*(0) + \int_{2b-1}^{2b-a_3} s_2 dF_2^*(s_2) \right\} + 
+ q \left\{ s_1 F_2^*(2b - s_1) + \int_{2b-s_1}^{2b-a_3} s_2 dF_2^*(s_2) \right\} = \alpha_3 - \frac{p\sqrt{a_3 - b}}{\sqrt{a}}.
\]

If \( s_2 = 0 \), then

\[
H_3 = H(F_1^*(s_1), s_2) = q \left\{ \int_{\alpha_3}^{1} s_1 dF_1^*(s_1) + 1 \cdot (1 - F_1^*(1)) \right\} = 2\sqrt{a}\sqrt{\alpha_3 - b} + 2b - \alpha_3 - p.
\]

If \( s_2 \in (2b-1; 2b-a] \), then

\[
H_4 = H(F_1^*(s_1), s_2) = ps_2 + q \left\{ \int_{\alpha_3}^{2b-s_2} s_1 dF_1^*(s_1) + s_2 (1 - F_1^*(2b - s_2)) \right\} = 2b - \alpha_3.
\]

\( F_1^*(s_1), F_2^*(s_2) \) be optimal iff

\[
\begin{cases} 
2b - a_3 = \alpha_3 - \frac{p\sqrt{a_3 - b}}{\sqrt{a}}, \\
2b - a_3 = 2\sqrt{a}\sqrt{a_3 - b} + 2b - \alpha_3 - p.
\end{cases}
\]

Solution of this system: \( \alpha_3 = b + \frac{p^2}{4a} \).

This form for \( H_2 - H_4 \) takes place, iff \( \alpha_3 \leq 1 \) or, equivalently, \( a > p/2 \). That determines the region \( D_4 \) on the plane \( (p, a) \).

**Theorem 6.** For \((p, a) \in D_4\) the equilibrium is \((F_1^*, F_2^*)\) of the form \((21-22)\). The value of the game: \( v = b - \frac{p^2}{4a} \).

The case \( a < p/2 \) was analysed in section 2.

## 5 Solution for \( p > 0.5 \)

At the begining we assumed \( p \leq 0.5 \). In case \( p > 0.5 \) the solution follows from the following theorem.

**Theorem 7.** Let for some fixed values of \( a \) and \( p \) we found the optimal strategies \( F_1^*(s_1, p, a) \) and \( F_2^*(s_2, p, a) \) in the game with \( P\{\alpha = a\} = p, \quad P\{\alpha = b\} = q, \quad a + b = 1, \quad p + q = 1, \quad a < b, \quad p \leq q. \)
Then the optimal strategies in the game for the same values $a, p$ and for

$$P\{\alpha = a\} = q, \quad P\{\alpha = b\} = p, \quad a + b = 1, \quad p + q = 1, \quad a < b, \quad p \leq q,$$

are

$$G_1^*(s_1, q, a) = 1 - F_2^*(1 - s_1, p, a), \quad G_2^*(s_2, q, a) = 1 - F_1^*(1 - s_2, p, a).$$

**Proof.** We have

$$G_1^*(s_1, q, a) = \begin{cases} 0, & \text{if } s_1 \leq 1 - 2b + \alpha_3, \\ 1 - \frac{\sqrt{\alpha_3 - b}}{\sqrt{1 - \alpha_3 - b}}, & \text{if } 1 - 2b + \alpha_3 < s_1 \leq 1 - \beta_3, \\ 1 - \frac{\sqrt{\alpha_3 - b}}{\sqrt{1 - \alpha_3 - b}}, & \text{if } 1 - \beta_3 < s_1 \leq 1 - \beta_2, \\ 1 + \frac{2}{p} \left( \frac{\sqrt{\alpha_3 - b}}{\sqrt{1 - \alpha_3 - b}} + \frac{q}{p} \right), & \text{if } 1 - \beta_2 < s_1 \leq 1 - 2a + \alpha_2, \\ 1, & \text{if } 1 - 2a + \alpha_2 < s_1, \end{cases}$$

and

$$G_2^*(s_2, q, a) = \begin{cases} 0, & \text{if } s_2 \leq 1 - 2b + \beta_3, \\ \frac{\sqrt{\alpha_3 - a}}{\sqrt{\alpha_3 - b}} + \frac{q}{p}, & \text{if } 1 - 2b + \beta_3 < s_2 \leq 1 - \alpha_3, \\ \frac{\sqrt{\alpha_3 - a}}{\sqrt{\alpha_3 - b}}, & \text{if } 1 - \alpha_3 < s_2 \leq 1 - \alpha_2, \\ \frac{\sqrt{\alpha_3 - a}}{\sqrt{\alpha_3 - b}}, & \text{if } 1 - \alpha_2 < s_2 \leq 1 - 2a + \beta_2, \\ 1, & \text{if } 1 - 2a + \beta_2 < s_2. \end{cases}$$

These functions will represent the optimal strategies, iff

$$H(s_1, G_2^*(s_2, q, a)) = \text{const for } s_1 \in (1 - 2b + \alpha_3; 1 - \beta_3] \cup (1 - \beta_2; 1 - 2a + \alpha_2],$$

$$H(G_1^*(s_1, q, a), s_2) = \text{const for } s_2 \in (1 - 2b + \beta_3; 1 - \alpha_3] \cup (1 - \alpha_2; 1 - 2a + \beta_2].$$

Denote $G_{1,12}^*(s_1)$ and $G_{1,34}^*(s_1)$ as the form of function $G_1^*(s_1, q, a)$ at the intervals $(1 - 2b + \alpha_3; 1 - \beta_3]$ and $(1 - \beta_2; 1 - 2a + \alpha_2]$ and $G_{2,12}^*(s_1), G_{2,34}^*(s_1)$ for the $G_2^*(s_1, q, a)$ at the intervals $(1 - 2b + \beta_3; 1 - \alpha_3], (1 - \alpha_2; 1 - 2a + \beta_2]$, respectively.

We obtain for $s_1 \in (1 - 2b + \alpha_3; 1 - \beta_3]$

$$H_1' = H(s_1, G_2^*(s_1, q, a)) = q \left\{ s_1 G_{2,12}^*(2a - s_1) + \int_{2a - s_1}^{1 - \alpha_3} s_2 dG_{2,12}^*(s_2) + \int_{1 - \alpha_3}^{1 - 2a + \beta_2} s_2 dG_{2,34}^*(s_2) \right\} +$$

$$+ p s_1 = q \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_3 - a}} ((\alpha_3 - 2b) - (\alpha_2 - 2a)) + p(\alpha_3 + 2a - 1) + q(1 - \beta_2).$$

If $s_1 \in (1 - \beta_2; 1 - 2a + \alpha_2],$ then

$$H_2' = H(s_1, G_2^*(s_1, q, a)) = q \left\{ 0 \cdot G_2^*(0, q, a) + \int_{1 - 2b + \beta_3}^{1 - \alpha_3} s_2 dG_{2,12}^*(s_2) + \int_{1 - \alpha_3}^{1 - 2a + \beta_2} s_2 dG_{2,34}^*(s_2) \right\} +$$

$$+ p \left\{ s_1 G_{2,34}^*(2b - s_1) + \int_{2b - s_1}^{1 - 2a + \beta_2} s_2 dG_{2,34}^*(s_2) \right\} =$$
\[
q \frac{\sqrt{a-b}}{\sqrt{a}} \left((\alpha_3 - 2b) - (\alpha_2 - 2a)\right) + p(\alpha_3 + 2a - 1) + q(1 - \beta_2) - q(1 - \beta_3) \frac{\sqrt{a-b}}{\sqrt{a}} + p(1 - \beta_2) - p(1 - \beta_3) \frac{\sqrt{a-b}}{\sqrt{b}}.
\]

If \(s_2 \in (1 - 2b + \beta_3; 1 - \alpha_3]\), then

\[
H'_3 = H(G_1^*(s_1, q, a), s_2) = q \left\{ \int_{1-2b+\alpha_3}^{2a-s_2} s_1 dG_{1,12}^*(s_1) + s_2 (1 - G_{1,12}^*(2a - s_2)) \right\} + p \left\{ \int_{1-2b+\alpha_3}^{1-\beta_3} s_1 dG_{1,12}^*(s_1) + \int_{1-\beta_3}^{1-2a+\alpha_2} s_1 dG_{1,34}^*(s_1) + 1 \cdot (1 - G_1^*(1)) \right\} =
\]

\[
p \frac{\sqrt{a-b}}{\sqrt{b-\beta_3}} ((2b - \beta_3) - (2a - \beta_2)) + p(1 - \alpha_3) - q(1 - 2b - \beta_2) - p(1 - \alpha_2) \frac{\sqrt{a-b}}{\sqrt{a-\beta_3}} + q(1 - \alpha_3) - q(1 - \alpha_2) \frac{\sqrt{a-b}}{\sqrt{a-\beta_3}} + p\eta.
\]

If \(s_2 \in (1 - \alpha_2; 1 - 2a + \beta_2]\), then

\[
H'_4 = H(G_1^*(s_1, q, a), s_2) = qs_2 + p \left\{ \int_{1-2b+\alpha_3}^{1-\beta_3} s_1 dG_{1,12}^*(s_1) + \int_{1-\beta_3}^{2b-s_2} s_1 dG_{1,34}^*(s_1) + s_2 (1 - G_{1,34}^*(2b - s_2)) \right\} =
\]

\[
p \frac{\sqrt{a-b}}{\sqrt{b-\beta_3}} ((2b - \beta_3) - (2a - \beta_2)) + p(1 - \alpha_3) - q(1 - 2b - \beta_2),
\]

where \(\eta = \begin{cases} 0, & \text{if } G_1^*(1) = 1, \\ -\frac{s}{p} + \left(\frac{\alpha_3 - b}{\sqrt{b-\beta_3}} + \frac{s}{p}\right) \frac{\sqrt{a-\beta_3}}{\sqrt{a}}, & \text{if } G_1^*(1) < 1. \end{cases}\)

We have

\[
\psi_1 = H'_2 - H'_1 = -q(1 - \beta_3) \sqrt{a-b} \frac{\sqrt{a}}{\sqrt{b-\beta_3}} + p(1 - \beta_2) - p(1 - \beta_3) \frac{\sqrt{a-b}}{\sqrt{b-\beta_3}},
\]

\[
\psi_2 = H'_3 - H'_4 = -p(1 - \alpha_2) \frac{\sqrt{a-b}}{\sqrt{b-\beta_3}} \sqrt{a} \frac{\sqrt{a}}{\sqrt{b-\beta_3}} + q(1 - \alpha_3) - q(1 - \alpha_2) \frac{\sqrt{a-b}}{\sqrt{a}} + p\eta.
\]

There are only four possible forms for the functions \(F_1^*(s_1, p, a)\) and \(F_2^*(s_2, p, a)\). With \(\chi_1 = \chi_2 = 0\), it gives:

1. For \(\alpha_2 = \alpha_3 = A, \beta_2 = \beta_3 = B\) take place \(\frac{\chi_1}{A} = \frac{\psi_1}{1-A}\) and \(\frac{\chi_2}{B} = \frac{\psi_2}{1-B}\), consequently, \(\psi_1 = \psi_2 = 0\).

2. For \(\alpha_2 = \alpha_3 = A, \beta_3 = 2b - 1\) take place \(\frac{\chi_1}{A} = \frac{\psi_1}{1-A}\) and \(\chi_2 = -\psi_1\), consequently, \(\psi_1 = \psi_2 = 0\).
For $\alpha_2 = 2a, \beta_3 = 1 - 2a$ take place $\chi_1 = -\psi_2$ and $\chi_2 = -\psi_1$, consequently, $\psi_1 = \psi_2 = 0$.

For $\alpha_1 = \alpha_2 = 2a, \alpha_4 = 1, \beta_1 = \beta_2 = 0, \beta_3 = 2b - 1$, the form of $G_1^*(s_1), G_2^*(s_2)$ is:

\[
G_1^*(s_1, q, a) = \begin{cases}
0, & \text{if } s_1 \leq a + \frac{p^2}{4a}, \\
1 - \frac{p}{2a}, & \text{if } 2a < s_1 \leq 1, \\
1, & \text{if } 1 < s_1,
\end{cases}
\]

\[
G_2^*(s_2, q, a) = \begin{cases}
0, & \text{if } s_2 \leq 0, \\
1 - \frac{1}{q} \left(1 - \frac{p}{2a} \right), & \text{if } 0 < s_2 \leq a - \frac{p^2}{4a}, \\
1, & \text{if } a - \frac{p^2}{4a} < s_2.
\end{cases}
\]

Then for $s_2 \in \left(0; a - \frac{p^2}{4a}\right]$:

\[
H(G_1^*(s_1, q, a), s_2) = q \left\{ \int_{a+\frac{p^2}{4a}}^{2a-s_2} s_1 dG_1^*(s_1, q, a) + s_2 (1 - G_1^*(2a-s_2, q, a)) \right\} + \\
+ p \left\{ \int_{a+\frac{p^2}{4a}}^{2a} s_1 dG_1^*(s_1, q, a) + 1 \cdot (1 - G_1^*(1, q, a)) \right\} = a + \frac{p^2}{4a}.
\]

For $s_1 \in \left(a + \frac{p^2}{4a}; 2a\right]$:

\[
H(s_1, G_2^*(s_2, q, a)) = q \left\{ s_1 G_2^*(a - s_1, q, a) + \int_{2a-s_1}^{a-\frac{p^2}{4a}} s_2 dG_2^*(s_2, q, a) \right\} + ps_1 = a + \frac{p^2}{4a}.
\]

Finally, for $s_1 = 1$

\[
H(s_1, G_2^*(s_2, q, a)) = q \left\{ \int_{0}^{a-\frac{p^2}{4a}} s_2 dG_2^*(s_2, q, a) + p = a + \frac{p^2}{4a} \right\}.
\]

In all cases the payoff is constant, and with $H_1 + H_4' = 1, H_4 + H_1' = 1$ and $H_1 = H_4$, \textit{gives} $H_1' = H_4'$, and all $H_i, i = 1, .., 4$ are equal. It proves the optimality $G_1^*(s_1, q, a)$ and $(s_2, q, a)$.

\textbf{Acknowledgements.}

The research was supported by the Russian Fund for Basic Research (projects 01-01-126, 01-01-00113). VM was supported also by the Japan Society for the Promotion of Science (grant L 01530).
References