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# ファジィ最適化における全順序関係について
# On $L-$fuzzy Optimization Problem and Total Order Relation

-ラムダファジィ順序関係はコンパクト $L$ ファジィ最適化問題において実最適値を実最適解で与える-

-Compact $L-$fuzzy Optimization Problems with the $\lambda-$fuzzy Max Order Relation Have Real Optimal Values at Real Optimal Solutions -

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## 1 Introduction

In this study we give some geometrical meaning of the parametric representation concerning fuzzy numbers with bounded supports as well as we show the representation of the addition, subtraction and product which are defined by the extensions principle due to Zadeh and many other theoreticians of fuzzy logic. Our aim of this research is to establish solving $L-$fuzzy optimization problems under which the $\lambda-$fuzzy max order relation, which is a total order one, is introduced over the set of $L-$fuzzy numbers with $0 \leq \lambda \leq 1$. In case that feasible sets of $L-$fuzzy optimization problems are uncompact we discuss criteria to guarantee the existence of optimal solutions by applying $L-$fuzzy analysis in which the subdifferential of $L-$fuzzy functions and the mimimax equality play an important role. Under that feasible sets are compact $L-$fuzzy optimization problems have real optimal values at real optimal solutions.

## 2 Parametric Representation

There are many fruitful results on representations of fuzzy numbers, differentials and integrals of fuzzy functions ( see, e.g., in [1, 2, 3, 4, 5, 6, 7, 8] etc). In this study we give some geometrical meaning concerning the parametric representation of fuzzy numbers.

Let $I = [0,1]$ and $\mathbf{R} = (-\infty, +\infty)$. A fuzzy number with a center is characterized by a membership function $\mu$ as follows:

**Definition 1** *Define a set of fuzzy numbers* *with bounded supports by*

$\mathcal{F}_{\mathbf{b}}^{st} = \{\mu : \mathbf{R} \to I$ *satisfying* (i) – (iv) *below*$\}$.

(i) *There exists a unique* $m \in \mathbf{R}$ *such that* $\mu(m) = 1$;

(ii) *The support set* $supp(\mu) = cl(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$ *is bounded in* $\mathbf{R}$;

(iii) *Let* $J = \{\xi \in \mathbf{R} : \mu(\xi) > 0\}$. *The membership function* $\mu$ *is strictly fuzzy convex on* $J$, *i.e.,* $\mu(\lambda\xi_1 + (1-\lambda)\xi_2) > \min[\mu(\xi_1), \mu(\xi_2)]$ *for* $0 < \lambda < 1$ *and* $\xi_1, \xi_2 \in J$ *such that* $\xi_1 \neq \xi_2$ ;

(iv) $\mu$ *is upper semi-continuous on* $\mathbf{R}$.

From the above definition the following theorem shows that fuzzy numbers mean bounded continuous curves in the two-dimensional space $\mathbf{R}^2$. Condition (iii) plays an important role in the proof (cf. [9]). Denote the following parametric representation of $\mu \in \mathcal{F}_{\mathbf{b}}^{st}$ by $x_1(\alpha) = \min L_\alpha(\mu), x_2(\alpha) = \max L_\alpha(\mu)$ for $0 < \alpha \leq 1$ and

$$\begin{aligned} L_\alpha(\mu) &= \{\xi \in \mathbf{R} : \mu(\xi) \geq \alpha\}, \\ x_1(0) &= \min cl(supp(\mu)), \\ x_2(0) &= \max cl(supp(\mu)). \end{aligned}$$

It follows that $L_\alpha(\mu) = [x_1(\alpha), x_2(\alpha)]$.

Denote fuzzy numbers $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_{\mathbf{b}}^{st}$. From the extension principle of Zadeh, it follows that

$$\begin{aligned} &\mu_{x+y}(\xi) \\ = &\max_{\xi=\xi_1+\xi_2} \min(\mu_x(\xi_1), \mu_y(\xi_2)) \\ = &\max\{\alpha \in I : \xi = \xi_1 + \xi_2, \\ &\quad \xi_1 \in L_\alpha(\mu_x), \xi_2 \in L_\alpha(\mu_y)\} \\ = &\max\{\alpha \in I : \\ &\quad \xi \in [x_1(\alpha) + y_1(\alpha), x_2(\alpha) + y_2(\alpha)]\}, \end{aligned}$$

where $\mu_x, \mu_y$ are membership functions of $x, y$, respectively. Thus we get $x+y = (x_1+y_1, x_2+y_2)$.

From the above addition and multiplication, it follows that $x - y = (x_1 - y_2, x_2 - y_1)$.

**Theorem 1** *Denote* $x = (x_1, x_2) \in \mathcal{F}_{\mathbf{b}}^{st}$, *where* $x_1, x_2$ *are functions from* $I$ *to* $\mathbf{R}$. *Then the following properties* (i)-(iii) *hold:*

(i) $x_i \in C(I), i = 1, 2$. *Here* $C(I)$ *is the set of all the continuous functions on* $I$;

(ii) *There exists a unique* $m \in \mathbf{R}$ *such that* $x_1(1) = x_2(1) = m$ *and* $x_1(\alpha) \leq m \leq x_2(\alpha)$ *for* $\alpha \in I$;

(iii) *One of the following statements* (a) *and* (b) *holds;*

(a) *Functions* $x_1, x_2$ *are strictly increasing, strictly decreasing on* $I$, *respectively, with* $x_1(\alpha) < x_2(\alpha)$ *for* $0 \leq \alpha < 1$;

(b) $x_1(\alpha) = x_2(\alpha) = m$ *for* $0 \leq \alpha \leq 1$.

*Conversely, under the above conditions* (i) -(iii), *if we denote*

$$\mu_x(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leq \xi \leq x_2(\alpha)\}$$

*then* $\mu_x$ *is the membership function of* $x$, *i.e.,* $x \in \mathcal{F}_{\mathbf{b}}^{st}$.

**Proof.** See [9].

By the above theorem we have the following theorem which means significance in proving the existence and solving of optimal solutions of fuzzy optimization problems by applying the generalized Newton method which can be proved by the contraction principle in the complete metric space (see [9]).

Denote a metric of $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_{\mathrm{b}}^{st}$ by

$$d(x,y) = \sup_{\alpha\in I}(|x_1(\alpha) - y_1(\alpha)| + |x_2(\alpha) - y_2(\alpha)|).$$

**Theorem 2** *It follows that statements* (i) *and* (ii) *hold.*

(i) *The metric space* $(\mathcal{F}_{\mathrm{b}}^{st}, d)$ *is complete.*

(ii) *The real set* $\mathbf{R}$ *is a subset in* $\mathcal{F}_{\mathrm{b}}^{st}$.

**Proof.** See [9].

Let $x = (x_1, x_2) \in \mathcal{F}_{\mathrm{b}}^{st}$. Denote $x \preceq y$, if

$$\min x_\alpha \le \min y_\alpha \text{ and } \max x_\alpha \le \max y_\alpha$$

for $\alpha \in I$. The relationship $\preceq$ is called **fuzzy-max order**, which is partially order relation. Immediately we get the following theorem.

**Theorem 3** ( [5]) *Let* $x = (x_1, x_2), y = (y_1, y_2)$ *be* $L - R$ *fuzzy numbers. If* $x \preceq y$, *then we have*

$$x_1(1) \le y_1(1), \quad x_1(0) \le y_1(0)$$
$$y_2(0) - y_2(1) - (x_2(0) - x_2(1)) \le y_1(1) - x_1(1).$$

Let $\mathcal{F}_L$ be the set of $L$-fuzzy numbers and let $\mathcal{F}_L \subset \mathcal{F}_{\mathrm{b}}^{st}$. In the case that $x \preceq y$ is false for $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_L$, then we have

$$y_1(1) - y_1(0) - (x_1(1) - x_1(0)) > |y_1(1) - x_1(1)|.$$

In order to decide the order relationship between $x$ and $y$ which satisfies the above inequality we consider some kind of order relationship over $\mathcal{F}_L$. Let $0 \le \lambda \le 1$.

**Definition 2** ( [5]) *Let* $x = (x_1, x_2), y = (y_1, y_2)$ *be* $L-$*fuzzy numbers. Denote* $x \preceq_\lambda y$, *if the only one of the following cases* (i)-(iii) *hold:*

(i) $|y_1(1) - y_1(0) - (x_1(1) - x_1(0))| \le y_1(1) - x_1(1)$ *for* $y_1(1) \ge x_1(1)$;

(ii) $\lambda|y_1(1) - y_1(0) - (x_1(1) - x_1(0))| \le y_1(1) - x_1(1) < |y_1(1) - y_1(0) - (x_1(1) - x_1(0))|$ *for* $y_1(1) > x_1(1)$ *and* $y_1(1) - y_1(0) \ne x_1(1) - x_1(0)$;

(iii) $|y_1(1) - x_1(1)| < \lambda[y_1(1) - y_1(0) - (x_1(1) - x_1(0))]$ *for* $y_1(1) - y_1(0) - (x_1(1) - x_1(0)) > 0$.

From the above defifnition the following theorem is immediately given.

**Theorem 4** (*See* [5]) *Let* $x = (x_1, x_2), y = (y_1, y_2)$ *be* $L-$*fuzzy numbers. The relation* $x \preceq_\lambda y$ *holds if and only if one of the following inequalities* (i) *or* (ii) *holds:*

(i) $\lambda[x_1(1) - x_1(0)] + x_1(1) < \lambda[y_1(1) - y_1(0)] + y_1(1)$ *for* $y_1(1) - y_1(0) > x_1(1) - x_1(0)$.

(ii) $\lambda[x_1(1) - x_1(0)] + x_1(1) \le \lambda[y_1(1) - y_1(0)] + y_1(1)$ *for* $y_1(1) - y_1(0) \le x_1(1) - x_1(0)$.

*Thus* $\preceq_\lambda$ *is a total order relationship over* $\mathcal{F}_L$.

By the above theorem we get the following statement which plays an important role in Section 4.

**Corollary 1** *Let $f^* \in \mathbf{R}$. Then there exists no $L-$fuzzy number $f \in \mathcal{F}_L \backslash \mathbf{R}$ such that $f = f^*$, i.e., $f \preceq_\lambda f^*$ and $f^* \preceq_\lambda f$.*

In [7] they consider the following relation in the sense of means defined by membership functions.

**Note.** In considering a relation $\leq_m$, i.e., $x \leq_m y$ means that

$$\int_0^1 \alpha(x_1(\alpha)+x_2(\alpha))d\alpha \leq \int_0^1 \alpha(y_1(\alpha)+y_2(\alpha))d\alpha$$

for$x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_{\mathbf{b}}^{st}$, we have the following statements (i) - (iii). Let $x, y, z \in \mathcal{F}_{\mathbf{b}}^{st}$.

(i) $x \leq_m x$.

(ii) If $x \leq_m y$, and $y \leq_m z$, then we have $x \leq_m z$.

(iii) If $x \leq_m y$, and $y \leq_m x$, then it follows that $x$ is equal to $y$ in the sense of mean. However they arn't necessarily equal each other in the sense of membership functions. Thus the relation $\leq_m$ isn't an oder relation over $\mathcal{F}_{\mathbf{b}}^{st}$.

In what follows we introduce an idea of $L-$fuzzied numbers generalied by $\mathcal{F}_{\mathbf{b}}^{st}$. Let $x \in \mathcal{F}_L$. The quadratic $x^2$ of an $L-$fuzzy number $x$ isn't necessarily $L-$fuzzy number but fuzzy number in $\mathcal{F}_{\mathbf{b}}^{st}$ (see [9]). For $x = (x_1, x_2) \in \mathcal{F}_L$ and $\alpha \in I$, we have $x^2 = (x_1^2, x_2^2)$ if $x_1(\alpha) \geq 0$; $x^2 = (x_1 x_2, \max[x_1^2, x_2^2])$ if $x_1(\alpha) \leq 0 \leq x_2(\alpha)$; $x^2 = (x_2^2, x_1^2)$ if $x_2(\alpha) \leq 0$. In this study we consider the left portion of the membership function $\mu_{x^2}$ is more significant than the right portion of $\mu_{x^2}$. Denote an operator $(\cdot)_L : \mathcal{F}_{\mathbf{b}}^{st} \to \mathcal{F}_L$ such that $(x)_L = (x_1(1), x_1(1) - x_1(0))_L$ for $x = (x_1, x_2) \in \mathcal{F}_{\mathbf{b}}^{st}$. We call that $(x)_L$ is an *$L-$fuzzized number*. Here the membership function of $x$ is $\mu_x(\xi) = L(\frac{x_1(1)-\xi}{x_1(1)-x_1(0)})_+$ for $\xi \in \mathbf{R}$, $L : \mathbf{R} \to \mathbf{R}_+$ is a shape function and $\xi_+ = \max(\xi, 0)$ if $\xi \in \mathbf{R}$. For $x \in \mathcal{F}_L$ we get the $L-$fuzzied number

$$(x^2)_L = (x_1(1)^2, x_1(1)^2 - x_i(0)x_j(0)))_L,$$

where $i = 1, j = 2$ if $x_1(0)x_2(0) \leq 0$, $i = j = 1$ if $x_1(0)x_2(0) > 0$ and $|x_1(0)| < |x_2(0)|$, $i = j = 2$ if $x_1(0)x_2(0) > 0$ and $|x_1(0)| \geq |x_2(0)|$.

Let a shape function be $L(\xi) = (1 - |\xi|)_+$. For an $L-$fuzzy number $x = (\xi_0, \ell)_L$ with $|\xi_0| \leq \ell$, which has the membership function $\mu_x(\xi) = L(\frac{\xi_0-\xi}{\ell})_+$ for $\xi \in \mathbf{R}$. Then we get the membership function

$$\mu_{x^2}(\xi) = \begin{cases} \left(1 - \frac{\sqrt{\xi_0^2-\xi}}{\ell}\right)_+ & \text{for } \xi < \xi_0^2; \\ \left(1 - \frac{\xi_0-\sqrt{\xi}}{\ell}\right)_+ & \text{for } \xi \geq \xi_0^2. \end{cases}$$

In this case we construct an $L-$fuzzy numbers $(x^2)_L$ with the same portion as the left one of $\mu_{x^2}$. It follows that $(x^2)_L = (\xi_0^2, \ell^2)_L$. For $x \in \mathcal{F}_L$ and $k \in \mathbf{R}$ we have $(kx)_L = kx$.

## 3 $L-$fuzzy Analysis

In this section we discuss general type of criteria for the existence of optimal solutions of $L-$fuzzy optimization problems.

Let $0 \leq \lambda \leq 1$, $F : \mathcal{F}_L \to \mathcal{F}_L$ an $L-$fuzzy function and $x \in \mathcal{F}_L$. Define

$$\partial F_\lambda(x) = \{p \in \mathcal{F}_L : F(x) + ph \preceq_\lambda F(x+h) \\ \forall h \in \mathcal{F}_L\}.$$

The set $\partial F_\lambda(x)$ is said to be a $\lambda-$subdifferential of $F$ at $x$. The $L-$fuzzy number $p$ is called a $\lambda-$subgradient of $F$ at $x$ if $p \in \partial F_\lambda(x)$. We illustrate the following example concerning the $\lambda-$subdifferential.

**Example 1** *Let $a = (a_1(1), \ell_a)_L \in \mathcal{F}_L$. Denote a function $F : \mathcal{F}_L \to \mathcal{F}_L$ by $F(x) = (ax)_L$. Then there exists a $\lambda-$subdifferential at $x$ $\partial F_\lambda(x) = \{(a_1(1), \rho\ell_a) \in \mathcal{F}_L : 0 \le \rho \le 1\}$ for $x \in \mathcal{F}_L$.*

Let a set

$$\mathcal{F}_L^n = \{z = (\tilde{z}^1, \tilde{z}^2, \cdots, \tilde{z}^n)^T : \tilde{z}^j \in \mathcal{F}_L, \quad j = 1, 2, \cdots, n\}$$

and elements
$$x = (\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n)^T \in \mathcal{F}_L^n;$$
$$y = (\tilde{y}^1, \tilde{y}^2, \cdots, \tilde{y}^n)^T \in \mathcal{F}_L^n.$$

with a metric $d(x, y) = \sum_{j=1}^n d(\tilde{x}^j, \tilde{y}^j)$. Then it can be seen that $\mathcal{F}_L^n$ is a complete metric space. Denote the addition of $x, y$ by

$$x + y = (\tilde{x}^1 + \tilde{y}^1, \tilde{x}^2 + \tilde{y}^2, \cdots, \tilde{x}^n + \tilde{y}^n)$$

and the multiplication of $x \in \mathcal{F}_L^n, k \in \mathbf{R}$ by

$$kx = (k\tilde{x}^1, k\tilde{x}^2, \cdots, k\tilde{x}^n)$$

where $k\tilde{x}^j = (kx_1^j, kx_2^j), \tilde{x}^j = (x_1^j, x_2^j) \in \mathcal{F}_L$ for $k \ge 0$ and $k\tilde{x}^j = (kx_2^j, kx_1^j) \in \mathcal{F}_L$ for $k < 0$.

In what follows we discuss an extension principle concerning the fuzzy function $F : \mathcal{F}_L^n \to \mathcal{F}_{\mathbf{b}}^{st}$. For example, an addition $F(z) = \tilde{z}^1 + \tilde{z}^2$, polynomial $F(z) = (\tilde{z}^1)^2$ where $z = (\tilde{z}^1, \tilde{z}^2) \in \mathcal{F}_L^2$. Denote the membership function

$$\mu_{F(z)}(\xi) = \sup_{\xi = f(t_1, \cdots, t_n)} \Big( \min_{j=1,\cdots,n \text{ and } C(z)} \mu_{\tilde{z}_j}(t_j) \Big).$$

Here some $f : \mathbf{R}^n \to \mathbf{R}, t = (t_1, t_2, \cdots, t_n) \in \mathbf{R}^n, C(z)$ is a condition on the membership functions $\mu_{\tilde{z}_j}, j = 1, 2, \cdots, n$, of $z = (\tilde{z}^1, \cdots, \tilde{z}^j, \cdots, \tilde{z}^n) \in \mathcal{F}_L^n$ under which $f(t_1, \cdots, t_n) = F(t_1, \cdots, t_n)$. In the case that $F(z) = \tilde{z}^1 + \tilde{z}^2, z = (\tilde{z}^1, \tilde{z}^2)$ we consider $f(t_1, t_2) = t_1 + t_2, C(z) = \emptyset$ and also

$$\mu_{F(z)}(\xi) = \sup_{\xi = t_1 + t_2} \min_{j=1,2} [\mu_{\tilde{z}^j}(t_j)].$$

When $F(z) = \tilde{z}^3, z = (\tilde{z}^1, \tilde{z}^2, \tilde{z}^3)$ then we have $f(t_1, t_2, t_3) = t_1^3, C(z) = \{\mu_{\tilde{z}^1} = \mu_{\tilde{z}^2} = \mu_{\tilde{z}^3}\}$ and also

$$\mu_{F(z)}(\xi) = \sup_{\xi = t_1 t_2 t_3} \Big( \min_{j=1,2,3, \text{ and} C(z)} [\mu_{\tilde{z}^j}(t_j)] \Big).$$

Let $F : \mathcal{F}_L^n \to \mathcal{F}_L$. The set $epi_\lambda(F) = \{(x, y) \in \mathcal{F}_L^n \times \mathcal{F}_L : F(x) \preceq_\lambda y\}$ is said to be a $\lambda-$epigraph of $F$.

**Definition 3** *Let $\mathcal{S}$ be a convex set in $\mathcal{F}_L^n$. A function $F : \mathcal{S} \to \mathcal{F}_L$ is convex if $epi_\lambda(F)$ is $\lambda-$convex.*

It follows that a function $F : \mathcal{S} \to \mathcal{F}_L^n$ is $\lambda-$convex if and only if $F(kx + (1-k)y) \preceq_\lambda kF(x) + (1-k)F(y)$ for $x, y \in \mathcal{F}_L^n$, and $0 \le k \le 1$.

In what follows we consider the following $L-$fuzzy optimization problem

$$\min F(z) \text{ subject to } g_j(z) \preceq_\lambda (0, \delta_j)_L \quad (P_\lambda^\delta)$$

where $\delta = (\delta_1, \delta_2, \cdots, \delta_m)^T \in \mathbf{R}^m$ with $\delta_j \ge 0$ for $j = 1, 2, \cdots, m$. Let $F : \mathcal{F}_L^n \to \mathcal{F}_L$ and $g_j : \mathcal{F}_L^n \to \mathcal{F}_L$ be $\lambda-$convex, respectively.

In order to give conditions for the existence of optimal solutions of the problem $(P_\lambda^\delta)$ we denote the following Lagrangian

$$\mathcal{L}(w) = F(x) + \sum_{j=1}^m \eta_j [g_j(z)^c + \lambda(\ell_{g_j(z)} - \delta_j)],$$

where $g_j(z)^c$ is the center, $\ell_{g_j(z)}$ is the spread of the $L-$fuzzy number $g_j(z)$, respectively, and $w = (z,\eta) \in \mathcal{F}_L^n \times \mathbf{R}_+^m, \eta = (\eta_1, \eta_2, \cdots, \eta_m) \in \mathbf{R}_+^m$. An element $(z^*, \eta^*) \in \mathcal{F}_L^n \times \mathbf{R}_+^m$ is called a saddle point of $\mathcal{L}$ if

$$\mathcal{L}(z^*, \eta) \preceq_\lambda \mathcal{L}(z^*, \eta^*) \preceq_\lambda \mathcal{L}(z, \eta^*)$$

for $\eta \in \mathbf{R}_+^m$ and $z \in \mathcal{C}_\lambda^\delta$, where the feasible set

$$\begin{aligned} \mathcal{C}_\lambda^\delta &= \{z \in \mathcal{F}_L^n : \\ &\quad g_j(z) \preceq_\lambda (0, \delta_j)_L \text{ for } j = 1, 2, \cdots, m\}. \end{aligned}$$

From now on it is necessary to establish existence criteria for fuzzy optimization problems by considering saddle points of the Lagrangian functions and to propose iteration method, for example generalized Newton method by applying the idea of the subdifferential of convex analysis. For example, the following results (I) and (II) are expected to hold:

(I) Let $\mathcal{S} \subset \mathcal{F}_L^n$ be convex and $F : \mathcal{S} \to \mathcal{F}_L$ be $\lambda-$convex. Then it follows that $\partial F_\lambda(x) \neq \emptyset$ for $x$.

(II) Assume that $F : \mathcal{F}_L^n \to \mathcal{F}_L$ and $g_j : \mathcal{F}_L^n \to \mathcal{F}_L, j = 1, 2, \cdots, m$, are $\lambda-$convex. It follows that statements (i) - (iv) are mutually equivalent.

- (i) An element $w^* = (z^*, \eta^*) \in \mathcal{C}_\lambda^\delta$ is the saddle point of $\mathcal{L}$;
- (ii) A point $z^* \in \mathcal{F}_L^n$ is an optimal solution of $(P_\lambda^\delta)$;
- (iii) The following relations (a) and (b) hold:
  - (a) $\eta_j g_j(x^*) = 0$ for $j = 1, 2, \cdots, m$;
  - (b) $\partial \mathcal{L}_\lambda(w^*) \ni 0$. Here

$$\begin{aligned} &\partial \mathcal{L}_\lambda(w) = \{p \in \mathcal{F}_L^n \times \mathbf{R}_+^m : \\ &\mathcal{L}(w) + ph \\ &\preceq_\lambda \mathcal{L}(w+h), \forall h \in \mathcal{F}_L^n \times \mathbf{R}_+^m\} \end{aligned}$$

    for $w \in \mathcal{F}_L^n \times \mathbf{R}_+^m$;
- (iv) It follows that

$$\mathcal{L}(w^*) = \max_z \min_\eta \mathcal{L}(w) = \min_\eta \max_z \mathcal{L}(w).$$

In case that there exists an optimal solution of $L-$fuzzy optimization problemsby the above theorems, the solution means a real number.

**Theorem 5** *Let $n = 1$. Denote*

$$f_1^\delta = \min\{f(z) : z \in \mathcal{C}_\lambda^\delta\}, f_2^\delta = \min\{f(z^c) : z \in \mathcal{C}_\lambda^\delta\},$$

$$f_3^\delta = \min\{f(z)^c : z \in \mathcal{C}_\lambda^\delta\}, f_4^\delta = \min\{f(z) : z \in \mathcal{C}_0^0\},$$

*where $z^c \in \mathbf{R}, f(z)^c \in \mathbf{R}$ are centers of $z, f(z)$, respectively, $\mathcal{C}_0^0 = \mathcal{C}_\lambda^\delta \cap \mathbf{R} = \{z^c \in \mathbf{R} : z \in \mathcal{C}_\lambda^\delta\}$*

*If there exist $f_i^\delta, i = 1, 2, 3, 4$, then it follows that $f_i^\delta \in \mathbf{R}, i = 1, 2, 3, 4$, and that*

$$f_1^\delta = f_2^\delta = f_3^\delta \leq f_4^\delta.$$

*If $\delta = 0$, then $f_1^0 = f_2^0 = f_3^0 = f_4^0$.*
*If $\delta \neq 0$, then $f_1^\delta = f_2^\delta = f_3^\delta < f_4^\delta$.*

# 4 Compact Feasible Sets

In this section we establish an criterion for the existence of optimal solutions of $L-$fuzzy optimization prolems with compact feasible sets. In the following example we consider $L-$fuzzy optimization problem with a fuzzy objective function and fuzzy constraints.

**Example 2** *Let $z = (u,v) \in \mathcal{F}_L^2$ and $\lambda \in I$. Fuzzy functions $F, g_j, j = 1,2,3$, are as follows $(P_\lambda^\delta)$:*

$$
\begin{aligned}
F(z) &= -u - v;\\
g_1(z) &= -u \preceq_\lambda (0,\delta_1)_L;\\
g_2(z) &= -v \preceq_\lambda (0,\delta_2)_L;\\
g_3(z) &= (u^2)_L + (v^2)_L \preceq_\lambda (1,\delta_3)_L.
\end{aligned}
$$

*Here $(0,\delta_1)_L, (0,\delta_2)_L, (1,\delta_3)_L$ are $L-$fuzzy numbers and $(u^2)_L = (u_1(1)^2, \ell_{u^2})_L, (v^2)_L = (v_1(1)^2, \ell_{v^2})_L$ are $L-$fuzzized numbers.*

In order to find an optimal solution $z^* = (u^*, v^*) \in \mathcal{F}_L^2$ we consider the Lagrangian function $\mathcal{L}(w) = -u - v + k(w)$, where $w = (z,\eta), \eta = (\eta_1,\eta_2,\eta_3)^T \in \mathbf{R}_+^3$, and $k(w) = \eta_1[-u_1(1) + \lambda(\ell_u - \delta_1)] + \eta_2[-v_1(1) + \lambda(\ell_v - \delta_2)] + \eta_3[u_1(1)^2 + v_1(1)^2 - 1 + \lambda(\ell_{u^2} + \ell_{v^2} - \delta_3)]$. Denote $w^* = (u^*, v^*, 0,0,0)$. We find conditions of $u^* = (u_1(1), \ell_{u^*})_L, v^* = (v_1(1), \ell_{v^*})_L$ satisfying the inequality $\mathcal{L}(z^*,\eta) \preceq_\lambda \mathcal{L}(w^*) \preceq_\lambda \mathcal{L}(z,0,0,0)$ for $\eta \in \mathbf{R}_+^3$ and $(u,v) \in \mathcal{C}$. Then it follows that

$$-u^* - v^* \preceq_\lambda -u - v.$$

Since saddle points of $\mathcal{L}$ are optimal solutions of $(P_\lambda^\delta)$, we get

(i) $-u_1^*(1) - v_1^*(1) + \lambda(\ell_{u^*} + \ell_{v^*}) \le -u_1(1) - v_1(1) + \lambda(\ell_u + \ell_v)$ for $\ell_{u^*} + \ell_{v^*} \ge \ell_u + \ell_v$,

(ii) $-u_1^*(1) - v_1^*(1) + \lambda(\ell_{u^*} + \ell_{v^*}) < -u_1(1) - v_1(1) + \lambda(\ell_u + \ell_v)$ for $\ell_{u^*} + \ell_{v^*} < \ell_u + \ell_v$.

From conditions of constraints we get the feasible set $\mathcal{C}_\lambda^\delta = \{(u,v) \in \mathcal{F}_L^2 : u = (u_1(1), \ell_u)_L, v = (v_1(1), \ell_v)_L$ satisfy the following conditions (c-i) - (c-iii) below $\}$.

(c-i) $u_1(1) \ge \lambda(\ell_u - \delta_1)$,

(c-ii) $v_1(1) \ge \lambda(\ell_v - \delta_2)$,

(c-iii) $u_1(1)^2 + v_1(1)^2 \le 1 + \lambda[\delta_3 - \ell_{u^2} - \ell_{v^2}]$.

Here

$$
\ell_{u^2} = \begin{cases} 2u_1(1)\ell_u - (\ell_u)^2 & (u_1(1) \ge \ell_u) \quad (U_1)\\ (\ell_u)^2 & (|u_1(1)| \le \ell_u) \quad (U_2)\\ -2u_1(1)\ell_u - (\ell_u)^2 & (u_1(1) \le -\ell_u) \quad (U_3) \end{cases}
$$

$$
\ell_{v^2} = \begin{cases} 2v_1(1)\ell_v - (\ell_v)^2 & (v_1(1) \ge \ell_v) \quad (V_1)\\ (\ell_v)^2 & (|v_1(1)| \le \ell_v) \quad (V_2)\\ -2v_1(1)\ell_v - (\ell_v)^2 & (v_1(1) \le -\ell_v) \quad (V_3) \end{cases}
$$

The set $\mathcal{C}_\lambda^\delta$ is non-empty since the point $(u_1(1), v_1(1))^T = (\lambda(\ell_u - \delta_1), \lambda(\ell_v - \delta_2))^T$ satisfies (c-iii) in case that $u_1(1) \ge \ell_u, v_1(1) \ge \ell_v$.

Conditions (c-i) - (c-iii) leads to

$(\mathrm{c-i})'$ $u_1(1) \ge -\lambda\delta_1$,

$(\mathrm{c-ii})'$ $v_1(1) \ge -\lambda\delta_2$,

$(\mathrm{c-iii})'$ $u_1(1)^2 + v_1(1)^2 \le 1 + \lambda\delta_3$.

So the set $C_e = \{(u_1(1), v_1(1))^T \in \mathbf{R}^2 : (\mathrm{c-i})' - (\mathrm{c-iii})'$ hold $\}$ is compact. It can be easily seen that the set $S_{pq} = \{(\ell_u, \ell_v) \in \mathbf{R}_+^2 : (U_p)$ and $(V_q)$ hold $\}, p = 1,3; q = 1,3$, are compact. In case that $p = 1$ and $q = 1,2,3$, it follows that

$\lambda(\ell_u)^2 \le 1 + \lambda\delta_3$ and $\lambda\ell_{v^2} \le 1 + \lambda\delta_3$.

The latter inequality means that $\ell_v \le |v_1(1)|$ or $\lambda(\ell_v)^2 \le 1 + \lambda\delta_3$, which show that $S_{pq}, p = 2; q = 1,2,3$, are compact. In the similar way it follows that $S_{pq}, p = 1,3; q = 2$, are compact. Thus, from the compactness of $C_e \subset \mathbf{R}^2$ and $S_{pq} \subset \mathbf{R}_+^2, p, q = 1,2,3$, the feasible set $\mathcal{C}_\lambda^\delta$ is compact in $\mathcal{F}_L$.

From $(\mathrm{c-i})$ and $(\mathrm{c-ii})$ we have

$-u_1(1) - v_1(1) + \lambda(\ell_u + \ell_v) \le \lambda(\delta_1 + \delta_2)$, so $f(z) \preceq_\lambda (0, \delta_1 + \delta_2)_L$. From $(\mathrm{c-iii})$ it follows that $-u_1(1) - v_1(1) \ge -\sqrt{2(1 + \lambda\delta_3)}$ and the mimimum is attained at $u_1(1) =$

$v_1(1) = (-\sqrt{\frac{1+\lambda\delta_3}{2}}, 0)_L$, which means that $\min_z f(z) = (-\sqrt{\frac{1+\lambda\delta_3}{2}}, 0)_L$ and $u^* = v^* = (-\sqrt{\frac{1+\lambda\delta_3}{2}}, 0)_L$. When $\lambda = 0$ and $\delta_j = 0, j = 1, 2, 3$, then the real type of optimization problem $(P_0^0)$ gives $-\sqrt{2} \leq f(z) \leq 0$ in $\mathbf{R}$ and $u^* = v^* = \sqrt{1/2} \in \mathbf{R}$.

This example shows that there exists a unique optimal solution of $L-$fuzzy number of fuzzy optimization problem $(P_\lambda^\delta)$ with a fuzzy coefficient, where $(P_\lambda^\delta)$ is an optimization problem with $\mathbf{R}-$valed coefficiets if $\ell_z = 0$ and $(P_\lambda^\delta)$ is fuzzy type if $\ell_z \neq 0$, where $\ell_z$ is the spread of $z \in C_\lambda^\delta$. Therefore the optimal solution to the real type $(P_0^0)$ is the same as solution to the fuzzy type $(P_\lambda^\delta)$ concerning $\lambda = 0$ and $\ell_z = 0$.

ij what follows we show an existence criteiron for $(P_\lambda^\delta)$ having compact feasible sets. By theorems in Section 3 we get the following existence criterion for real optimal solutions of $L-$fuzzy optimazation problems.

**Theorem 6** *Let $n = 1$. If $f$ and $g_j, j = 1, 2, \cdots, m$, are $\lambda-$convex and the feasible set $C_\lambda^\delta$ is compact in $\mathcal{F}_L^n$, then $(P_\lambda^\delta)$ has a real optimal value at a real optimal solution.*

Moreover the following minimax criterion is expected to be proved in the same way as the minimax theorems in the real analysis. Let $C_\lambda^\delta$ be a convex and compact set in $\mathcal{F}_L^n$ and let $\lambda \in I$. A function $\mathcal{L} : C_\lambda^\delta \times \mathbf{R}_+^m \to \mathcal{F}_L$ satisfies (i) and (ii). (i) $\mathcal{L}(\cdot, \eta)$ is $\lambda-$lower semi-continuous and $\lambda-$quasiconvex on $C_\lambda^\delta$ for $\eta \in \mathbf{R}_+^m$; (ii) $\mathcal{L}(z, \cdot)$ is $\lambda-$concavelike on $\mathbf{R}_+^m$ for $z \in C_\lambda^\delta$. Then there exists an optimal solution $z^* \in C_\lambda^\delta$ of $(P_\lambda^\delta)$.

Here we mean that definitions of $\lambda-$semicontinuous, $\lambda-$quasiconvex or $\lambda-$concavelike of $L-$fuzzy functions are as follows: It is said that $F : \mathcal{F}_L^n \to \mathcal{F}_L$ is $\lambda-$lower semi-continuous at $z \in \mathcal{F}_L^n$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $F(z) \preceq_\lambda F(z+h) + \varepsilon$ where $d(z, z+h) < \delta$. It is said that $F : \mathcal{F}_L^n \to \mathcal{F}_L$ is $\lambda-$quasiconvex if for each $z_1, z_2 \in \mathcal{F}_L^n$ and $k \in I$, $kF(z_1) + (1-k)F(z_2) \preceq_\lambda \max[F(z_1), F(z_2)]$. It is said that $Y : \mathbf{R}_+^m \to \mathcal{F}_L$ is $\lambda-$concavelike if for each $\eta_1, \eta_2 \in \mathbf{R}_+^m$ and $0 < k < 1$, there exists an $\eta_0 \in \mathbf{R}_+^m$ such that $kY(\eta_1) + (1-k)Y(\eta_2) \preceq_\lambda Y(\eta_0)$.

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