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COMPLEMENT TO "WHAT DIFFERENTIATES STATIONARY STOCHASTIC PROCESSES FROM ERGODIC ONES: A SURVEY"

MICHEL VALADIER

These pages complete my paper entitled "What differentiates stationary stochastic processes from ergodic ones: a survey" published in *Publication series of the Research Institute of Mathematical Sciences* (Kyoto University, ed. Toru Maruyama) (2001), 33-52.

Rohlin [Ro] and Mackey [Mck1] are among the most important references I forgot. Note that the papers on the subject are tremendously numerous. I selected some ones excluding those treating differential geometry (flows) and C^*-algebras. Now follow some comments.

1. I wrote my paper before reading of [Mck1–2]. About prediction of stationary processes and the identification of the law of an ergodic process, [Mck1] exposes another way to reconstruct the process law from the complete past of one trajectory; see specially the bottom of page 204 till the end of Section 5 and the top of page 223. Many comments in [Mck1] go in the same direction as my paper.

2. Among recent books, see Kifer [Ki], Kallenberg [Ka] and McCutcheon [McC] for their short proofs. Moreover Kallenberg [Ka, Theorem 9.12] proves an ergodic decomposition theorem for a finite number of measurable transformations $T_1, \dots, T_d$ which commute; he uses a mean spatial ergodic theorem he proved before [Ka, Theorem 9.9].

3. If I had now to rewrite my paper, I would emphazise the method of Kryloff and Bogoliouboff [KB] and then the possibility to going to some abstract measurable spaces.

In [KB], [Ox], [DGS], the space K in which the process takes its values is compact metrizable and this is the easiest case to handle. But most processes are unbounded $\mathbb{R}$-valued ones, so the proofs of these papers do not directly apply. One can consider $\mathbb{R}$ as a subspace of the compact $\overline{\mathbb{R}}$: this "respects" the topology and introduces a compact over-space, but the big drawback is that $\mathbb{R}$ is not closed in $\overline{\mathbb{R}}$.

This leads to say some words about isomorphisms. Note that in Ergodic Theory maybe three notions of isomorphisms can be used[1]: one-to-one map between sets, one-to-one map between subsets of full measure and isomorphism of the quotient σ-algebras such as $\mathcal{F}/P$ (the set A and B in $\mathcal{F}$ are equivalent if $P(A \triangle B) = 0$). See Petersen [Pe, pp.15–17] and (only for the two first notions) [DGS, pp.3–5].

A general hypothesis encountered in most papers is: the space $(K, \mathcal{K})$ in which the process takes its values is a *Borel standard* (or *Lusin*) measurable space, that is

[1] Isomorphism with $[0, 1]$ is an essential tool in [Ro]. See also [Mah, Th.6 p.157].

a measurable space isomorphic to a Borel subset of a Polish topological space. Any Borel standard space $(K, \mathcal{K})$ is either countable, either has the cardinality of $\mathbb{R}$. In the first case it is isomorphic to $\{1, \ldots, n\}$ or to $\mathbb{N} \cup \{\infty\}$ (the tribe being that of all subsets). In the second case it is isomorphic to $([0,1], \mathcal{B}([0,1]))$. As a reference see [DeM, Appendice au chapitre III, Th.80 p.249] (from many authors all properties of Borel standard spaces are proved in Kuratowski's book [Ku]). Hence if $(K, \mathcal{K})$ is Borel standard there exists a compact metrizable topology on K whose Borel tribe coincide with $\mathcal{K}$.

For example $\mathbb{R}$ is Borel standard. A direct way to check that $\mathbb{R}$ is isomorphic as a measurable space to $\overline{\mathbb{R}}$ is the following: let $\varphi\colon \mathbb{R} \to \overline{\mathbb{R}}$ defined by $\varphi(x) = x$ on $\mathbb{R} \backslash \mathbb{N}$ and any bijection from $\mathbb{N}$ onto $\mathbb{N} \cup \{-\infty, +\infty\}$.

Among all works treating Borel standard spaces I single out Chersi [Che] and Dynkin [Dy]. Maybe only Chersi succeeded proving narrow convergence of the sequence $(Q_n^\omega)_n$; surely this is thanks to the notion he used of Daniell integral. Dynkin, whose method is summarized in my paper, develops his idea of sufficient statistic, and covers with a unified approach several other notions: Gibbs states, symmetric laws (de Finetti-Hewitt-Savage — on this question see Aldous [Ald]), superharmonic functions...

The work of Lauritzen [Lau] could have some connections with [Dy] (this author, in a preliminary work, his thesis, does not quote Dynkin's paper. I did not see the book [Lau]).

4. For applications to homogenization the acting group is $\mathbb{R}^d$ where d is the dimension of the domain under consideration. In 1962 Farrell [Fa] and Varadarajan [Var1–2] worked simultaneously and independently in the case when G is a locally compact group (G is not necessarily commutative but it should admit a countable dense subgroup); see also [Dy, Remark p.717]. Both use limit theorems about powers of compositions of conditional expectations. Note that the case of flows (that is $G = \mathbb{R}$) was already treated by von Neumann and Kryloff-Bogoliouboff and that Fomin gave in 1950 some results in this line ([Fo] is in Russian).

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