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Discrete final-offer arbitration model

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Abstract

A bargaining problem with two players Labor (player L) and Management (player M) is considered.
The players must decide the monthly wage payed to L by M. At the begining players L and M
submit their offers $s_1$ and $s_2$. If $s_1 \leq s_2$ there is an agreement at $(s_1 + s_2)/2$. If not, the arbitrator
is called in and he chooses the offer which is nearest for his solution $\alpha$. We suppose that a solution
$\alpha$ is concentrated in two points $a, 1 - a$ at the interval $[0,1]$ with probabilities $p, q = 1 - p$. The
equilibrium in the arbitration game among pure and mixed strategies is derived.

Key words: bargaining problem, arbitration, equilibrium strategy.
AMS Subject Classification: 91A05, 91A80, 91B26.

1 Introduction

We consider a zero-sum game related with a model of the labor-management negotiations
using an arbitration procedure. Imagine that two players: Labor (player $L$) and Management
(player $M$) bargain on a wage bill which has to be in the range $[0,1]$ where the current wage
bill is normalised at zero, and the known maximum management ability to pay is at 1.
Player $L$ is interested to maximize a wage bill as much as possible and the player $M$ has the
opposite goal.

At the begining the players $L$ and $M$ submit their offers $s_1$ and $s_2$ respectively, $s_1, s_2 \in
[0,1]$. If $s_1 \leq s_2$ there is an agreement at $(s_1 + s_2)/2$. If not, the arbitrator $A$ is called in
and he has to choose one of the decisions.

There are different approaches in analyzing the arbitration models [1-6]. We consider
here the final-offer arbitration procedure [3] which allows the arbitrator only to choose one
of the two final offers made by the players. We suppose here that the arbitrator imposes a
solution $\alpha$ which is random variable being concentrated in two points $a$ and $b = 1 - a$ with
different probabilities $p$ and $q = 1 - p$, $0 \leq a, p \leq 1$. The arbitrator chooses the offer which
is nearest for his solution $\alpha$. The solution of this game with equal $p = q = 1/2$ was obtained
in [6]. In this paper we obtain the solution of this game where $p$ and $q$ can be non-equal.
So, we have a zero-sum game determined in the unit square where the strategies of players L and M are the real numbers $s_1, s_2 \in [0,1]$ and payoff function in this game has form $H(s_1, s_2) = EH_\alpha(s_1, s_2)$, where

$$H_\alpha(s_1, s_2) = \begin{cases} 
(s_1 + s_2)/2, & \text{if } s_1 \leq s_2 \\
 s_1, & \text{if } s_1 > s_2, |s_1 - \alpha| < |s_2 - \alpha| \\
 s_2, & \text{if } s_1 > s_2, |s_1 - \alpha| > |s_2 - \alpha| \\
 \alpha, & \text{if } s_1 > s_2, |s_1 - \alpha| = |s_2 - \alpha| 
\end{cases}$$

(1)

Below we show that the equilibrium in this game in dependence on value $a$ can be among pure (section 2) and mixed (sections 3-4) strategies.

2 Solution of the game. Pure strategies

**Theorem 1.** Let $p \in (0,0.5]$ and $a \in [0,p/2]$. Equilibrium consists of pure strategies and has form $s_1^* = 1$, $s_2^* = 0$. The value of the game $v = q$.

**Proof.** Let player II uses $s_2 = 0$. The payoff of player I is equal to:
for $s_1 \in [0,2a)$ $H(s_1,0) = ps_1 + qs_1 = s_1 < 2a \leq p \leq q$,
for $s_1 = 2a$ $H(2a,0) = pa + (1-p)2a = (2-p)a < 2a \leq p \leq q$,
for $s_1 \in (2a,1]$ $H(s_1,0) = 0 + qs_1 = q$s_1$.

The maximum of the function is reached for $s_1 = 1$ and equals to $q$. Now, suppose that player I uses $s_1 = 1$. For $s_2 \in [0,1-2a) H(1,s_2) = ps_2 + qs_2$. Minimum of this function lies in $s_2 = 0$ and equal to $q$. For $s_2 = 1-2a H(1,2a) = pa + (1-p)2a = (2-p)a < 2a \leq p \leq q$.

So, for all $s_2$ $H(1,s_2) \geq q$ and $H(s_1,0) \leq q$ for all $s_1$. Hence, $\{s_1 = 1, s_2 = 0\}$ is an equilibrium in the game and $v = q$.

Analogous arguments leads to

**Theorem 2.** Let $p \in (0.5,1)$ and $a \in [0,q/2]$. Equilibrium consists of pure strategies and has form $s_1^* = 1$, $s_2^* = 0$, and value of the game $v = q$.

3 Method for obtaining the equilibrium among mixed strategies

In case $a > \min\{p/2, q/2\}$ equilibrium consists of mixed strategies, i.e. randomised strategies of players L and M. Denote $F_1(s_1)$ and $F_2(s_2)$ distribution functions of the strategies for L and M, respectively. Suppose, that $F_1(s_1) F_2(s_2)$ is continuous and its support consists of two intervals $(\alpha_1;\alpha_2)$ and $(\alpha_3;\alpha_4)$ at the $[0;1]$ with $\alpha_2 \leq \alpha_3 \leq \beta_2 \leq \beta_3$. 
In extreme points of the interval [0; 1] functions $F_1(s_1)$ and $F_2(s_2)$ can have a gap. Let also $\beta_4 \leq \alpha_1$, $F_1(\alpha_1) = 0$ and $F_2(\beta_4) = 1$.

Let $F_{1,12}(s_1)$ and $F_{1,34}(s_1)$ denote the form of $F_1(s_1)$ at the intervals $(\alpha_1; \alpha_2]$ and $(\alpha_3; \alpha_4]$; and, respectively, $F_{2,12}(s_2)$ and $F_{2,34}(s_2)$ – for the function $F_2(s_2)$ at $(\beta_1; \beta_2]$ and $(\beta_3; \beta_4]$.

Firstly, consider the case $p \leq 0.5$. Admit, that the intervals $(\alpha_1; \alpha_2]$ and $(\beta_1; \beta_2]$ are symmetric in respect on the point $a$ and the intervals $(\alpha_3; \alpha_4]$ and $(\beta_3; \beta_4]$ are symmetric in respect on $b$. Otherwords,

$$\alpha_1 = 2a - \beta_2, \quad \beta_1 = 2a - \alpha_2, \quad \alpha_4 = 2b - \beta_3, \quad \beta_4 = 2b - \alpha_3. \quad (2)$$

Suppose, that player L (M) uses a mixed strategy $F_1(s_1)$ ($F_2(s_2)$) and consider the payoffs of the players.

For $s_1 \in (\alpha_1; \alpha_2]$, 

$$H(s_1, F_2(s_2)) = p \{s_1 F_{2,12}(2a - s_1) + \int_{2a-s_1}^{\beta_2} s_2 dF_{2,12}(s_2) + \int_{\beta_3}^{2b-a_s} s_2 dF_{2,34}(s_2)\} + qs_1. \quad (3)$$

For $s_1 \in (\alpha_3; \alpha_4]$, 

$$H(s_1, F_2(s_2)) = p \{0 \cdot F_2(0) + \int_{2a-\alpha_2}^{\beta_2} s_2 dF_{2,12}(s_2) + \int_{\beta_3}^{2b-a_3} s_2 dF_{2,34}(s_2)\}$$

$$+ q \left\{s_1 F_{2,34}(2b - s_1) + \int_{2b-s_1}^{2b-a_3} s_2 dF_{2,34}(s_2)\right\}. \quad (4)$$

For $s_2 \in (\beta_1; \beta_2]$, 

$$H(F_1(s_1), s_2) = p \left\{\int_{2a-s_2}^{2a} s_1 dF_{1,12}(s_1) + s_2 (1 - F_{1,12}(2a - s_2))\right\}$$

$$+ q \left\{\int_{2a-\beta_2}^{\alpha_2} s_1 dF_{1,12}(s_1) + \int_{\alpha_3}^{2b-s_2} s_1 dF_{1,34}(s_1) + 1 \cdot (1 - F_{1}(1))\right\}. \quad (5)$$

For $s_2 \in (\beta_3; \beta_4]$, 

$$H(F_1(s_1), s_2) = ps_2 + q \left\{\int_{2a-\beta_2}^{\alpha_2} s_1 dF_{1,12}(s_1) + \int_{\alpha_3}^{2b-s_2} s_1 dF_{1,34}(s_1) + s_2 (1 - F_{1,34}(2b - s_2))\right\}. \quad (6)$$

If $F_1^*(s_1), F_2^*(s_2)$ are optimal then the equations $H(s_1, F_2^*(s_2)) = v$ and $H(F_1^*(s_1), s_2) = v$, must be satisfied in the support-intervals where $v$-value of the game. Hence,

$$H(s_1, F_2^*(s_2)) = v, \quad s_1 \in (\alpha_1; \alpha_2] \cup (\alpha_3; \alpha_4],$$
\[ H(F_1^*(s_1), s_2) = v, \ s_2 \in (\beta_1; \beta_2] \cup (\beta_3; \beta_4]. \]

From here,
\[
\frac{\partial H(s_1, F_2^*(s_2))}{\partial s_1} = 0, \ s_1 \in (\alpha_1; \alpha_2] \cup (\alpha_3; \alpha_4],
\]
\[
\frac{\partial H(F_1^*(s_1), s_2)}{\partial s_2} = 0, \ s_2 \in (\beta_1; \beta_2] \cup (\beta_3; \beta_4].
\]

Finding the derivative of (3-4) in \( s_1 \) and putting it equal to 0, and using the admission that \( F_2^*(\beta_4) = 1 \) and \( F_2^*(s_2) \) is continuous at \( [\beta_2; \beta_3] \), consequently, \( F_2^*(\beta_2) = F_2^*(\beta_3) \), we obtain the system of differential equations with boundary conditions:
\[
\begin{align*}
\frac{\partial}{\partial t_1} \{2(s_1 - a)F_{2,12}^*(2a - s_1) - F_{2,12}^*(2a - s_1)\} - q &= 0, \ s_1 \in (\alpha_1; \alpha_2], \\
\frac{\partial}{\partial t_2} \{2(b - s_1)F_{2,34}^*(2b - s_1) + F_{2,34}^*(2b - s_1)\} &= 0, \ s_1 \in (\alpha_3; \alpha_4], \\
F_{2,34}^*(\beta_4) &= 1, \ F_{2,12}^*(\beta_2) = F_{2,12}^*(\beta_3).
\end{align*}
\]

Changing the arguments \( t_1 = 2a - s_1, t_1 \in (\beta_1; \beta_2] \) in the first equation and \( t_2 = 2b - s_1, t_2 \in (\beta_3; \beta_4] \) in the second one we obtain the system:
\[
\begin{align*}
\frac{dt_1}{2(a - t_1)} &= \frac{dF_{2,12}^*}{F_{2,12}^* + p/q}, \\
\frac{dt_2}{2(b - t_2)} &= \frac{dF_{2,34}^*}{1 + p/q + F_{2,34}^*}.
\end{align*}
\]

The solution which satisfies the boundary conditions has the following form
\[
F_2^*(s_2) = \begin{cases} 
0, & \text{if } s_2 \leq 2a - \alpha_2, \\
\frac{\sqrt{a - s_2} + \sqrt{a - \beta_2}}{\sqrt{a - \alpha_2}}, & \text{if } 2a - \alpha_2 < s_2 \leq \beta_2, \\
\sqrt{b - s_2}, & \text{if } \beta_2 < s_2 \leq \beta_3, \\
1, & \text{if } \beta_3 < s_2 \leq 2b - \alpha_3, \\
1 - \frac{\sqrt{a - \beta_3} - \sqrt{a - s}}{\sqrt{a - \alpha_3} + \sqrt{a - s}} & \text{if } 2b - \alpha_3 < s_2.
\end{cases}
\]

Finding the derivative of (5-6) in \( s_2 \) and putting it equal to 0, and using the admission \( F_1^*(\alpha_1) = 0 \) and \( F_1^*(\alpha_2) = F_1^*(\alpha_3) \), we obtain the system:
\[
\begin{align*}
p \{1 - F_{1,12}^*(2a - s_2) - 2(a - s_2)F_{1,12}^*(2a - s_2)\} &= 0, \ s_2 \in (\beta_1; \beta_2], \\
p + q \{1 - F_{1,34}^*(2b - s_2) - 2(b - s_2)F_{1,34}^*(2a - s_2)\} &= 0, \ s_2 \in (\beta_3; \beta_4], \\
F_{1,12}^*(\alpha_1) &= 0, \ F_{1,12}^*(\alpha_2) = F_{1,34}^*(\alpha_3).
\end{align*}
\]

Let change the arguments \( t_1 = 2a - s_2, t_1 \in (\alpha_1; \alpha_2] \) in the first equation, and \( t_2 = 2b - s_2, t_2 \in (\alpha_3; \alpha_4] \) in the second equation:
\[
\begin{align*}
\frac{dt_1}{2(t_1 - a)} &= \frac{dF_{1,12}^*}{1 - F_{1,12}^*}, \\
\frac{dt_2}{2(t_2 - b)} &= \frac{dF_{1,34}^*}{1 + p/q + F_{2,34}^*}.
\end{align*}
\]

The solution of the system:
\[
F_1^*(s_1) = \begin{cases} 
0, & \text{if } s_1 \leq 2a - \beta_2, \\
1 - \frac{\sqrt{a - \beta_2}}{\sqrt{a - s}}, & \text{if } 2a - \beta_2 < s_1 \leq \alpha_2, \\
1 - \frac{\sqrt{a - \beta_2}}{\sqrt{a - \alpha_2}}, & \text{if } \alpha_2 < s_1 \leq \alpha_3, \\
1 + \frac{\sqrt{a - \beta_3}}{\sqrt{a - \alpha_3} + 1} & \text{if } \alpha_3 < s_1 \leq 2b - \beta_3, \\
1, & \text{if } 2b - \beta_3 < s_1.
\end{cases}
\]
Now let us substitute the functions (7)–(8) to (3) – (6). For $s_1 \in (\alpha_1; \alpha_2]$,

$$H_1 = H(s_1, F_2^*(s_2)) = \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}((2a - \beta_2) - (2b - \beta_3)) + p\alpha_3 + q(2a - \beta_2).$$

For $s_1 \in (\alpha_3; \alpha_4]$,

$$H_2 = H(s_1, F_2^*(s_2)) = \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}((2a - \beta_2) - (2b - \beta_3)) + p\alpha_3 + q(2a - \beta_2) - \frac{p\alpha_2}{\sqrt{\alpha_3 - b}} \cdot \frac{\sqrt{a - \beta_2}}{\sqrt{a - \alpha_2}} - \frac{q\alpha_2}{\sqrt{\alpha_3 - b}} \cdot \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + q\alpha_3.$$

For $s_2 \in (\beta_1; \beta_2]$,

$$H_3 = H(F_1^*(s_1), s_2) = \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}}((\alpha_2 - 2a) - (\alpha_3 - 2b)) + q\beta_2 - p(\alpha_3 - 2b) - \frac{q\beta_3}{\sqrt{\alpha_3 - b}} \cdot \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} - \frac{p\beta_3}{\sqrt{\alpha_3 - b}} \cdot \frac{\sqrt{a - \beta_2}}{\sqrt{b - \beta_3}} + p\beta_2 + q\theta.$$

For $s_2 \in (\beta_2; \beta_3]$,

$$H_4 = H(F_1^*(s_1), s_2) = \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}}((\alpha_2 - 2a) - (\alpha_3 - 2b)) + q\beta_2 - p(\alpha_3 - 2b),$$

where

$$\theta = \begin{cases} 0, & \text{if } F_1^*(1) = 1, \\ \frac{p - \beta_2}{q} + \left(\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + \frac{p - \beta_2}{q}\right), & \text{if } F_1^*(1) < 1. \end{cases}$$

So, take place

$$H_2 = H_1 + \chi_1,$$

$$H_3 = H_4 + \chi_2,$$

where

$$\chi_1 = -p\alpha_2 \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} \cdot \frac{\sqrt{a - \beta_2}}{\sqrt{a - \alpha_2}} - q\alpha_2 \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + q\alpha_3,$$

$$\chi_2 = -q\beta_3 \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} \cdot \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} - p\beta_3 \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} + p\beta_2 + q\theta.$$
Denote \( \frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2^2-a}} = \frac{1}{x}, \alpha_{ovalbox 3} = y \). After symplifications (9) can be rewritten:

\[
-\frac{\alpha_2}{x}(py+q) + q\alpha_3 = 0, \\
-\beta_3y(q/x+p) + p\beta_2 + q\theta = 0,
\]

(11)

\[
p(y(2a-2b-\beta_2+\beta_3) + 2\alpha_3 - 2b) = q \left( \frac{1}{x}(\alpha_2 - \alpha_3 - 2a + 2b) + 2\beta_2 - 2a \right).
\]

If \( F_1^*(1) = 1 \) (or, \( F_{1,34}^*(2b-\beta_3) = 1 \), or \( \theta = 0 \)), then \( y(q/x+p) = p \). Substituting it to (11) we receive \( \beta_2 = \beta_3 \).

Varying different collections of the values \( F_1^*(1) \) and \( F_2^*(0+) \) and demanding that the support of optimal strategies belongs to \([0;1]\), we will obtain the form of optimal strategies depending on values of \( a \) and \( p \) (see Fig. 1).

### 4 Solution of the game. Mixed Strategies

#### 4.1 Equilibrium for \((p,a) \in D_1\)

Suppose that \( F_1^*(1) = 1 \) and \( F_2^*(0+) = 0 \) (i.e. \( \alpha_2 = \alpha_3 = A, \beta_2 = \beta_3 = B \)). From the equations

\[
\frac{1}{x} = \frac{\sqrt{a-\beta_2}}{\sqrt{A-\beta_3}}, \quad y = \frac{\sqrt{A-\beta_3}}{\sqrt{B-\alpha_2}}
\]

it follows

\[
\alpha_2 = \alpha_3 = A = \frac{bx^2(1+y^2) - ay^2(1+x^2)}{x^2-y^2}, \quad \beta_2 = \beta_3 = B = \frac{a(1+x^2) - b(1+y^2)}{x^2-y^2}.
\]

(12)

The first two equations in (11) give

\[
\begin{aligned}
q x &= py + q, \\
y \left( \frac{q}{x} + p \right) &= p,
\end{aligned}
\]

which positive solution is

\[
x = \frac{p^2 + pq - q^2 + \sqrt{p^4 + 2p^3q - p^2q^2 + 2pq^3 + q^4}}{2pq},
\]

(13)

\[
y = \frac{p^2 - pq - q^2 + \sqrt{p^4 + 2p^3q - p^2q^2 + 2pq^3 + q^4}}{2p^2}.
\]

(14)

It is not difficult to check that it satisfies to the third equation in (11).

The values \( x, y \) and (12) give the solution of the game iff the following system of inequalities be satisfied

\[
\beta_1 \geq 0, \quad \alpha_4 \leq 1,
\]
Theorem 3. For \((p,a) \in D_1\) the equilibrium is \((F_1^*, F_2^*)\) of the form (7–8) with parameters determined by (12–14). The value of the game: \(v = q(2a - \beta_2) + p\alpha_3 - 2p(2b - 1)\sqrt{\alpha_3 - 5} / \sqrt{5} - \beta_2\).

Notice some properties of the solution:

\[
\lim_{p \to 0} a_1(p) = 0.4, \quad \lim_{p \to 0.5} a_1(p) = z^2,
\]

where \(z\) is the "golden section" of the interval [0, 1]. It follows from

\[
\lim_{p \to 0+} x = 1, \quad \lim_{p \to 0+} y = 0, \quad \lim_{p \to 0.5-} x = \frac{\sqrt{5} + 1}{2}, \quad \lim_{p \to 0.5-} y = z = \frac{\sqrt{5} - 1}{2}.
\]

Notice also, that for fixed \(p\) if \(a\) decreases then \(\alpha_4\) increases to 1 and reaches it for \(a = a_1(p)\) (to obtain it we can substitute \(a_1(p)\) instead of \(a\) to \(\alpha_4 = 2 - 2a - \beta_3\)). For values \(a \leq a_1(p)\), the solution of the game is different.

### 4.2 Equilibrium for \((p,a) \in D_2\)

If \(F_1^*(1) < 1\) and \(F_2^*(0+) = 0\) (or, equivalently, \(\alpha_2 = \alpha_3 = A, \beta_2 = B, \beta_3 = 2b - 1\)), then from the equations \(\frac{1}{x} = \frac{\sqrt{a-B}}{A-a}\) and \(y = \frac{\sqrt{A+5}}{a}\) we obtain

\[
\alpha_2 = \alpha_3 = A = ay^2 + b, \quad \beta_2 = B = \frac{a(1 + x^2) - (ay^2 + b)}{x^2}.
\]
The first two equations of (11) take form

\[
\begin{aligned}
qx &= py + q, \\
2ay \left( \frac{q}{x} + p \right) &= p(1 - B).
\end{aligned}
\]  

(16)

From the first equation it follows \( x = \frac{py + q}{q} \). Substituting it to the second equation we receive after symplification

\[
(2y^3ap^3 + (-p^3 + 4ap^2q + paq^2 + ap^3)y^2 + 
+ (2aq^3 - 2p^2q + 2paq^2 + 2ap^2q)y + 3paq^2 - 2pq^2) / (py + q)^2 = 0.
\]

(17)

Substituting it to the third equation in (11) we obtain

\[
y (2y^3ap^3 + (-p^3 + 4ap^2q + paq^2 + ap^3)y^2 + 
+ (2aq^3 - 2p^2q + 2paq^2 + 2ap^2q)y + 3paq^2 - 2pq^2) / (py + q)^2 = 0.
\]

It is sufficient to find only positive roots of (17).

Denoting \( \lambda = p/q \) we have

\[
2a\lambda^3y^3 + \lambda(a + 4a\lambda - \lambda^2 + a\lambda^2)y^2 + 2(a + a\lambda - \lambda^2 + a\lambda^2)y + \lambda(3a - 2) = 0.
\]

(18)

Denote the cubic polynomial at the left side of (18) as \( \nu(y) \), \( \nu(0) = \lambda(3a - 2) < 0, a \in [0;0.5] \). The coefficient in higher degree of \( y \) in (18) is positive, hence, at least one postive root exists. From here also follows that the maximum lies before minimum. The function \( \nu = \nu(y) \) has two extreme points \( y_1 = \frac{1}{3} (\frac{1}{a} - \frac{1 + \lambda + \lambda^2}{\lambda^2}) \) and \( y_2 = -\frac{1}{\lambda} < 0 \). With \( \nu(0) < 0 \) it gives the uniqueness of the positive root of (18).

The solution takes place in case of \( \beta_1 \geq 0 \), or \( a(3 - y^2) \geq 1 \). It determines the lower border \( a_2(p) \) of the region \( D_2 \) on the plane \((p,a)\).

**Theorem 4.** For \((p,a) \in D_2\) the equilibrium is \((F_1^*, F_2^*)\) of the form (7-8) with parameters determined by (15-17). The value of the game: \( v = q(2a - \beta_2) + p\alpha_3 - p(2b - 1 + \beta_2)^{\alpha} \frac{\alpha_3 - \beta_2}{\alpha} \).

In case \( a < a_2(p) \) the following solution will take place.

### 4.3 Equilibrium for \((p,a) \in D_3\)

If \( F_1^*(1) < 1 \) and \( F_1^*(0+) > 0 \) (or, equivalently, \( \alpha_2 = 2a, \beta_2 = 2b - 1, \alpha_3 = A, \beta_2 = B \)), the first two equations in (11) with \( 1/x = \frac{\sqrt{a-B}}{\sqrt{a}} \) and \( y = \frac{\sqrt{A-\delta}}{\sqrt{a}} \) (or, \( \beta_2 = B = a - a/x^2 \) and \( \alpha_3 = A = ay^2 + b \)) take the form

\[
\begin{aligned}
2a(py + q) &= q(ay^2 + b)x, \\
2ay \left( \frac{q}{x} + p \right) &= p \left( b + \frac{a}{x^2} \right).
\end{aligned}
\]

(19)

From the first equation in (19) it follows \( x = \frac{2a(py + q)}{q(ay^2 + b)} \). Substituting it to the second equation in (19) and the third equation in (11) we obtain

\[
(3a^2y^4q^2p + (8a^2p^3 + 4a^2q^3)y^3 + (-2pa^2q^2 - 4p^3a + 16a^2p^2q + 4a^2p^3 + 2paq^2) y^2 +
\]
\[ + (8 a^2 p^2 q + 8 p^2 a^2 - 4 a^2 q^3 - 8 p^2 a q + 4 q^3) y + 3 p a^2 q^2 - p q^2 - 2 p a q^2) / (4 a (p y + q)^2) = 0. \]  

(20)

and

\[ y (3 a^2 y^4 + (8 a^2 p^2 + 4 a^2 q^3) y^3 + (-2 p a^2 q^2 - 4 p^3 a + 16 a^2 p q + 4 q^3 a) y + 3 p a^2 q^2 - p q^2 - 2 p a q^2) / (4 a (p y + q)^2) = 0. \]

It is sufficient to find only positive solutions of (20).

Denoting \( \lambda = p / q \) we rewrite (20) in the form

\[ 3 a^2 \lambda y^4 + 4 a^2 (1 + 2 \lambda^3) y^3 + 2 a \lambda (1 - a + 8 a \lambda - 2 \lambda^2 + 2 a \lambda^2) y^2 + 4 a (1 - a + 2 a \lambda - 2 \lambda^2 + 2 a \lambda^2) y - (1 - a)(1 + 3 a) \lambda = 0. \]

Denote \( \nu(y) \) polynomials at the left side of the equation. Then \( \nu(0) = -(1 - a)(1 + 3 a) \lambda < 0 \), and because the coefficient in higher degree of \( y \) is positive then there exists at least one positive root of the equation. Let us show that it is unique. It follows from the fact that the points where \( \nu'(y) = 0 \) are negative.

\[ \nu'(y) = 36 a^2 \lambda y^2 + 24 a^2 (1 + 2 \lambda^3) y + 4 a \lambda ((1 - a)(1 - 2 \lambda^2) + 8 a \lambda). \]

If this parabola has no roots then \( \nu(y) \) is concave and the positive root is unique. Let there are two roots

\[ y_{1, 2} = \frac{-a(1 + 2 \lambda^3) \pm \sqrt{a(4 a \lambda^6 - 2 a \lambda^4 + 2 \lambda^4 - 4 a \lambda^3 + a \lambda^2 - \lambda^2 + a)}}{3 a \lambda}. \]

The root \( y_1 \) is negative. Coefficient in higher degree of \( y \) of \( \nu''(y) \) is positive, hence, the largest root \( y_2 \) is negative, iff the coefficient in lower degree of \( \nu(y) \) is positive. It is equal to \( \xi(a, \lambda) = (1 - a)(1 - 2 \lambda^2) + 8 a \lambda \). We have: \( \xi(a, 0) = 1 - a > 0 \), the function \( \xi(a, \lambda) \) is convex in \( \lambda \), \( \xi(a, 1) = 9 a - 1 \). If \( a > \frac{1}{9} \), then \( \xi(a, \lambda) > 0 \), coefficient in lower degree in \( \nu''(y) \) is positive, \( y_2 \) is negative, hence, the positive root of the equation is unique.

The solution takes place, iff \( \beta_2 \geq 0 \) or \( \frac{a \beta^2 + b}{2 a (1 + \lambda p)} \leq 1 \). This inequality determines the lower border \( a_3(p) \) of the region \( D_3 \) on the plane \( (p, a) \). Notice, that in \( D_3 \) the inequality \( a < \frac{1}{9} \) is satisfied automatically.

**Theorem 5.** For \( (p, a) \in D_3 \) the equilibrium is \( (F_1^*, F_2^*) \) of the form (7–8) with parameters determined by (19–20). The value of the game: \( v = q(2 a - \beta_2) + p a_3 - p(2 b - 1 + \beta_2) \frac{\sqrt{a_3 - \beta}}{\sqrt{a}} \).

For fixed \( p \), if \( a \) decreases from \( a_2(p) \) to \( a_3(p) \), then \( \beta \) decreases to zero. Finally, consider the case \( a < a_3(p) \).

### 4.4 Equilibrium for \( (p, a) \in D_4 \)

For \( \alpha_1 = \alpha_2 = 2 a \), \( \alpha_4 = 1 \), \( \beta_1 = \beta_2 = 0 \), \( \beta_3 = 2 b - 1 \) the optimal strategies are

\[ F_1^*(s_1) = \begin{cases} 
0, & \text{if } s_1 \leq \alpha_3, \\
\frac{1}{q} \left(1 - \sqrt{\frac{\alpha_3 - \beta}{s_1 - \beta}}\right), & \text{if } \alpha_3 < s_1 \leq 1, \\
1, & \text{if } 1 < s_1, 
\end{cases} \]

(21)
\[
F_2^*(s_2) = \begin{cases} 
0, & \text{if } s_2 \leq 0, \\
\frac{\sqrt{\alpha_3 - b}}{\sqrt{a}}, & \text{if } 0 < s_2 \leq 2b - 1, \\
\frac{\sqrt{\alpha_3 - b}}{\sqrt{2b-s_2}}, & \text{if } 2b - 1 < s_2 \leq 2b - \alpha_3, \\
1, & \text{if } 2b - \alpha_3 < s_2.
\end{cases}
\]

Then, for \( s_1 \in (\alpha_3; 1] \)

\[
H_2 = H(s_1, F_2^*(s_2)) = p \left\{ 0 \cdot F_2^*(0) + \int_{2b-1}^{\alpha_3} s_2 dF_2^*(s_2) \right\} + \\
+ q \left\{ s_1 F_2^*(2b - s_1) + \int_{2b-s_1}^{\alpha_3} s_2 dF_2^*(s_2) \right\} = \alpha_3 - \frac{p\sqrt{\alpha_3 - b}}{\sqrt{a}}.
\]

If \( s_2 = 0 \), then

\[
H_3 = H(F_1^*(s_1), s_2) = q \left\{ \int_{\alpha_3}^{1} s_1 dF_1^*(s_1) + 1 \cdot (1 - F_1^*(1)) \right\} = 2\sqrt{a}\sqrt{\alpha_3 - b} + 2b - \alpha_3 - p.
\]

If \( s_2 \in (2b - 1; 2b - \alpha] \), then

\[
H_4 = H(F_1^*(s_1), s_2) = ps_2 + q \left\{ \int_{\alpha_3}^{2b-s_2} s_1 dF_1^*(s_1) + s_2(1 - F_1^*(2b - s_2)) \right\} = 2b - \alpha_3.
\]

\( F_1^*(s_1), F_2^*(s_2) \) be optimal iff

\[
\begin{align*}
2b - \alpha_3 &= \alpha_3 - \frac{p\sqrt{\alpha_3 - b}}{\sqrt{a}}, \\
2b - \alpha_3 &= 2\sqrt{a}\sqrt{\alpha_3 - b} + 2b - \alpha_3 - p.
\end{align*}
\]

Solution of this system: \( \alpha_3 = b + \frac{p^2}{4a} \).

This form for \( H_2-H_4 \) takes place, iff \( \alpha_3 \leq 1 \) or, equivalently, \( a > p/2 \). That determines the region \( D_4 \) on the plane \((p, a)\).

Theorem 6. For \((p, a) \in D_4\) the equilibrium is \((F_1^*, F_2^*)\) of the form \((21-22)\). The value of the game: \( v = b - \frac{p^2}{4a} \).

The case \( a < p/2 \) was analysed in section 2.

5 Solution for \( p > 0.5 \)

At the beginning we assumed \( p \leq 0.5 \). In case \( p > 0.5 \) the solution follows from the following theorem.

Theorem 7. Let for some fixed values of \( a \) and \( p \) we found the optimal strategies \( F_1^*(s_1, p, a) \) and \( F_2^*(s_2, p, a) \) in the game with

\[
P\{\alpha = a\} = p, \quad P\{\alpha = b\} = q, \quad a + b = 1, \quad p + q = 1, \quad a < b, \quad p \leq q.
\]
Then the optimal strategies in the game for the same values $a$, $p$ and for

$$P\{\alpha = a\} = q, \quad P\{\alpha = b\} = p, \quad a + b = 1, \quad p + q = 1, \quad a < b, \quad p \leq q,$$

are

$$G_1^*(s_1, q, a) = 1 - F_2^*(1 - s_1, p, a), \quad G_2^*(s_2, q, a) = 1 - F_1^*(1 - s_2, p, a).$$

**Proof.** We have

$$G_1^*(s_1, q, a) = \begin{cases} 
0, & \text{if } s_1 \leq 1 - 2b + \alpha_3, \\
\frac{1 - \sqrt{\alpha_3 - b}}{1 - \sqrt{\alpha_3 - b}}, & \text{if } 1 - 2b + \alpha_3 < s_1 \leq 1 - \beta_2, \\
\frac{1 - \sqrt{\alpha_3 - b}}{1 - \sqrt{\alpha_3 - b}} + \frac{2}{p} - \frac{\sqrt{\alpha_3 - b}}{1 - \sqrt{\alpha_3 - b}}, & \text{if } 1 - \beta_2 < s_1 \leq 1 - 2a + \alpha_2, \\
1 - \frac{1}{s_1}, & \text{if } 1 - 2a + \alpha_2 < s_1,
\end{cases}$$

$$G_2^*(s_2, q, a) = \begin{cases} 
0, & \text{if } s_2 \leq 1 - 2b + \beta_3, \\
\sqrt{\alpha_3 - b} - \alpha_3, & \text{if } 1 - 2b + \beta_3 < s_2 \leq 1 - \alpha_3, \\
\frac{\sqrt{\alpha_3 - b}}{\sqrt{\alpha_3 - b} - \alpha_3}, & \text{if } 1 - \alpha_3 < s_2 \leq 1 - \alpha_2, \\
\frac{\sqrt{\alpha_3 - b}}{\sqrt{\alpha_3 - b} - \alpha_3} - \frac{2}{p} - \frac{\sqrt{\alpha_3 - b}}{1 - \sqrt{\alpha_3 - b}}, & \text{if } 1 - \alpha_2 < s_2 \leq 1 - 2a + \beta_2, \\
1, & \text{if } 1 - 2a + \beta_2 < s_2.
\end{cases}$$

These functions will represent the optimal strategies, iff

$$H(s_1, G_2^*(s_2, q, a)) = \text{const for } s_1 \in (1 - 2b + \alpha_3; 1 - \beta_3] \cup (1 - \beta_2; 1 - 2a + \alpha_2],$$

$$H(G_1^*(s_1, q, a), s_2) = \text{const for } s_2 \in (1 - 2b + \beta_3; 1 - \alpha_3] \cup (1 - \alpha_2; 1 - 2a + \beta_2].$$

Denote $G_{1,12}^*(s_1)$ and $G_{1,34}^*(s_1)$ as the form of function $G_1^*(s_1, q, a)$ at the intervals $(1 - 2b + \alpha_3; 1 - \beta_3]$ and $(1 - \beta_2; 1 - 2a + \alpha_2]$ and $G_{2,12}^*(s_1)$, $G_{2,34}^*(s_1)$ for the $G_2^*(s_1, q, a)$ at the intervals $(1 - 2b + \beta_3; 1 - \alpha_3]$, $(1 - \alpha_2; 1 - 2a + \beta_2]$, respectively.

We obtain for $s_1 \in (1 - 2b + \alpha_3; 1 - \beta_3]$

$$H_1' = H(s_1, G_2^*(s_1, q, a)) = q \left\{ s_1 G_{2,12}^*(2a - s_1) + \int_{2a - s_1}^{1 - \alpha_3} s_2 dG_{2,12}^*(s_2) + \int_{1 - \alpha_3}^{1 - 2a + \beta_2} s_2 dG_{2,34}^*(s_2) \right\} +$$

$$+ p s_1 = \frac{\sqrt{\alpha_3 - b} - \alpha_3}{\sqrt{\alpha_3 - b} - \alpha_2 - a} \left( (\alpha_3 - 2b) - (\alpha_2 - 2a) \right) + p(\alpha_3 + 2a - 1) + q(1 - \beta_2).$$

If $s_1 \in (1 - \beta_2; 1 - 2a + \alpha_2]$, then

$$H_2' = H(s_1, G_2^*(s_1, q, a)) = q \left\{ 0 \cdot G_2^*(0, q, a) + \int_{1 - 2b + \beta_3}^{1 - \alpha_3} s_2 dG_{2,12}^*(s_2) + \int_{1 - \alpha_3}^{1 - 2a + \beta_2} s_2 dG_{2,34}^*(s_2) \right\} +$$

$$+ p \left\{ s_1 G_{2,34}^*(2b - s_1) + \int_{2b - s_1}^{1 - 2a + \beta_2} s_2 dG_{2,34}^*(s_2) \right\} =$$
\[
= q \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} ((\alpha_3 - 2b) - (\alpha_2 - 2a)) + p(\alpha_3 + 2a - 1) + q(1 - \beta_2) - q(1 - \beta_3) \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} \cdot \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} + p(1 - \beta_2) - p(1 - \beta_3) \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}.
\]

If \( s_2 \in (1 - 2b + \beta_3; 1 - \alpha_3] \), then

\[
H'_3 = H(G_1^*(s_1, q, a), s_2) = q \left\{ \int_{1-2b+\alpha_3}^{1-2b+s_2} s_1 dG_1^{*}(s_1) + s_2 (1 - G_1^{*}(2a - s_2)) \right\} + p \left\{ \int_{1-2b+\alpha_3}^{1-2b+s_2} s_1 dG_1^{*}(s_1) + s_2 (1 - G_1^{*}(2a - s_2)) \right\} =
\]

\[
= p \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} ((2b - \beta_3) - (2a - \beta_2)) + p(1 - \alpha_3) - q(1 - 2b - \beta_2) - p(1 - \alpha_2) \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + q(1 - \alpha_3) - q(1 - \alpha_2) \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + p\eta.
\]

If \( s_2 \in (1 - \alpha_2; 1 - 2a + \beta_2] \), then

\[
H'_4 = H(G_1^*(s_1, q, a), s_2) = qs_2 + p \left\{ \int_{1-2b+\alpha_3}^{1-b+s_2} s_1 dG_1^{*}(s_1) + \int_{1-\beta_2}^{2b-s_2} s_1 dG_1^{*}(s_1) + s_2 (1 - G_1^{*}(2b - s_2)) \right\} =
\]

\[
= p \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} ((2b - \beta_3) - (2a - \beta_2)) + p(1 - \alpha_3) - q(1 - 2b - \beta_2),
\]

where \( \eta = \begin{cases} 0, & \text{if } G_1^*(1) = 1, \\ -q + \left( \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} + q \right) \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}}, & \text{if } G_1^*(1) < 1. \end{cases} \)

We have

\[
\psi_1 = H'_2 - H'_1 = -q(1 - \beta_3) \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} \cdot \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} + p(1 - \beta_2) - p(1 - \beta_3) \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}},
\]

\[
\psi_2 = H'_3 - H'_4 = -p(1 - \alpha_2) \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + q(1 - \alpha_3) - q(1 - \alpha_2) \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + p\eta.
\]

There are only four possible forms for the functions \( F_1^*(s_1, p, a) \) and \( F_2^*(s_2, p, a) \). With \( \chi_1 = \chi_2 = 0 \), it gives:

1. For \( \alpha_2 = \alpha_3 = A, \beta_2 = \beta_3 = B \) take place \( \frac{\chi_1}{A} = \frac{\psi_1}{1-A} \) and \( \frac{\chi_2}{B} = \frac{\psi_2}{1-B} \), consequently, \( \psi_1 = \psi_2 = 0 \).

2. For \( \alpha_2 = \alpha_3 = A, \beta_3 = 2b - 1 \) take place \( \frac{\chi_1}{A} = \frac{\psi_1}{1-A} \) and \( \chi_2 = -\psi_1 \), consequently, \( \psi_1 = \psi_2 = 0 \).
For $\alpha_2 = 2a$, $\beta_3 = 1 - 2a$ take place $\chi_1 = -\psi_2$ and $\chi_2 = -\psi_1$, consequently, $\psi_1 = \psi_2 = 0$.

For $\alpha_1 = \alpha_2 = 2a$, $\alpha_4 = 1$, $\beta_1 = \beta_2 = 0$, $\beta_3 = 2b - 1$, the form of $G_1^*(s_1)$, $G_2^*(s_2)$ is:

$$G_1^*(s_1, q, a) = \begin{cases}  
0, & \text{if } s_1 \leq a + \frac{p^2}{4a}, \\
1 - \frac{p}{2a}, & \text{if } 2a < s_1 \leq 1,
1, & \text{if } 1 < s_1,
\end{cases}$$

$$G_2^*(s_2, q, a) = \begin{cases}  
0, & \text{if } s_2 \leq 0,
1 - \frac{1}{q} \left( 1 - \frac{p}{2\sqrt{a} \sqrt{a-s_2}} \right), & \text{if } 0 < s_2 \leq a - \frac{p^2}{4a},
1, & \text{if } a - \frac{p^2}{4a} < s_2.
\end{cases}$$

Then for $s_2 \in (0; a - \frac{p^2}{4a}]$

$$H(G_1^*(s_1, q, a), s_2) = q \left\{ \int_{a+\frac{p^2}{4a}}^{2a-s_2} s_1 dG_1^*(s_1, q, a) + s_2 (1 - G_1^*(2a-s_2, q, a)) \right\} +$$

$$+ p \left\{ \int_{a+\frac{p^2}{4a}}^{2a} s_1 dG_1^*(s_1, q, a) + 1 \cdot (1 - G_1^*(1, q, a)) \right\} = a + \frac{p^2}{4a}.$$

For $s_1 \in (a + \frac{p^2}{4a}; 2a]$

$$H(s_1, G_2^*(s_2, q, a)) = q \left\{ s_1 G_2^*(2a-s_1, q, a) + \int_{2a-s_1}^{a+\frac{p^2}{4a}} s_2 dG_2^*(s_2, q, a) \right\} + ps_1 = a + \frac{p^2}{4a}.$$

Finally, for $s_1 = 1$

$$H(s_1, G_2^*(s_2, q, a)) = q \int_{0}^{a-\frac{p^2}{4a}} s_2 dG_2^*(s_2, q, a) + p = a + \frac{p^2}{4a}.$$

In all cases the payoff is constant, and with $H_1 + H_4 = 1$, $H_4 + H'_1 = 1$ and $H_1 = H_4$, gives $H'_1 = H'_4$, and all $H'_i, i = 1,.., 4$ are equal. It proves the optimality $G_1^*(s_1, q, a)$ and $(s_2, q, a)$.

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