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Risk-Aversion, Intertemporal Substitution, and the Investment-Uncertainty Relationship: A Continuous-Time Model

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Abstract

This paper investigates the roles of risk-aversion and intertemporal substitution in the investment-uncertainty relationship. To distinguish the effect of intertemporal substitution from that of risk-aversion, we utilize a non-expected utility maximization approach. It is shown that not only the degree of risk-aversion but also the elasticity of intertemporal substitution plays a crucial role in determining the sign of the investment-uncertainty relationship for a competitive firm in a continuous-time dynamic model. Also, the non-expected utility approach gets rid of undesirable properties of the investment function derived in the standard state- and time-separable expected-utility setup.

JEL classification: D92, E22

Key words: Risk-aversion, Intertemporal substitution, Uncertainty, Investment, Non-expected utility maximization

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1. Introduction

Beginning with the influential contribution of Hartman (1972), which was in turn related to the seminal work of Oi (1961), a large number of theoretical studies have been done on the investment-uncertainty relationship. Hartman showed that a mean-preserving spread in the distribution of the price of output leads a competitive risk-neutral firm to increase investment in a discrete-time dynamic model of investment. Abel (1983) verified this finding in a continuous-time setting. This somewhat paradoxical result depends crucially on the fact that the marginal product of capital is a convex function of the random variable(s) and therefore is due to Jensen's inequality. However, such recent empirical studies as Calgagnini and Saltari (2001), Ferderer (1993), Guiso and Parigi (1999), Leahy and Whited (1995), and Price (1996) find evidence for a negative relationship between investment and uncertainty.[1]

In order to reconcile the theoretical predictions with the empirical findings, we need an element of concavity and/or asymmetry. A natural way to introduce asymmetry is to consider irreversible investment. The literature on irreversible investment (e.g., Pindyck, 1988) has shown that increased uncertainty reduces the optimal rate of investment. The asymmetry in the investment process arises not only from the strict irreversibility but also when the cost of adjusting capital stock downward is much larger than the upward adjustment cost.[2] However, as Caballero (1991) correctly points out, asymmetric adjustment costs are not sufficient to yield the result. Another important condition is required that ensures some linkage between current and future investment like decreasing returns to scale or downward sloping demand. Only when the aforementioned two conditions are met, the irreversibility effect can dominate the convexity effect. This implies that under the assumption of the competitive firm with linearly homogenous technology, such as Hartman and Abel, irreversibility does not play a crucial role.

Risk-aversion is another line to invalidate the convexity of the marginal product of capital of the competitive firm with linearly homogenous technology. In the case of a risk-averse firm, although its cash flow is still a convex function of the output price, its expected utility is a concave function of its cash flow. In other words, the convex profit function is passed through a concave utility function. As Nakamura (1999) shows, with enough risk aversion, the convexity argument can be turned around.[3] However, the investment function derived there has a couple of strange properties: a very risk-averse firm behaves like a risk neutral one and a rise in a capital depreciation rate may increase investment.

Recently, Saltari and Ticchi (2001) shows that the above strange features can be gotten rid of by distinguishing the intertemporal substitution from the risk-aversion in a discrete-time setup with i.i.d. uncertainty of a stochastic variable. They use Kreps-Porteus non-expected utility preferences instead of time- and state-separable isoelastic preferences. This paper shows that their results hold in a continuous-time setting with uncertainty that follows a stochastic Brownian motion.

The organization of the rest of this paper is as follows. Section 2 presents a simple investment model of a firm with a non-expected utility preference. Section 3 investigates the role of intertemporal substitution in the investment-uncertainty relationship. The final section provides some concluding remarks.

2. The model

Consider a competitive firm using labor $L(t)$ and capital $K(t)$ to produce output $Y(t)$ according to a Cobb-Douglas production function:

$$Y(t) = L(t)^{\alpha} K(t)^{1-\alpha} \text{ with } 0 < \alpha < 1. \tag{1}$$

The firm hires labor at a fixed wage rate w and adjust labor input within each period. Therefore, the instantaneous profit function takes the form:

$$hp(t)^{1/(1-\alpha)} K(t) \equiv \max_{L(t)} \{p(t)L(t)^{\alpha} K(t)^{1-\alpha} - wL(t)\}, \tag{2}$$

where $p(t)$ is the output price and $h = (1-\alpha)(\alpha/w)^{\alpha/(1-\alpha)}$. Suppose that $I(t)$ is the rate of investment and $p_I(t)$ the investment goods price. Then the firm's cash flow $\pi(t)$ becomes

$$\pi(t) = hp(t)^{1/(1-\alpha)}K(t) - p_I(t)I(t). \tag{3}$$

The investment goods price is considered to be related with the profitability of the existing capital stock or the marginal revenue product of capital $hp(t)^{1/(1-\alpha)}$. For analytical tractability, we assume the ratio of $p_I(t)$ to $hp(t)^{1/(1-\alpha)}$ is constant at q over time,

$$p_I(t)/hp(t)^{1/(1-\alpha)} = q \text{ or } p_I(t) = qhp(t)^{1/(1-\alpha)}. \tag{4}$$

This assumption implies that the price of the investment good is a nonlinear function of the output price. Since the output price does not have a trend, however, this assumption might not be strong, especially if uncertainty (σ) is not big.[4]

The output price evolves according to the following equation:

$$dp(t)/p(t) = \sigma dz(t), \tag{5}$$

where $dz(t)$ is a Wiener process with mean zero and unit variance, and σ is a positive constant. Also, the capital accumulation equation is

$$dK(t) = \{I(t) - \delta K(t)\}dt, \tag{6}$$

where δ is the constant capital depreciation rate. Let us define $W(t) = hp(t)^{1/(1-\alpha)}K(t)$, which is the value of capital stock evaluated by the current profitability.[5] Applying Ito's lemma to obtain

$$dW = \frac{\partial W}{\partial K}dK + \frac{\partial W}{\partial p}dp + \frac{1}{2}\frac{\partial^2 W}{\partial K^2}(dK)^2 + \frac{1}{2}\frac{\partial^2 W}{\partial p^2}(dp)^2 + \frac{\partial^2 W}{\partial K \partial P}(dK)(dp). \tag{7}$$

For notational convenience, time arguments are suppressed as long as no ambiguity results. Substituting (3), (4), (5), and (6) for (7), and recognizing that $(dt)^2 = (dt)(dz) = 0$ and $(dz)^2 = dt$, we have

$$dW = r_W W dt + \sigma_W W dz - (\pi/q)dt, \tag{8}$$

where $r_W = [\alpha\sigma^2/2(1-\alpha)^2] + q^{-1} - \delta$ and $\sigma_W = \sigma/(1-\alpha)$, both of which are constant over time. We can interpret that r_W is the expected rate of return of "risky" asset $W(t)$, and σ_W^2 is its instantaneous variance.

To distinguish the effect of intertemporal substitution from that of risk-aversion, we employ a non-expected utility maximization setup. We assume that at point in time t the firm maximizes the intertemporal objective $V(t)$ by recursion,

$$f([1-\gamma]V(t)) = \left(\frac{1-\gamma}{1-1/\varepsilon}\right)\pi(t)^{1-1/\varepsilon}h + e^{-\rho h}f([1-\gamma]E_tV(t+h)), \tag{9}$$

where the function $f(x)$ is given by

$$f(x) = \left(\frac{1-\gamma}{1-1/\varepsilon}\right)x^{(1-1/\varepsilon)/(1-\gamma)}. \tag{10}$$

In (9), h is the economic decision interval, E_t is a mathematical expectation conditional on time-t information, and $\rho > 0$ the subjective discount rate. The parameter $\gamma > 0$ measures the relative risk-aversion while the parameter $\varepsilon > 0$ is the intertemporal substitution elasticity.[6] When $\gamma = 1/\varepsilon$, so that $f(x) = x$, our setup is the standard state- and time-separable expected-utility setup, which does not allow independent variation in risk aversion and intertemporal substitutability over time.[7]

Let $J(W(t))$ denote the maximum feasible level of the expected sum of discounted cash flows. The value function $J(W(t))$ depends on the contemporaneous variable $W(t)$ only. Applying Ito's lemma to the maximization of $V(t)$ in (9), we get the following stochastic Bellman equation:

$$0 = \max_{\pi}\{[(1-\gamma)/(1-1/\varepsilon)]\pi^{1-1/\varepsilon} - \rho f([1-\gamma]J(W))$$
$$+ (1-\lambda)f'([1-\lambda]J(W))[J'(W)(r_W W - \pi/q) + (1/2)J''(W)\sigma_W^2 W^2]\}. \tag{11}$$

From (11), the first-order condition with respect to π is

$$\pi^{-1/\varepsilon} - f'([1-\gamma]J(W))J'(W)/q = 0. \tag{12}$$

Eq. (9)'s form suggests that $J(W)$ is given by

$$J(W) = (aW)^{1-\gamma}/(1-\gamma), \tag{13}$$

where a is a positive constant to be determined. Eq. (12) becomes

$$\pi = \mu W. \tag{14}$$

where $\mu \equiv a^{1-\varepsilon}q^{\varepsilon}$. Substituting Eq. (14) for Eq. (11) gives

$$a = \{\varepsilon[\rho - (1-1/\varepsilon)(r_W - \gamma\sigma_W^2/2)]\}^{1/(1-\varepsilon)}q, \tag{15}$$

and therefore

$$\mu = \varepsilon\{\rho - (1-1/\varepsilon)(r_W - \gamma\sigma_W^2/2)\}q, \tag{16}$$

where $r_W - \gamma\sigma_W^2/2$ is the risk-adjusted rate of return of asset W. From Eqs. (3), (4), (14) and (16), we have

$$I(t) = \frac{hp(t)^{1/(1-\alpha)}K(t) - \pi(t)}{p_I(t)} = \Big[\frac{1}{q} - \varepsilon\Big\{\rho - \Big(1-\frac{1}{\varepsilon}\Big)\Big(r_W - \frac{\gamma\sigma_W^2}{2}\Big)\Big\}\Big]K(t). \tag{17}$$

Finally, substituting the definitions of r_W and σ_W for Eq. (17), we have the following investment function:

$$I(t) = \varepsilon\Big[\frac{1}{q} - \Big\{\rho - \Big(1-\frac{1}{\varepsilon}\Big)\Big(\frac{(\alpha-\gamma)\sigma^2}{2(1-\alpha)^2} - \delta\Big)\Big\}\Big]K(t). \tag{18}$$

We must notice that q^{-1} in Eqs. (17) and (18) corresponds to $hp(t)^{1/(1-\alpha)}/p_I(t)$ in Eq. (17) in our normalization. Therefore, Eq. (18) implies

$$hp(t)^{1/(1-\alpha)} \gtreqless \Big\{\rho - \Big(1-\frac{1}{\varepsilon}\Big)\Big(\frac{(\alpha-\gamma)\sigma^2}{2(1-\alpha)^2} - \delta\Big)\Big\}p_I(t) \Leftrightarrow I(t) \gtreqless 0. \tag{19}$$

Realizing that $\{\rho - (1-\frac{1}{\varepsilon})(\frac{(\alpha-\gamma)\sigma^2}{2(1-\alpha)^2} - \delta)\}$ is the risk-adjusted discount rate, the investment function has plausible nature in which the marginal revenue product of capital $hp(t)^{1/(1-\alpha)}$ is larger than the user cost of capital $\{\rho-(1-\frac{1}{\varepsilon})(\frac{(\alpha-\gamma)\sigma^2}{2(1-\alpha)^2}-\delta)\}p_I(t)$, the firm executes investment, and in the reverse case, it sells its capital equipment.

3. The role of intertemporal substitution

From (18), we have the following relationship:

$$sign(dI(t)/d\sigma)=sign((1-1/\varepsilon)(\alpha-\gamma))=sign((\varepsilon-1)(\alpha-\gamma)). \qquad (20)$$

It is evident that the sign of the investment-uncertainty relationship depends both the degree of risk-aversion and the elasticity of intertemporal substitution. In principle, risk-aversion affects the investment-uncertainty relationship via changing the risk-adjusted rate of return $r_W-\gamma\sigma_W^2/2$ while intertemporal substitution affects the relationship through the choice between current and future cash flows.

To make this clear, let us imagine a two-period model in which the firm maximizes its utility: $u(\pi_1,\pi_2)$, subject to two budget constraints: $W_1-\pi_1=I$ and $\{1+(r_W-\gamma\sigma_W^2/2)\}I=\pi_2$, or the corresponding intertemporal budget constraint: $\pi_1+\pi_2/\{1+(r_W-\gamma\sigma_W^2/2)\}=W_1$. It is obvious that an increase in uncertainty raises the risk-adjusted rate of return if $\alpha>\gamma$ and vice versa, or

$$sign(d(r_W-\gamma\sigma_W^2/2)/d\sigma)=sign(\alpha-\gamma). \qquad (21)$$

[Fig. 1 is around here.]

Hence, as Fig. 1 shows, the intertemporal budget constraint shifts inside when $\alpha<\gamma$. Since this makes the firm poorer than before, the firm's utility level becomes lower. (The new budget constraint (BC') is now tangent to indifference curve IC_2.) By the income effect, therefore, both π_1 and π_2 decrease. [8] At the same time, however, the substitution effect increases π_1 and decreases π_2 because a decrease in $r_W - \gamma\sigma_W^2/2$ implies an increase in the *price* of π_2. If the substitution effect dominates the income effect, then increased uncertainty increases π_1 even if the budget constraint shifts inwards, and therefore decrease investment, $I = W_1 - \pi_1$, as Fig.1 (a) demonstrates. In this case we have the negative investment-uncertainty relationship when $\alpha < \gamma$. This is the result in Nakamura (1999). But it is not always true.

If the income effect dominates the substitution effect, then there appears the case that both π_1 and π_2 decrease as is shown in Fig. 1 (b). Therefore, when the substitution effect is relatively small, an increase in uncertainty raises investment even if $\alpha < \gamma$. This clearly shows the important role of intertemporal substitution in the investment-uncertainty relationship.

In our continuous-time model, since ε is the elasticity of intertemporal substitution, a fall in the risk-adjusted rate of return $r_W - \gamma\sigma_W^2/2$ raises the ratio of the current profits to the wealth $\mu = \pi/W$ when $\varepsilon > 1$, but lowers μ when $\varepsilon < 1$,

$$sign(d\mu/d(r_W - \gamma\sigma_W^2/2)) = sign(1-\varepsilon), \tag{22}$$

which is obvious from (16). It is also evident

$$\frac{d\mu}{d\sigma} = \frac{d\mu}{d(r_W - \gamma\sigma_W^2/2)} \cdot \frac{d(r_W - \gamma\sigma_W^2/2)}{d\sigma}, \tag{23a}$$

and therefore,

$$sign\left(\frac{d\mu}{d\sigma}\right) = sign\left(\frac{d\mu}{d(r_W - \gamma\sigma_W^2/2)}\right) \cdot sign\left(\frac{d(r_W - \gamma\sigma_W^2/2)}{d\sigma}\right). \tag{23b}$$

Substituting (21) and (22) for (23b), we have

$$sign(d\mu/d\sigma) = sign(1-\varepsilon) \cdot sign(\alpha - \gamma) = sign((1-\varepsilon)(\alpha - \gamma)). \tag{24}$$

Since $I(t) = (1-\mu)W(t)$, $dI(t)/d\sigma$ and $d\mu/d\sigma$ have the opposite signs, and hence we have the relationship in (20).

In Nakamura the degree of risk-aversion and the elasticity of intertemporal substitutability collapse into one parameter $\gamma(=1/\varepsilon)$. In this case, the relationship (20) becomes

$$sign(dI(t)/d\sigma) = sign((1-\gamma)(\alpha-\gamma)). \tag{25}$$

If we cannot distinguish the effect of intertemporal substitutability from that of risk-aversion, we may infer from the above that a very risk-averse firm ($\gamma > 1 > \alpha$) behaves like a risk-neutral firm ($\gamma = 0$) since investment increases with uncertainty for both types of firms. As we have shown, this is not true. Not risk-aversion but intertemporal substitutability plays a crucial role. Because of low intertemporal substitutability (a small ε or a large γ), investment increases with uncertainty for a risk-averse firm.

In our model as well as in Nakamura's, there is the possibility that a rise in a capital depreciation rate increases investment. This seems implausible if we do not consider the role of intertemporal substitutability. However, it is quite natural in our model. A rise in δ surely decreases the mean rate of return $r_W = [\alpha\sigma^2/2(1-\alpha)^2] + q^{-1} - \delta$ and hence the risk-adjusted rate of return $r_W - \gamma\sigma_W^2/2$, which in turn raises the ratio of the current profits to the wealth $\mu = \pi/W$ when $\varepsilon > 1$, but lowers μ when $\varepsilon < 1$. As a result, we have the following relationship:

$$sign(dI/d\sigma) = sign(1-\varepsilon), \tag{26}$$

which is directly derived from (18).

4. Concluding Remarks

This paper has analyzed the investment decision of a risk-averse firm with a constant return to scale technology using a continuous-time model. Appealing to a non-expected utility preference, it is shown that not only the risk aversion but also the intertemporal substitution plays a crucial role in determining the sign of the investment-uncertainty relationship. If the degree of risk-aversion is large, the sign may be negative. However, this is true only with a large intertemporal substitution elasticity. If the elasticity is low, we have the positive relationship even for a risk-averse firm.

One way to relate the intertemporal substitution in this paper with a plausible assumption is to consider the firm's owners' portfolio in which the substitution means consumption substitution over time. However, it may be more relevant to analyze the interaction in capital markets between risk-averse consumers and risk neutral firms in a dynamic framework. This deserves the subject of future research.

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Endnotes

[1] Carruth, Dickerson and Henley (2000) neatly summarizes the recent theoretical and empirical developments in investment under uncertainty.

[2] Irreversibility can be considered as a special case of asymmetric costs where the downward cost is infinite.

[3] Femminis (2000) also analyzes the investment decision of a risk-averse firm that can borrow outside resources at a risk-free rate and shows that the firm's portfolio considerations lead a negative investment-uncertainty relationship.

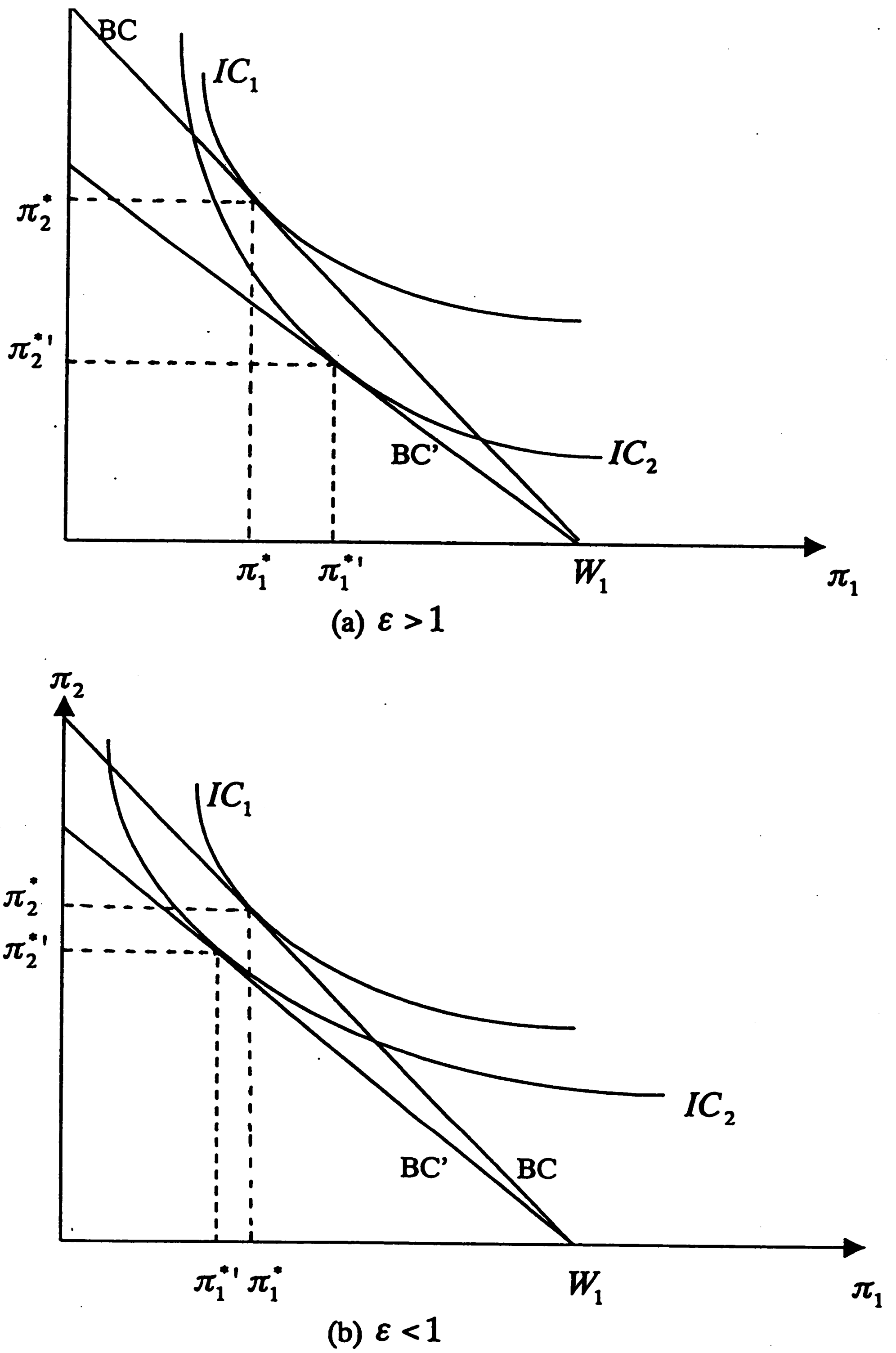
[4] Also, we should notice that in our model a risk-neutral firm increases investment with when uncertainty increases due to the convexity effect as in the traditional models of investment under uncertainty.

[5] It does not introduce any problem that the capital $K(t)$ is evaluated by current profitability $hp(t)^{1/(1-\alpha)}$. To characterize the solution, the absolute level of the total asset is not important. Only the rate of return of each asset and its variance are required, which becomes clear later.

[6] For a detailed discussion on the roles of these parameters and more general preference setups, see, for example, Kreps and Porteus (1979, 1979), Epstein and Zin (1989, 1991), Weil (1989), and Obstfeld (1994a, 1994b).

[7] This paper analyzes the firm's behavior in the limit as h becomes infinitesimally small. When $\gamma = 1/\varepsilon$, (8) implies that as $h \to 0$, $V(t)$ becomes the preference setup defined in Nakamura (1999), $V(t) = E_t\left\{(1-\gamma)^{-1}\int_t^\infty \pi(s)^{1-\gamma} e^{-\rho(s-t)} ds\right\}$.

[8] It is assumed that both π_1 and π_2 are normal goods.



e of Intertemporal Substitutability in a Two-Period Example When $\alpha < \gamma$

Aversion, Intertemporal Substitutability, and the Investment-Uncertainty Relationship: A Continuous-Time Model