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Risk Aversion and Wealth Effects in Portfolio Selection Problems with Two Assets

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Abstract

This paper examines the risk aversion and initial wealth effects for an optimal selection problem with two risky assets. It is assumed that a risk averse investor wishes to maximize the expected utility from his/her final wealth. Many comparative static results are obtained for the situations when the risk aversion of investor increases or decreases in the sense of Arrow-Pratt or in that of Ross, and when the level of investor's initial wealth increases or decreases. Especially, we investigate in more detail the cases when the return rates of the two risky assets are stochastically independent, and when they have bivariate normal distribution.

Key word: optimal portfolio, two risky assets, Arrow-Pratt measure of risk aversion, Ross ordering of risk aversion, bivariate normal distribution, total positivity of order 2.

1 Introduction

This paper considers an optimal selection problem with two risky assets, where a risk averse investor wishes to maximize the expected utility from his/her final wealth. We examine the risk aversion and initial effects on the optimal portfolio, that is, comparative statics on how the optimal portfolio is affected by the change of the investors risk aversion and/or the level of his/her initial wealth.

There have been a great number of studies on the risk aversion and initial effects for various optimal selection problems. The seminal studies were done independently by Arrow [1] and Pratt [26] in 1960s. They proposed a measure of risk aversion, which is now very common and called Arrow-Pratt measure of risk aversion, and some notions on individual's attitudes toward risk such as IARA (Increasing Absolute Risk Aversion), DARA (Decreasing Absolute Risk Aversion), IRRA (Increasing Relative Risk Aversion), DRRA (Decreasing Relative Risk Aversion), and others, and showed that these notions play important roles in various decision making problems under risk, including an optimal selection problem with one risk free asset and one risky asset. Their interesting and important results are summarized with new proofs in the next section. Since then, there have been extensive studies concerning such problems (see, e.g., [5, 10, 11, 20, 28], and references therein).

On the other hand, for an optimal selection problem with two risky assets, Cass and Stiglitz [4] and Ross [27] showed that, by counterexamples, the Arrow-Pratt measure does not yield any clear comparative statics result, and, in some cases, implies rather counterintuitive effects on the optimal portfolios. Further, Ross [27] introduced a new ordering risk aversion, so called Ross ordering, which strengthens that of Arrow-Pratt, and obtained a distribution-free comparative statics results for an optimal selection problem with two risky assets. Rubinstein [28] and Li and Ziemba [22] the cases when return rates of two risky assets have a bivariate normal distribution under another measure of risk aversion, so called Rubinstein measure.

In this paper, we first derive well-known comparative statics results of the risk aversion and initial effects for an optimal selection problem with one risk free and one risky assets, based on the Arrow-Pratt measure of risk aversion. The proofs given here are new and based on variation diminishing properties of TP_2 (Totally Positive of order 2) functions (see Appendix A). Then, their results are partially extended to an optimal selection problem with two risky assets. Analyses based on the Ross ordering of risk aversion are also done for two risky assets problems. Especially, we investigate in more detail the cases when the return rates of the two risky assets are stochastically independent, and when they have a bivariate normal distribution.

2 Ordering of Risk Aversion

2.1 Arrow–Pratt Ordering of Risk Aversion

Many measures of risk aversion for utility functions have been proposed with the object of expressing individual's risk aversion in the economic behavior under uncertainty. Among them, the following Arrow–Pratt measures of risk aversion have been used most widely (Arrow [1], Pratt [26]).

Definition 2.1 (Arrow–Pratt Measures of Risk Aversion). Let u ($u' > 0$, $u'' \leq 0$) be a twice differentiable von Neumann–Morgenstern (vN–M) utility function of a *risk averse* individual, defined on an open interval of the real line $\mathbb{R} := (-\infty, \infty)$. Define the Absolute Risk Aversion (ARA) and the Relative Risk Aversion (RRA, or Proportional Risk Aversion (PRA)) of u (or the individual) by

$$R_{\mathrm{A}}(x; u) := -\frac{u''(x)}{u'(x)} \ (\geq 0); \tag{2.1}$$

$$R_{\mathrm{R}}(x; u) := -\frac{xu''(x)}{u'(x)}, \tag{2.2}$$

respectively. □

The above measures of risk aversion are functions of the wealth level x. We further introduce some notions of their functional behavior with respect to the wealth level x.

Definition 2.2 (IARA, DARA, IRRA, DRRA (in the Sence of Arrow–Pratt)).

(1) We say that a risk averse vN–M utility function u displays IARA (Increasing Absolute Risk Aversion) in the sense of Arrow–Pratt if and only if its absolute risk aversion $R_{\mathrm{A}}(x; u)$ is increasing in the wealth level x, and DARA (Decreasing Absolute Risk Aversion) in the sense of the Arrow–Pratt if and only if it is decreasing in the wealth level x.[1]

(2) We say that a risk averse vN–M utility function u displays IRRA (Increasing Relative Risk Aversion) in the sense of Arrow–Pratt if and only if its relative risk aversion $R_{\mathrm{R}}(x; u)$ is increasing in the wealth level x, and DRRA (Decreasing Relative Risk Aversion) in the sense of Arrow–Pratt if and only if it is decreasing in the wealth level x. □

Arrow [1] and Pratt [26] examined the following hypothesis or claim concerning with the risk attitude of typical investors.

Hypothesis 2.1.

(H1) The ARA (Absolute Risk Aversion) $R_{\mathrm{A}}(x; u)$ of a typical u (or a typical investor) is decreasing in the wealth level x (DARA).

(H2) The RRA (Relative Risk Aversion) $R_{\mathrm{R}}(x; u)$ of a typical u (or a typical investor) is increasing in the wealth level x (IRRA). □

[1] In this paper, the terms "increasing" and "decreasing" are used in the weak sense, that is, "increasing" means "nondecreasing" and "decreasing" means "nonincreasing".

Arrow [1] and Pratt [26] found the following facts which support the above hypothesis 2.1: Consider an optimal portfolio selection problem with one risk-free asset and one risky asset, where a risk averse investor with a vN–M utility function u seeks to maximize his/her expected utility from the final wealth.

(1) If the utility function u displays DARA, then the optimal "amount" of the wealth to be invested in the risky asset is increasing in the initial wealth level x;

(2) If the utility function u displays IRRA, then the optimal "proportion" of the wealth to be invested in the risky asset is decreasing in the initial wealth level x.

In the next section 3, we will give a "new proof" of the above fact by applying a variation diminishing property of a TP_2 function (see Appendix A).

Definition 2.3 ($\geq_{\mathrm{APRA}}$: Arrow–Pratt Ordering of Risk Aversion). Let u_1, u_2 ($u_i' > 0$, $u_i'' \leq 0$, $i = 1, 2$) be twice differentiable vN–M utility functions of two risk averse individuals, defined on a common open interval of the real line $\mathbb{R}$. If it holds that

$$R_{\mathrm{A}}(x; u_1) = -\frac{u_1''(x)}{u_1'(x)} \geq -\frac{u_2''(x)}{u_2'(x)} = R_{\mathrm{A}}(x; u_2) \quad \text{for all } x, \tag{2.3}$$

then (the individual with) u_1 is said to be *more risk averse* than (the individual with) u_2 (or (the individual with) u_2 is said to be *more risk tolerant* than (the individual with) u_1) in the sense of Arrow–Pratt, and in this case, we write as

$$u_1 \geq_{\mathrm{APRA}} u_2 \quad (\text{or} \quad u_2 \leq_{\mathrm{APRA}} u_1). \tag{2.4}$$

□

The following equivalence among (1), (2), and (3) are well known in economics under uncertainty and incomplete information (see Laffont [21], Hirshleifer and Riley [7], Gollier [6]). For (4), see Appendix A.2.

Theorem 2.1. For twice differentiable vN–M utility functions u_1, u_2 ($u_i' > 0$, $u_i'' \leq 0$, $i = 1, 2$) of two risk averse individuals, defined on a common open interval of the real line $\mathbb{R}$, the following four statements are mutually equivalent:

(1) $u_1 \geq_{\mathrm{APRA}} u_2$, that is,

$$R_{\mathrm{A}}(x; u_1) = -\frac{u_1''(x)}{u_1'(x)} \geq -\frac{u_2''(x)}{u_2'(x)} = R_{\mathrm{A}}(x; u_2) \quad \text{for all } x; \tag{2.5}$$

(2) For some increasing and concave function G, it holds that

$$u_1(x) = G(u_2(x)) \quad \text{for all } x; \tag{2.6}$$

(3) If we let $\pi_i(w; X)$ be an insurance premium paid by the individual u_i, $i = 1,\ 2$ with wealth level w for any *fair gamble* X ($\mathbb{E}[X] = 0$) (in other words, $\pi_i(w; X)$ be a certainty equivalent value π such that $\mathbb{E}[u_i(w + X)] = u_i(w - \pi)$), then

$$\pi_1(w; X) \geq \pi_2(w; X); \tag{2.7}$$

(4) The marginal utility $u_i'(x)$ is TP_2 (Totally Positive of Order 2) with respect to $i = 1, 2$ and possible x, that is,

$$\begin{vmatrix} u_1'(x) & u_1'(y) \\ u_2'(x) & u_2'(y) \end{vmatrix} \geq 0 \quad \text{for all } x < y, \tag{2.8}$$

or, equivalently,

$$\frac{u_2'(x)}{u_1'(x)} \quad \text{is increasing in } x. \tag{2.9}$$

2.2 Ross Ordering of Risk Aversion

The following ordering of risk aversion is a strengthened one of that of Arrow–Pratt (Ross [27]).

Definition 2.4 ($\geq_{\mathrm{RRA}}$: Ross Ordering of Risk Aversion). Let u ($u' > 0$, $u'' \leq 0$) be a twice differentiable vN–M utility function of a risk averse individual, defined on an open interval of the real line $\mathbf{R}$. If it holds that

$$\inf_x \frac{u_1''(x)}{u_2''(x)} \geq \sup_x \frac{u_1'(x)}{u_2'(x)}, \tag{2.10}$$

then (the individual with) u_1 is said to be *more risk averse* than (the individual with) u_2 (or (the individual with) u_2 is said to be *more risk tolerant* than (the individual with) u_1) in the sense of Ross, and in this case, we write as

$$u_1 \geq_{\mathrm{RRA}} u_2 \quad (\text{or} \quad u_2 \leq_{\mathrm{RRA}} u_1). \tag{2.11}$$

□

Obviously, $u_1 \geq_{\mathrm{RRA}} u_2$ implies $u_1 \geq_{\mathrm{APRA}} u_2$, however, it could be shown by a counterexample that the converse does not necessarily hold.

Theorem 2.2 (Ross [27]). For twice differentiable vN–M utility functions u_1, u_2 ($u_i' > 0$, $u_i'' \leq 0$, $i = 1, 2$) of two risk averse individuals, defined on a common open interval of the real line $\mathbf{R}$, the following four statements are mutually equivalent:

(1) $u_1 \geq_{\mathrm{RRA}} u_2$, that is,

$$\inf_x \frac{u_1''(x)}{u_2''(x)} \geq \sup_x \frac{u_1'(x)}{u_2'(x)}; \tag{2.12}$$

(2) For some positive number a (> 0), it holds that

$$\frac{u_1''(x_1)}{u_2''(x_1)} \geq a \geq \frac{u_1'(x_2)}{u_2'(x_2)} \quad \text{for all } x_1, x_2; \tag{2.13}$$

(3) For some positive number a (> 0) and some decreasing and concave function G, it holds that

$$u_1(x) = a u_2(x) + G(x) \quad \text{for all } x; \tag{2.14}$$

(4) If we let $\pi_i(W; X)$ be an insurance premium paid by the individual u_i, $i = 1, 2$ with random initial wealth level W for any *fair gamble* X ($\mathbf{E}[X|W] = 0$, a.s.) (in other words, $\pi_i(W; X)$ be a certainty equivalent value π such that $\mathbf{E}[u_i(W + X)] = \mathbf{E}[u_i(W - \pi)]$), then

$$\pi_1(W; X) \geq \pi_2(W; X). \tag{2.15}$$

□

Correspondingly to the notions IARA, DARA, IRRA, DRRA in the sense of Arrow–Pratt defined in Definition 2.2, we define IARA, DARA, IRRA, DRRA in the sense of Ross as follows:

Definition 2.5 (IARA, DARA, IRRA, DRRA in the Sense of Ross).

(1) We say that a risk averse vN–M utility function u displays IARA (Increasing Absolute Risk Aversion) in the sense of Ross if and only if

$$u(\cdot + y) \geq_{\mathrm{RRA}} u(\cdot) \quad \text{for all } y > 0, \tag{2.16}$$

while we say that it displays DARA (Decreasing Absolute Risk Aversion) in the sense of Ross if and only if

$$u(\cdot) \geq_{\mathrm{RRA}} u(\cdot + y) \quad \text{for all } y > 0. \tag{2.17}$$

(2) We say that a risk averse vN–M utility function u displays IRRA (Increasing Relative Risk Aversion) in the sense of Ross if and only if

$$u((1+y)\ \cdot) \geq_{\mathrm{RRA}} u(\cdot) \quad \text{for all } y > 0, \tag{2.18}$$

while we say that it displays DRRA (Decreasing Relative Risk Aversion) in the sense of Ross if and only if

$$u(\cdot) \geq_{\mathrm{RRA}} u((1+y)\ \cdot) \quad \text{for all } y > 0. \tag{2.19}$$

□

Obviously, the above notions of Ross are stronger than the corresponding ones of Arrow–Pratt.

3 Portfolio Selection Problem with One Risk–Free and One Risky Assets

We consider a risk averse investor with a vN–M utility function u ($u' > 0$, $u'' \leq 0$) who allocates his/her initial wealth w (> 0) between one risk–free and one risky assets. Let r (> 0) be an interest rate ($+1$) of the risk–free asset, and X denote a random variable representing the rate of return ($+1$) on the risky asset, whose cumulative distribution function is denoted by F_X. Let w_X denote the "amount" of wealth invested in the risky asset X. Then, the investor's optimization problem is to maximize the expected utility from the random final wealth, which is described as follows:

$$\max_{w_X \in \mathcal{F}(w)} \mathbb{E}[u((w-w_X)r + w_X X)], \tag{3.1}$$

where $\mathcal{F}(w)$ ($\subset \mathbb{R}$) denotes the set of all feasible solutions (e.g., $\mathcal{F}(w) = \mathbb{R}_+ := [0,\infty)$ if a short sale of the risky asset is not allowed).

Let $w_X^*(w;u)$ denote the (or an) optimal solution of the portfolio selection problem (3.1) in order to represent explicitly the dependence on the utility function u and the initial wealth w.

Now, define the objective function of the problem (3.1) to be maximized as

$$U(w_X, w; u) := \mathbb{E}[u((w-w_X)r + w_X X)], \quad w_X \in \mathcal{F}(w). \tag{3.2}$$

Differentiating it with respect to w_X, we have

$$\begin{aligned} U'(w_X, w; u) := \frac{\partial}{\partial w_X} U(w_X, w; u) &= \mathbb{E}[u'((w-w_X)r + w_X X)\{X - r\}] \\ &= \int_{-\infty}^{\infty} u'((w-w_X)r + w_X x)\{x-r\}\,\mathrm{d}F_X(x), \end{aligned} \tag{3.3}$$

$$U''(w_X, w; u) := \frac{\partial^2}{\partial {w_X}^2} U(w_X, w; u) = \mathbb{E}\left[u''((w-w_X)r + w_X X)\{X-r\}^2\right]. \tag{3.4}$$

Since $u'' \leq 0$ in (3.4), we have

$$U''(w_X, w; u) \leq 0,$$

which implies $U(w_X, w; u)$ is a concave function of w_X. It is noted that, in (3.3), the function

$$g(x) := x - r$$

is increasing in x, so that it changes its sign at most once, and its possible sign change is from negative to

For an arbitrarily fixed $w_X \in \mathbb{R}$, define a function by

$$k(w,x;u) := \log u'\left((w - w_X)\, r + w_X x\right), \quad w \in \mathbb{R}, \tag{3.5}$$

then its differentiation yields

$$\begin{aligned} \frac{\partial}{\partial w} k(w,x;u) &= \frac{1}{u'\left((w-w_X)\,r + w_X x\right)} u''\left((w-w_X)\,r + w_X x\right) r \\ &= -R_{\mathrm{A}}\left((w-w_X)\,r + w_X x; u\right) r, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \frac{\partial}{\partial x} k(w,x;u) &= \frac{1}{u'\left((w-w_X)\,r + w_X x\right)} u''\left((w-w_X)\,r + w_X x\right) w_X \\ &= -R_{\mathrm{A}}\left((w-w_X)\,r + w_X x; u\right) w_X. \end{aligned} \tag{3.7}$$

Theorem 3.1. As the risk aversion, in the sense of the Arrow–Pratt, of (the utility function u of) the investor increases, the optimal amount $w_X^*(w;u)$ invested in the risky asset decreases. □

Theorem 3.2. Suppose that (the utility function u of) the investor displays DARA (IARA, respectively) in the sense of Arrow–Pratt. Then, as the initial wealth w increases, the optimal amount $w_X^*(w;u)$ invested in the risky asset increases (decreases, respectively). □

Next, we rewrite the portfolio selection problem (3.1) as follows:

$$\max_{\lambda_X \in \mathcal{F}} \mathbf{E}[u(w\{(1-\lambda_X)\, r + \lambda_X X\})] = \max_{\lambda_X \in \mathcal{F}} \mathbf{E}[u(w\{r + \lambda_X (X - r)\})], \tag{3.8}$$

where $\lambda_X := w_X/w$ is the "proportion" of the wealth invested in the risky asset X and $\mathcal{F}$ is the set of all feasible solutions.

Let $\lambda_X^*(w;u)$ denote the (or an) optimal solution of the portfolio selection problem (3.8) in order to represent explicitly the dependence on the utility function u and the initial wealth w.

Now, define the objective function of the problem (3.8) to be maximized as

$$U(\lambda_X, w; u) := \mathbf{E}[u(w\{r + \lambda_X (X - r)\})], \quad \lambda_X \in \mathcal{F}. \tag{3.9}$$

Differentiating it with respect to λ_X, we have

$$\begin{aligned} U'(\lambda_X, w; u) := \frac{\partial}{\partial \lambda_X} U(\lambda_X, w; u) &= w\mathbf{E}[u'(w\{r + \lambda_X (X-r)\})\{X - r\}] \\ &= w \int_{-\infty}^{\infty} u'(w\{r + \lambda_X (x - r)\})\{x - r\}\, \mathrm{d}F_X(x), \end{aligned} \tag{3.10}$$

$$U''(\lambda_X, w; u) := \frac{\partial^2}{\partial {\lambda_X}^2} U(\lambda_X, w; u) = w^2 \mathbf{E}\left[u''(w\{r + \lambda_X (X-r)\})\{X-r\}^2\right]. \tag{3.11}$$

Since $u'' \le 0$ in (3.11), we have

$$U''(w_X, w; u) \le 0,$$

which implies that $U(\lambda_X, w; u)$ is a concave function of λ_X.

Noting again that, in (3.10), the function

$$g(x) := x - r$$

is increasing in x, so that it changes its sign at most once, and its possible sign change is from negative to positive.

For an arbitrarily fixed $\lambda_X \in \mathbb{R}$, define a function by

$$k(w,x;u) := \log u'(w\{r + \lambda_X (x - r)\}), \tag{3.12}$$

then its differentiation yields

$$\begin{aligned}\frac{\partial}{\partial w}k(w,x;u) &= \frac{1}{u'\left(w\left\{r+\lambda_X(x-r)\right\}\right)}u''\left(w\left\{r+\lambda_X(x-r)\right\}\right)\left\{r+\lambda_X(x-r)\right\} \\ &= -R_{\mathrm{R}}\left(w\left\{r+\lambda_X(x-r)\right\};u\right)\frac{1}{w}. \end{aligned} \tag{3.13}$$

Theorem 3.3. Suppose that (the utility function u of) the investor displays DRRA (IRRA, respectively) in the sense of Arrow–Pratt. Then, as the initial wealth w increases, the optimal proportion $\lambda_X^*(w;u)$ invested in the risky asset increases (decreases, respectively). □

4 Portfolio Selection Problem with Two Risky Assets

We consider a risk averse investor with a vN–M utility function u who allocates his/her positive initial wealth w (>0) between two risky assets. Let possibly dependent random variables X and Y denote the rates of returns $(+1)$ on the two risky assets, and denote their joint distribution function by $F_{X,Y}$ and their marginal distributions by F_X and F_Y, respectively. For convenience, we call these assets as X, Y throughout this paper. If we denote the proportion of his/her initial wealth w invested in the asset X by λ_X $(\in[0,1])$, and that in the asset Y by $1-\lambda_X$ $(\in[0,1])$, then the expected utility from his/her final wealth is given by

$$\begin{aligned}U(\lambda_X,w;u) &:= \mathbb{E}[u(w\{\lambda_X X+(1-\lambda_X)Y\})] \\ &= \mathbb{E}[u(w\{\lambda_X(X-Y)+Y\})]. \end{aligned} \tag{4.1}$$

Assuming that the investor's objective is to maximize the expected utility of his/her final wealth, then the portfolio selection problem with the two risky assets is to find or characterize the (or an) optimal solution $\lambda_X^*(w;u)$ of the following optimization problem:

$$\max_{\lambda_X\in[0,1]} U(\lambda_X,w;u). \tag{4.2}$$

By differentiating $U(\lambda_X,w;u)$ with respect to λ_X, we have

$$\begin{aligned}U'(\lambda_X,w;u) &:= \frac{\partial}{\partial\lambda_X}U(\lambda_X,w;u) \\ &= w\mathbb{E}[u'(w\{\lambda_X X+(1-\lambda_X)Y\})\{X-Y\}], \qquad (4.3)\\ U''(\lambda_X,w;u) &:= \frac{\partial^2}{\partial{\lambda_X}^2}U(\lambda_X,w;u) \\ &= w^2\mathbb{E}\left[u''(w\{\lambda_X X+(1-\lambda_X)Y\})\{X-Y\}^2\right]. \qquad (4.4)\end{aligned}$$

Since $u''\le 0$ in (4.4), we have

$$U''(\lambda_X,w;u)\le 0,$$

which implies that $U(\lambda_X,w;u)$ is a concave function of λ_X. Accordingly, the (or an) optimal solution of the problem (4.2) could be characterized as follows: for a solution λ_X $(\in[0,1])$,

$$U'(\lambda,w;u)\ge 0 \iff \lambda_X^*(w;u)\ge\lambda; \tag{4.5}$$

$$U'(\lambda,w;u)\le 0 \iff \lambda_X^*(w;u)\le\lambda. \tag{4.6}$$

For an example, if we set $\lambda=1/2$ in eqs. (4.5) and (4.5), then the investor demands the risky asset X more than the risky asset Y, that is,

$$\lambda_X^*(w;u)\ge\frac{1}{2}\quad(\ge 1-\lambda_X^*(w;u)) \tag{4.7}$$

if and only if

$$U'\left(\frac{1}{2},w;u\right)\ge 0. \tag{4.8}$$

Remark 4.1.

(1) By the no–short–sale constraint $\mathcal{F} = [0,1]$ in the portfolio selection problem (4.2), i.e.,

$$\lambda_X \in [0,1] \quad (1-\lambda_X \in [0,1]),$$

if we set $\lambda = 1$ and $\lambda = 0$ in eqs. (4.5) and (4.6), then the (or an) optimal proportion $\lambda_X^*(w;u)$ invested in the risky asset X is characterized as follows:

$$U'(1,w;u) \geq 0 \iff \lambda_X^*(w;u) = 1; \tag{4.9}$$

$$U'(0,w;u) \leq 0 \iff \lambda_X^*(w;u) = 0. \tag{4.10}$$

(2) The constraint set $\mathcal{F}$ could be generalized to the case when

$$\mathcal{F} = [a,b] \quad (\supset [0,1],\ -\infty \leq a \leq 0 < 1 \leq b \leq \infty).$$

In this case, the above characterizations of the optimal proportions $\lambda_X^*(w;u)$ and $1-\lambda_X^*(w;u)$ invested in the risky assets X and Y would be modified as follows:

$$U'(1,w;u) \geq 0 \iff 1-\lambda_X^*(w;u) \leq 0; \tag{4.11}$$

$$U'(0,w;u) \leq 0 \iff \lambda_X^*(w;u) \leq 0, \tag{4.12}$$

and accordingly the presented results in the sequel could be modified in obvious ways. □

4.1 Analysis Based on Arrow–Pratt Measure of Risk Aversion

By writing down $U'(1,w;u)$ in eq. (4.9), we have

$$\begin{aligned} U'(1,w;u) &= w\mathbf{E}[u'(wX)\{X-Y\}] \\ &= w\mathbf{E}_X\left[\mathbf{E}_Y\left[u'(wX)\{X-Y\}|X\right]\right] \\ &= w\mathbf{E}_X\left[u'(wX)\{X-\mathbf{E}_Y[Y|X]\}\right] \\ &= w\int_{-\infty}^{\infty} u'(wx)\left\{x - m_{Y|X}(x)\right\} \mathrm{d}F_X(x), \end{aligned} \tag{4.13}$$

where $\mathbf{E}_X[\cdot]$ and $\mathbf{E}_Y[\cdot]$ are the expectation operators with respect to the random variables X and Y, respectively, and we define

$$m_{Y|X}(x) := \mathbf{E}_Y[Y|X=x]. \tag{4.14}$$

By writing down $U'\left(\frac{1}{2},w;u\right)$ in eq. (4.8), we have

$$\begin{aligned} U'\left(\frac{1}{2},w;u\right) &= w\mathbf{E}\left[u'\left(w\left\{\frac{X+Y}{2}\right\}\right)\{X-Y\}\right] \\ &= 2w\mathbf{E}[u'(wZ)\{Z-Y\}] \\ &= 2w\mathbf{E}_Z\left[\mathbf{E}_Y\left[u'(wZ)\{Z-Y\}|Z\right]\right] \\ &= 2w\mathbf{E}_Z\left[u'(wZ)\{Z-\mathbf{E}_Y[Y|Z]\}\right] \\ &= 2w\int_{-\infty}^{\infty} u'(wz)\left\{z - m_{Y|Z}(z)\right\} \mathrm{d}F_Z(z), \end{aligned} \tag{4.15}$$

where, we define as

$$Z := \frac{X+Y}{2},$$

F_Z is the cumulative distribution function of the random variable Z, $\mathbf{E}_Z[\cdot]$ is the expectation operator with respect to the random variable Z, and

$$m_{Y|Z}(z) := \mathbf{E}_Y[Y|Z=z] = \mathbf{E}_Y\left[Y \,\middle|\, \frac{X+Y}{2} = z\right]. \tag{4.16}$$

4.1.1 Risk Aversion Effects

First, let us investigate the risk aversion effects on the (or an) optimal portfolio for an arbitrarily fixed positive initial wealth w (> 0).

By eq. (4.13), we have the following theorem.

Theorem 4.1. Let a positive initial wealth w (> 0) be arbitrarily fixed.

(1) Suppose that $x - m_{Y|X}(x)$ changes its sign at most once in x and its possible sign change is from "negative to positive." If an investor does not invest all of his/her wealth exclusively in X, then neither does a more risk averse investor in the sense of Arrow–Pratt (if an investor invests a positive proportion of his/her initial wealth in Y, then so does a more risk averse investor in the sense of Arrow–Pratt).

(2) Suppose that $x - m_{Y|X}(x)$ changes its sign at most once in x and its possible sign change is from "positive to negative." If an investor invests all of his/her wealth exclusively in X, then so does a more risk averse investor in the sense of Arrow–Pratt. □

Furthermore, by eq. (4.15), we have the following theorem.

Theorem 4.2. Let a positive initial wealth w (> 0) be arbitrarily fixed.

- Suppose that $z - m_{Y|Z}(z)$ changes its sign at most once in z and its possible sign change is from "negative to positive." If an investor invests more of his/her initial wealth in Y than in X, then so does a more risk averse investor in the sense of Arrow–Pratt. □

4.1.2 Initial Wealth Effects

Next, let us investigate the initial wealth effects on the (or an) optimal portfolio, when a (vN–M utility function u of) a risk averse investor is arbitrarily fixed.

Theorem 4.3. Let (a vN–M utility function u of) a risk averse investor be fixed. Suppose that his/her vN–M utility function u displays IRRA (DRRA, respectively).

(1) Suppose that $x - m_{Y|X}(x)$ changes its sign at most once in x and its possible sign change is from "negative to positive." If an investor does not invest all of his/her wealth w_1 exclusively in X, then neither does he/she all of his/her larger (smaller, respectively) initial wealth w_2 exclusively in X (if an investor invests a positive proportion of his/her initial wealth w_1 in Y, then so does he/she a positive proportion of his/her larger (smaller, respectively) initial wealth w_2 in Y).

(2) Suppose that $x - m_{Y|X}(x)$ changes its sign at most once in x and its possible sign change is from "positive to negative." If an investor invests all of his/her wealth w_1 exclusively in X, then so does he/she all of his/her larger (smaller, respectively) initial wealth w_2 exclusively in X □

Theorem 4.4. Let (a vN–M utility function u of) a risk averse investor be fixed. Suppose that his/her vN–M utility function u displays IRRA (DRRA, respectively).

- Suppose that $z - m_{Y|Z}(z)$ changes its sign at most once in z and its possible sign change is from "negative to positive." If an investor invests more of his/her initial wealth w_1 in Y than in X, then so does he/she more of his/her larger (smaller, respectively) initial wealth w_2 in Y than in X. □

4.1.3 Sufficient Conditions

Now, let us examine sufficient conditions for

$$z - m_{Y|Z}(z) := z - \mathbf{E}_Y[Y|Z=z] = \mathbf{E}\left[Z - Y \,\middle|\, \frac{X+Y}{2} = z\right] = \mathbf{E}\left[\frac{X-Y}{2} \,\middle|\, \frac{X+Y}{2} = z\right] \tag{4.17}$$

to change its sign at most once in z, from negative to positive. It suffices for this that $\mathbf{E}\left[\frac{X-Y}{2} \,\middle|\, \frac{X+Y}{2} = z\right]$ is increasing in z. Further, for the latter, it is sufficient that the following conditional random variable is stochastically increasing in z in a sense of a suitable stochastic dominance relation (or stochastic ordering relation):

$$\left[\frac{X-Y}{2} \,\middle|\, \frac{X+Y}{2} = z\right]. \tag{4.18}$$

For a candidate of such a stochastic dominance relation, we consider the likelihood rate dominance (or likelihood ratio ordering), which is known to be rather strong but easily verifiable stochastic dominance relation. A necessary and sufficient condition for the conditional random variable (4.18) to be stochastically increasing in z with respect to the likelihood rate dominance is in the followings: the joint probability density function

$$f_{\frac{X-Y}{2},\frac{X+Y}{2}}(w,z)$$

of the bivariate random vector

$$\left(\frac{X-Y}{2}, \frac{X+Y}{2}\right)$$

is TP_2 (Totally Positive of order 2) with respect w and z (see, Appendix A and, e.g., Tong [34]). On the other hand, since

$$f_{\frac{X-Y}{2},\frac{X+Y}{2}}(w,z) = 2f_{X,Y}(z+w, z-w), \tag{4.19}$$

we have, by Theorem 4.2, the following corollary.

Corollary 4.1. Let a positive initial wealth w (> 0) be arbitrarily fixed.

- Assume that $f_{X,Y}(z+w, z-w)$ is TP_2 with respect to w and z, that is,

$$\begin{vmatrix} f_{X,Y}(z_1+w_1, z_1-w_1) & f_{X,Y}(z_2+w_1, z_2-w_1) \\ f_{X,Y}(z_1+w_2, z_1-w_2) & f_{X,Y}(z_2+w_2, z_2-w_2) \end{vmatrix} \geq 0 \quad \text{for all } w_1 \leq w_2, \; z_1 \leq z_2. \tag{4.20}$$

  Then, if an investor invests more of his/her initial wealth in Y than in X, then so does a more risk averse investor in the sense of Arrow–Pratt. □

Corollary 4.2. Let (a vN–M utility function u of) a risk averse investor be fixed. Suppose that his/her vN–M utility function u displays IRRA (DRRA, respectively).

- Assume that $f_{X,Y}(z+w, z-w)$ is TP_2 with respect to w and z. If an investor invests more of his/her initial wealth w_1 in Y than in X, then so does he/she more of his/her larger (smaller, respectively) initial wealth w_2 in Y than in X. □

4.1.4 Independent Cases

When two random variables X and Y are stochastically independent, since

$$m_{Y|X}(x) = \mu_Y \quad (\text{: the mean of } Y = \text{a constant}), \tag{4.21}$$

$x - m_{Y|X}(x) = x - \mu_Y$ is increasing in x, so that it changes its sign at most once in x, and its possible sign change is from negative to positive. Accordingly, by Theorem 4.1, we have the following corollary.

Corollary 4.3. Let a positive initial wealth w (> 0) be arbitrarily fixed.

- Assume that two risky assets X and Y are stochastically independent. If an investor does not invest all of his/her wealth exclusively in X, then neither does a more risk averse investor in the sense of Arrow–Pratt (if an investor invests a positive proportion of his/her initial wealth in Y, then so does a more risk averse investor in the sense of Arrow–Pratt). □

Similarly, by Theorem 4.3, we obtain the following corollary.

Corollary 4.4. Let (a vN–M utility function u of) a risk averse investor be fixed. Suppose that his/her vN–M utility function u displays IRRA (DRRA, respectively).

- Assume that two risky assets X and Y are stochastically independent. If an investor does not invest all of his/her wealth w_1 exclusively in X, then neither does he/she all of his/her larger (smaller, respectively) initial wealth w_2 exclusively in X (if an investor invests a positive proportion of his/her initial wealth w_1 in Y, then so does he/she a positive proportion of his/her larger (smaller, respectively) initial wealth w_2 in Y). □

For examples where the sufficient condition in Theorem 4.2, 4.4 is easily verifiable, there is a case when the random variable X and Y are independently distributed according to Gamma distributions with a common scale parameter as follows:

Example 4.1. Consider the case when the random variables X and Y are independently distributed according to Gamma distributions with a common scale parameter λ (> 0), and possibly distinct shape parameters α_X and α_Y (> 0), respectively. That is, their probability density functions f_X are f_Y are given by

$$f_X(x) = \frac{\lambda^{\alpha_X} x^{\alpha_X - 1} e^{-\lambda x}}{\Gamma(\alpha_X)}; \quad f_Y(y) = \frac{\lambda^{\alpha_Y} y^{\alpha_Y - 1} e^{-\lambda y}}{\Gamma(\alpha_Y)}$$

and their means and variances by

$$\mathbb{E}[X] = \frac{\alpha_X}{\lambda}, \quad \mathrm{Var}[X] = \frac{\alpha_X}{\lambda^2}; \quad \mathbb{E}[Y] = \frac{\alpha_Y}{\lambda}, \quad \mathrm{Var}[Y] = \frac{\alpha_Y}{\lambda^2}.$$

Then, their sum $X + Y$ is also Gamma distributed with scale parameter λ and shape parameter $\alpha_X + \alpha_Y$, that is, its probability density function f_{X+Y} is given by

$$f_{X+Y}(z) = \frac{\lambda^{\alpha_X + \alpha_Y} z^{\alpha_X + \alpha_Y - 1} e^{-\lambda z}}{\Gamma(\alpha_X + \alpha_Y)}.$$

Further, since the bivariate random vector $(Y, X + Y)$ has its probability density function $f_{Y,X+Y}$ given by

$$f_{Y,X+Y}(y,z) = f_X(z-y) f_Y(y) = \frac{\lambda^{\alpha_X + \alpha_Y} (z-y)^{\alpha_X - 1} y^{\alpha_Y - 1} e^{-\lambda z}}{\Gamma(\alpha_X)\Gamma(\alpha_Y)},$$

the probability density function $f_{Y|X+Y}$ of the conditional random variable $[Y|X + Y = z]$, i.e., the conditional probability density function of the random variable Y given the event $\{X + Y = z\}$ is

$$\begin{aligned} f_{Y|X+Y}(y|z) &= \frac{f_{Y,X+Y}(y,z)}{f_{X+Y}(z)} \\ &= \frac{\Gamma(\alpha_X + \alpha_Y)}{\Gamma(\alpha_X)\Gamma(\alpha_Y)} \left(1 - \frac{y}{z}\right)^{\alpha_X - 1} \left(\frac{y}{z}\right)^{\alpha_Y - 1} \frac{1}{z}. \end{aligned}$$

Therefore, since

$$\begin{aligned}
\mathbb{E}[Y|X+Y=z] &= \int_0^z y f_{Y|X+Y}(y|z)\mathrm{d}y \\
&= \frac{\Gamma(\alpha_X+\alpha_Y)}{\Gamma(\alpha_X)\Gamma(\alpha_Y)} \int_0^z \left(1-\frac{y}{z}\right)^{\alpha_X-1} \left(\frac{y}{z}\right)^{\alpha_Y} \mathrm{d}y \\
&= \frac{\Gamma(\alpha_X+\alpha_Y)}{\Gamma(\alpha_X)\Gamma(\alpha_Y)}\, z \int_0^1 (1-v)^{\alpha_X-1} v^{\alpha_Y} \mathrm{d}v \\
&= z\, \frac{\Gamma(\alpha_X+\alpha_Y)}{\Gamma(\alpha_X)\Gamma(\alpha_Y)} \frac{\Gamma(\alpha_X)\Gamma(\alpha_Y+1)}{\Gamma(\alpha_X+\alpha_Y+1)} \\
&= z\, \frac{\alpha_Y}{\alpha_X+\alpha_Y}
\end{aligned}$$

and

$$z - m_{Y|Z}(z) = z - \mathbb{E}\left[Y \left| \frac{X+Y}{2} = z \right.\right] = z - \mathbb{E}[Y|X+Y=2z] = z - \left(2z\frac{\alpha_Y}{\alpha_X+\alpha_Y}\right) = \frac{\alpha_X-\alpha_Y}{\alpha_X+\alpha_Y} z, \tag{4.22}$$

we have the following equivalence

$$\alpha_X > \alpha_Y \Longleftrightarrow z - m_{Y|Z}(z): \quad \text{incrasing in } z. \tag{4.23}$$

Further, it is well-known that, if $\alpha_X > \alpha_Y$, then X is larger than Y in the sense of increasing and convex ordering, that is, for any increasing and convex function g, we have

$$\mathbb{E}[g(X)] \geq \mathbb{E}[g(Y)]$$

(see Chapter 4, and e.g., Kijima and Ohnishi [17, 19], Stoyan [32], Shaked and Shanthikumar [31]). □

For an example of discrete probability distribution, we have the case when each of X and Y is Poisson distributed.

Example 4.2. Let us consider the case when X and Y are Poisson distributed with parameters λ_X and λ_Y (> 0), respectively, that is, their probability mass functions p_X and p_Y are given by

$$p_X(x) = \mathbb{P}(X=x) = \frac{\lambda_X^x e^{-\lambda_X}}{x!}, \quad x \in \mathbf{Z}_+; \quad p_Y(y) = \mathbb{P}(Y=y) = \frac{\lambda_Y^y e^{-\lambda_Y}}{y!}, \quad y \in \mathbf{Z}_+,$$

and their means and variances by

$$\mathbb{E}[X] = \lambda_X, \quad \mathrm{Var}[X] = \lambda_X; \quad \mathbb{E}[Y] = \lambda_Y, \quad \mathrm{Var}[Y] = \lambda_Y.$$

Then, their sum $X+Y$ is also Poisson distributed with parameter $\lambda_X + \lambda_Y$, that is, its probability mass function p_{X+Y} is given by

$$p_{X+Y}(z) = \mathbb{P}(X+Y=z) = \frac{(\lambda_X+\lambda_Y)^z e^{-(\lambda_X+\lambda_Y)}}{z!}.$$

Further, since the bivariate random vector $(Y, X+Y)$ has its probability mass function $p_{Y,X+Y}$ given by

$$p_{Y,X+Y}(y,z) = \mathbb{P}(Y=y, X+Y=z) = \mathbb{P}(X=z-y)\mathbb{P}(Y=y) = p_X(z-y)p_Y(y) = \frac{\lambda_X^{z-y}\lambda_Y^y e^{-(\lambda_X+\lambda_Y)}}{(z-y)!y!},$$

the probability mass function of the conditional random variable $[Y|X+Y=z]$, i.e., the conditional probability mass function $p_{Y|X+Y}$ of the random variable Y given the event $\{X+Y=z\}$ is the following binomial distribution:

$$\begin{aligned}
p_{Y|X+Y}(y|z) &= \mathbb{P}(Y=y|X+Y=z) \\
&= \frac{\mathbb{P}(Y=y, X+Y=z)}{P(X+Y=z)} \\
&= \frac{p_{Y,X+Y}(y,z)}{p_{X+Y}(z)} \\
&= \frac{\dfrac{\lambda_X^{z-y}\lambda_Y^{y}e^{-(\lambda_X+\lambda_Y)}}{(z-y)!y!}}{\dfrac{(\lambda_X+\lambda_Y)^z e^{-(\lambda_X+\lambda_Y)}}{z!}} \\
&= \binom{z}{y}\left(\frac{\lambda_X}{\lambda_X+\lambda_Y}\right)^{z-y}\left(\frac{\lambda_Y}{\lambda_X+\lambda_Y}\right)^{y}.
\end{aligned}$$

Therefore, we have

$$\mathbb{E}[Y|X+Y=z] = \sum_{y=0}^{z} y\, p_{Y|X+Y}(y|z) = z\,\frac{\lambda_Y}{\lambda_X+\lambda_Y},$$

so that

$$z - m_{Y|Z}(z) = z - \mathbb{E}\left[Y\,\middle|\,\frac{X+Y}{2}=z\right] = z - \mathbb{E}[Y|X+Y=2z] = z - \left(2z\,\frac{\lambda_Y}{\lambda_X+\lambda_Y}\right) = \frac{\lambda_X-\lambda_Y}{\lambda_X+\lambda_Y}\,z. \tag{4.24}$$

Accordingly, we have the following equivalence

$$\lambda_X > \lambda_Y \iff z - m_{Y|Z}(z): \quad \text{increasing in } z. \tag{4.25}$$

□

Next, let us examine the condition given in Corollaries 4.1 and 4.2. When X and Y are stochastically independent,

$$f_{X,Y}(z+w, z-w) = f_X(z+w)f_Y(z-w). \tag{4.26}$$

Therefore, in order for $f_{X,Y}(z+w, z-w)$ to be TP_2 with respect to w and z, it suffices that

(1) $f_X(z+w)$ is TP_2 with respect to w and z;

(2) $f_Y(z-w)$ is TP_2 with respect to w and z.

Hence, from Corollaries 4.1 and 4.2, we have the following corollary.

Corollary 4.5. Assume that X and Y are stochastically independent, and

(1) $f_X(z+w)$ is TP_2 with respect to w and z;

(2) $f_Y(z-w)$ is TP_2 with respect to w and z.

Then, if an investor invests more of his/her initial wealth in Y than in X, then so does a more risk averse investor in the sense of Arrow–Pratt. □

Corollary 4.6. Let (a vN–M utility function u of) a risk averse investor be fixed. Suppose that his/her vN–M utility function u displays IRRA (DRRA, respectively). Assume that X and Y are stochastically independent, and

(1) $f_X(z+w)$ is TP_2 with respect to w and z;

(2) $f_Y(z-w)$ is TP_2 with respect to w and z.

If the investor invests more of his/her initial wealth w_1 in Y than in X, then so does more of his/her larger (smaller, respectively) initial wealth w_2 in Y than in X. □

Remark 4.2.

(1) If $f_X(z+w)$ is TP_2 with respect to w and z, then random variable X is said to be DLR (Decreasing Likelihood Ratio). In this case, it is well known that the coefficient of variation of X satisfies

$$\mathrm{C}[X] := \frac{\sigma[X]}{\mathbf{E}[X]} \geq 1. \tag{4.27}$$

(2) If $f_Y(z-w)$ is TP_2 with respect to w and z, then random variable Y is said to be ILR (Increasing Likelihood Ratio). In this case, it is well known that the coefficient of variation of Y satisfies

$$\mathrm{C}[Y] := \frac{\sigma[Y]}{\mathbf{E}[Y]} \leq 1. \tag{4.28}$$

Generally, a function f $(: \mathbf{R} \to \mathbf{R}_+)$ is called PF_2 (Pólya Frequency of Order 2) if $f(z-w)$ is TP_2 with respect to w and z (see Barlow and Proschan [2, 3] and Karlin [12]). □

From the above, if

(1) $f_X(z+w)$ is TP_2 with respect to w and z;

(2) $f_Y(z-w)$ is TP_2 with respect to w and z;

(3) $\mathbf{E}[X] \geq \mathbf{E}[Y]$,

then we have

$$\sigma[X] \geq \mathbf{E}[X] \geq \mathbf{E}[Y] \geq \sigma[Y], \tag{4.29}$$

that is, asset X is more "high risk and high return" than asset Y.

4.1.5 Bivariate Normal Cases

We consider the case when the random vector (X,Y) has a bivariate normal distribution, that is,

$$(X,Y) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{4.30}$$

where $\boldsymbol{\mu} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$ is the mean vector, and $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_X^2 & \sigma_{X,Y} \\ \sigma_{X,Y} & \sigma_Y^2 \end{pmatrix}$ is the variance–covariance matrix. Further, the correlation coefficient is defined as

$$\rho := \frac{\sigma_{X,Y}}{\sigma_X \sigma_Y}. \tag{4.31}$$

In this case, the joint density function of (X,Y) is given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{Q(x,y)}{2}}, \tag{4.32}$$

where

$$Q(x,y) := \frac{1}{1-\rho^2}\left\{\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right\}, \tag{4.33}$$

and the conditional density function of Y given $\{X = x\}$, written $f_{Y|X}(y|x)$, is a probability density function of a univariate normal distribution

$$N\left(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X),\ \sigma_Y^2(1 - \rho^2)\right) \tag{4.34}$$

(see, e.g., Tong [34]).

By this result, we have

$$m_{Y|X}(x) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) \tag{4.35}$$

so that

$$\begin{aligned} x - m_{Y|X}(x) &= x - \left(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)\right) \\ &= \left(1 - \rho\frac{\sigma_Y}{\sigma_X}\right)x - \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}\mu_X. \end{aligned} \tag{4.36}$$

Therefore, if we set $c := 1 - \rho\frac{\sigma_Y}{\sigma_X}$, then $x - m_{Y|X}(x)$ changes its sign at most once in x and its possible sign change is from negative to positive for $c > 0$, and from positive to negative for $c < 0$.

Since the correlation coefficient satisfies $-1 \leq \rho \leq 1$, the conditions for the sign of c are characterized as follows:

(1) If $\sigma_X \geq \sigma_Y$ then $c \geq 0$;

(2) If $\sigma_X < \sigma_Y$ then

  (2.1) $c > 0$ for $-1 \leq \rho < \dfrac{\sigma_X}{\sigma_Y}$,

  (2.2) $c < 0$ for $\dfrac{\sigma_X}{\sigma_Y} < \rho \leq 1$.

Accordingly, let us use standard deviation (or variance) of return rate as a "risk" measure of risky asset, and say that "X is riskier than Y" when $\sigma_X \geq \sigma_Y$, Then, the following corollary is obtained from Theorem 4.1.

Corollary 4.7. Assume that the random vector (X, Y) has a bivariate normal distribution. Let a positive initial wealth w (> 0) be arbitrarily fixed.

(1) Suppose that X is riskier than Y, that is, $\sigma_X \geq \sigma_Y$. If an investor does not invest all of his/her initial wealth exclusively in X, then neither does a more risk averse investor in the Arrow–Pratt sense (if an investor invests a positive proportion of his/her initial wealth in Y, then so does a more risk averse investor in the sense of Arrow–Pratt.)

(2) Suppose that Y is riskier than X, that is, $\sigma_X \leq \sigma_Y$.

  (2.1) When $-1 \leq \rho < \dfrac{\sigma_X}{\sigma_Y}$, if an investor does not invest all of his/her initial wealth exclusively in X, then so does not a more risk averse investor in the Arrow–Pratt sense (if an investor invests a positive proportion of his/her initial wealth in Y, then so does a more risk averse investor in the sense of Arrow–Pratt);

  (2.2) When $\dfrac{\sigma_X}{\sigma_Y} < \rho \leq 1$, if an investor invests all of his/her initial wealth exclusively in X, then so does a more risk averse investor in the Arrow–Pratt sense. □

Similarly, from Theorem 4.3, we have the following corollary.

Corollary 4.8. Assume that the random vector (X, Y) has a bivariate normal distribution. Let (a vN–M utility function u of) a risk averse investor be fixed. Suppose that his/her vN–M utility function u displays IRRA (DRRA, respectively).

(1) Suppose that X is riskier than Y, that is, $\sigma_X \geq \sigma_Y$. If an investor does not invest all of his/her wealth w_1 exclusively in X, then neither does he/she all of his/her larger (smaller, respectively) initial wealth w_2 exclusively in X (if an investor invests a positive proportion of his/her initial wealth w_1 in Y, then so does he/she a positive proportion of his/her larger (smaller, respectively) initial wealth w_2 in Y).

(2) Suppose that Y is riskier than X, that is, $\sigma_X \leq \sigma_Y$.

(2.1) When $-1 \leq \rho < \dfrac{\sigma_X}{\sigma_Y}$, if an investor does not invest all of his/her initial wealth exclusively in X, then so does not a more risk averse investor in the Arrow–Pratt sense (if an investor invests a positive proportion of his/her initial wealth in Y, then so does a more risk averse investor in the sense of Arrow–Pratt;)

(2.2) When $\dfrac{\sigma_X}{\sigma_Y} < \rho \leq 1$, if an investor invests all of his/her wealth w_1 exclusively in X, then so does he/she all of his/her larger (smaller, respectively) initial wealth w_2 exclusively in X. □

In the sequel, the following lemma plays important roles.

Lemma 4.1 (Covariance Operator of Stein–Rubinstein). Assume that the random vector (X, Y) has a bivariate normal distribution, and a function g $(: \mathbb{R} \to \mathbb{R})$ is a differentiable function. Then, under a suitable integrability condition, we have

$$\mathrm{Cov}(X, g(Y)) = \mathrm{Cov}(X, Y)\mathbf{E}[g'(Y)]. \tag{4.37}$$

□

Theorem 4.5. Assume that the random vector (X, Y) has a bivariate normal distribution. If $\mu_X \leq \mu_Y$ and $\sigma_X \geq \sigma_Y$ then any risk averse investor invests more of his/her initial wealth in Y than in X. □

Now, if the random vector (X, Y) has a bivariate normal distribution, then the random vector $(Y, Z) = \left(Y, \dfrac{X+Y}{2}\right)$ has another bivariate normal distribution, and its mean vector and variance–covariance matrix are as follows (see, e.g., Tong [34]):

$$\boldsymbol{\mu}^\dagger := \begin{pmatrix} \mu_Y \\ \mu_Z \end{pmatrix} = \begin{pmatrix} \mu_Y \\ \dfrac{\mu_X + \mu_Y}{2} \end{pmatrix}, \tag{4.38}$$

$$\boldsymbol{\Sigma}^\dagger := \begin{pmatrix} \sigma_Y^2 & \sigma_{Y,Z} \\ \sigma_{Y,Z} & \sigma_Z^2 \end{pmatrix} = \begin{pmatrix} \sigma_Y^2 & \dfrac{\sigma_{X,Y} + \sigma_Y^2}{2} \\ \dfrac{\sigma_{X,Y} + \sigma_Y^2}{2} & \dfrac{\sigma_X^2 + 2\sigma_{X,Y} + \sigma_Y^2}{4} \end{pmatrix}. \tag{4.39}$$

Further, if we define the correlation coefficient between Y and Z as

$$\rho^\dagger := \frac{\sigma_{Y,Z}}{\sigma_Y \sigma_Z}, \tag{4.40}$$

then, similarly to the previous argument, the conditional distribution of Y given $\{Z = z\}$ is the following normal distribution:

$$N\left(\mu_Y + \rho^\dagger \frac{\sigma_Y}{\sigma_Z}(z - \mu_Z),\ \sigma_Y^2\left(1 - {\rho^\dagger}^2\right)\right). \tag{4.41}$$

From the above results, we have

$$m_{Y|Z}(z) = \mu_Y + \rho^\dagger \frac{\sigma_Y}{\sigma_Z}(z - \mu_Z), \tag{4.42}$$

$$
\begin{aligned}
z - m_{Y|Z}(z) &= z - \left(\mu_Y + \rho^{\dagger}\frac{\sigma_Y}{\sigma_Z}(z-\mu_Z)\right) \\
&= \left(1 - \rho^{\dagger}\frac{\sigma_Y}{\sigma_Z}\right) z - \mu_Y + \rho^{\dagger}\frac{\sigma_Y}{\sigma_Z}\mu_Z.
\end{aligned} \tag{4.43}
$$

Therefore, if we let $c^{\dagger} := 1 - \rho^{\dagger}\frac{\sigma_Y}{\sigma_Z}$, then $z - m_{Y|Z}(z)$ changes its sign in z at most once, and its possible sign change is from negative to positive for $c^{\dagger} > 0$, and from positive to negative for $c^{\dagger} < 0$. Rewriting $c^{\dagger}$ as

$$
c^{\dagger} = 1 - \rho^{\dagger}\frac{\sigma_Y}{\sigma_Z} = \frac{\sigma_X^2 - \sigma_Y^2}{\sigma_X^2 + 2\sigma_{X,Y} + \sigma_Y^2}, \tag{4.44}
$$

we have, by Theorem 4.2, the following corollary.

Corollary 4.9. Assume that the random vector (X, Y) has a bivariate normal distribution. Let a positive initial wealth w (> 0) be arbitrarily fixed.

- Suppose that X is riskier than Y, that is, $\sigma_X \geq \sigma_Y$. Then, if an investor invests more of his/her initial wealth in Y than in X, then so does a more risk averse investor in the sense of Arrow–Pratt. □

Similarly, by Theorem 4.4, we obtain the following corollary.

Corollary 4.10. Assume that the random vector (X, Y) has a bivariate normal distribution. Let (a vN–M utility function u of) a risk averse investor be fixed. Suppose that his/her vN–M utility function u displays IRRA (DRRA, respectively).

- Suppose that X is riskier than Y, that is, $\sigma_X \geq \sigma_Y$. Then, if the investor invests more of his/her initial wealth w_1 in Y than in X, then so does more of his/her larger (smaller, respectively) initial wealth w_2 in Y than in X. □

4.2 Analysis Based on Ross Ordering of Risk Aversion

In this subsection, we examine the risk aversion and initial wealth effects on the optimal portfolio based on the ordering of risk aversion proposed by Ross, S. A. which is a stronger notion than that of Arrow–Pratt.

Ross [27] proved the following comparative statics results.

Theorem 4.6 (Ross [27]). Let a positive initial wealth w (> 0) be arbitrarily fixed. Assume that $m_{Y|X}(x) \geq x$ for all possible x. Then, the (or an) optimal proportion of the initial wealth invested in X is larger for a more risk averse investor in the sense of Ross. □

Theorem 4.7 (Ross [27]). Let an investor be fixed. Assume that $m_{Y|X}(x) \geq x$ for all possible x, and the investor's utility function displays IRRA (DRRA, respectively) in the sense of Ross. Then, the (or an) optimal proportion of his/her initial wealth invested in X increases (decreases, respectively) in his/her initial wealth. □

Notice that the statement "$m_{Y|X}(x) \geq x$ for all possible x" implies that Y is riskier and offers a higher return than X in a sense.

Above two theorems are very interesting since they don't assume the distribution form of returns rates on the assets X, Y. However, the condition "$m_{Y|X}(x) \geq x$ for all possible x" does not hold in some important cases, for an example, in the case when the random vector (X, Y) has a bivariate normal distribution. Hence, in this section, we will discuss the case of bivariate normal distribution.

Theorem 4.8. Let a positive initial wealth w (> 0) be arbitrarily fixed. Assume that the random vector (X, Y) has a bivariate normal distribution, and that the mean μ_X of X is smaller than the mean μ_Y of Y. Then, the (or an) optimal proportion of the initial wealth invested in X is larger for a more risk averse investor in the sense of Ross. □

Theorem 4.9. Let (a vN–M utility function u of) a risk averse investor be fixed. Suppose that the investor's utility function u displays IRRA (DRRA, respectively) in the sense of Ross. Assume that the random vector (X, Y) has a bivariate normal distribution, and the mean μ_X of X is smaller than the mean μ_Y of Y. Then, the optimal proportion of his initial wealth invested in X increases (decrease, respectively) in his initial wealth. □

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A Appendix

A.1 Total Positivity

In this appendix, we provide the information needed in this paper about total positivity. The theory of totally positive functions is very rich and the results provided here is indeed only "the tip of the iceberg." More detailed discussions of the theory of total positivity are in Karlin [12].

Definition A.1 (Total Positivity of Order n). A real valued function $K(x,y)$ defined on a rectangle subset $X \times Y$ in $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, is said to be *Totally Positive of order n* (TP$_n$) in x and y if and only if, for all possible $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$, we have

$$K(x_1,y_1) \geq 0, \tag{A.1}$$

and for each $k = 2, \cdots, n$,

$$\begin{vmatrix} K(x_1,y_1) & \cdots & K(x_1,y_k) \\ \vdots & \ddots & \vdots \\ K(x_k,y_1) & \cdots & K(x_k,y_k) \end{vmatrix} \geq 0. \tag{A.2}$$

□

For a function $K(x,y)$ defined on a rectangle subset $X \times Y$ in $\mathbb{R}^2$, we denote

$$K\begin{pmatrix} x_1, & x_2 \\ y_1; & y_2 \end{pmatrix} = \det \begin{pmatrix} K(x_1,y_1) & K(x_1,y_2) \\ K(x_2,y_1) & K(x_2,y_2) \end{pmatrix}, \quad x_1 < x_2, \quad y_1 < y_2.$$

Then, for $n = 2$, the above definition is reduced to the following.

Definition A.2. A nonnegative function $K(x,y)$ defined on a rectangle subset $X \times Y$ in $\mathbb{R}^2$, is said to be *Totally Positive of order* 2 or simply TP$_2$, denoted by $K \in \mathrm{TP}_2(X \times Y)$, if and only if

$$K\begin{pmatrix} x_1, & x_2 \\ y_1; & y_2 \end{pmatrix} = K(x_1,y_1)K(x_2,y_2) - K(x_1,y_2)K(x_2,y_1) \geq 0, \quad x_1 < x_2, \quad y_1 < y_2.$$

□

Assuming the twice differentiability of the function, its TP$_2$ property is easily verified by its 2nd order derivative.

Lemma A.1. Continuously twice differentiable positive valued function $K(x,y)$ defined on a rectangle subset $X \times Y$ in $\mathbb{R}^2$, is TP$_2$ in x and y if and only if

$$\frac{\partial^2 \log K(x,y)}{\partial x \partial y} \geq 0 \quad \text{for all } (x,y) \in X \times Y. \tag{A.3}$$

□

The property (A.3) is called as *log-super-modularity* of function K.

For two nonnegative functions $K(x,z)$, $L(z,y)$ defined on rectangle subsets $X \times Z$ and $Z \times Y$ in $\mathbb{R}^2$, respectively, let

$$M(x,y) := \int_{-\infty}^{\infty} K(x,z)L(z,y)\mathrm{d}z, \quad x \in X, \quad y \in Y.$$

The next result is a special case of the well known *composition formula* (see page 17 of Karlin [12])).

Proposition A.1 (Composition Formula). We have

$$M\begin{pmatrix} x_1, & x_2 \\ y_1, & y_2 \end{pmatrix} = \int\int_{z_1<z_2} K\begin{pmatrix} x_1, & x_2 \\ z_1, & z_2 \end{pmatrix} L\begin{pmatrix} z_1, & z_2 \\ y_1, & y_2 \end{pmatrix} \mathrm{d}z_1\mathrm{d}z_2.$$

As a consequence, if $K \in \mathrm{TP}_2(X \times Z)$ and $L \in \mathrm{TP}_2(Z \times Y)$, then $M \in \mathrm{TP}_2(X \times Y)$. □

Definition A.3 (Pólya Frequency of Order 2). A nonnegative function $f(x)$ defined on a subset of $\mathbb{R}$ is said to be *Pólya Frequency of order* 2 (PF_2) in x if and only if $f(x-y)$ is TP_2 in x and y, i.e.,

$$f(x_1-y_1)f(x_2-y_2) \geq f(x_1-y_2)f(x_2-y_1); \quad x_1 < x_2, \quad y_1 < y_2.$$

□

The class of PF_2 functions is important and has many applications in various fields. A key property that every PF_2 function possesses is the characterization that it has the form $f(x) = \mathrm{e}^{-\phi(x)}$ where $\phi(x)$ is a convex function.

Proposition A.2. Every PF_2 function on a subset of $\mathbb{R}$ is log–concave. □

The next result is found in page 128 of Karlin [12].

Proposition A.3. For cumulative distribution functions F and G, let

$$H(x) := \int_{-\infty}^{\infty} F(x-y)\mathrm{d}G(y) = \int_{-\infty}^{\infty} G(x-y)\mathrm{d}F(y), \quad x \in \mathbb{R}.$$

If both F and G are PF_2 then so is H (H is a cumulative distribution function too). □

The next result is called the ***variation–diminishing property*** of Karlin [12] (the details are found in Section 3 of Chapter 5).

Theorem A.1 (Variation Diminishing Property). Suppose that $K(x,y)$ is TP_n in x and y, and $f(y)$ has at most $n-1$ sign changes. Let

$$g(x) := \int K(x,y)f(y)\mathrm{d}\mu(y) \tag{A.4}$$

where μ is a σ–finite measure on the Borel measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then $g(x)$ has at most $n-1$ sign changes. Moreover, if $g(x)$ has exactly $n-1$ sign changes, then they occur with same pattern as that of $f(y)$. □

A.2 TP_2 Functions and Theory of Risk Aversion

Let u_1, u_2 ($u_i' > 0$, $u_i'' \leq 0$, $i = 1,2$) be twice differentiable vN–M utility functions of two risk averse individuals, defined on a common open interval of the real line $\mathbb{R}$. Jewitt [11] showed that $u_1 \geq_{\mathrm{RA}} u_2$ if and only if

$$\frac{u_2'(x)}{u_1'(x)} \quad \text{is increasing in } x \tag{A.5}$$

or, equivalently,

$$\begin{vmatrix} u_1'(x) & u_1'(y) \\ u_2'(x) & u_2'(y) \end{vmatrix} \geq 0 \quad \text{for all } x < y. \tag{A.6}$$

The property in (A.6) can be stated in terms of total positivity. Namely, $u_i'(x)$ is TP_2 in x and $i = 1,2$ (see Karlin [12]).

Lemma A.2. Suppose that utility function u displays IRRA (DRRA, respectively) and $0 < w_1 \leq w_2$. Define $u_i(x) = u(w_i x)$, $i = 1,2$. Then, $u_1 \leq_{\mathrm{RA}} u_2$ ($u_2 \leq_{\mathrm{RA}} u_1$). □

Theorem A.2. For utility function u on $\mathbb{R}$, suppose that $u' > 0$, $u'' \leq 0$ and u''' exists. If u displays IRRA, then $u'(-ax)$ is TP_2 in $a > 0$ and $x \in \mathbb{R}$. □