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Monte Carlo Method for pricing of Bermuda type derivatives

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1 Introduction

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, \infty)}, P)\) be a filtered space with the usual condition, and \(\{B_t\}_{t \in [0, \infty)}\) be a \(d\)-dimensional Brownian motion. Let \(T > 0\), and let \(\sigma : [0, T] \times \mathbb{R}^D \to \mathbb{R}^D \times \mathbb{R}^d\) and \(b : [0, T] \times \mathbb{R}^D \to \mathbb{R}^D\) be continuous functions. For each \(s \in [0, T]\) and \(x \in \mathbb{R}^D\), let \(X(t; s, x), t \in [s, T]\) be a solution of the following SDE.

\[
X(t; s, x) = x + \int_s^t \sigma(r, X(r; s, x))dB_r + \int_s^t b(r, X(r; t, x))dr, \quad t \in [s, T].
\]

We assume that the above SDE 1 has a path-wise unique solution for every \((s, x) \in [0, T] \times \mathbb{R}^D\).

Concerning the pricing of American derivatives, we are interested in computing the following value function,

\[
u(s, x) = \sup \{E[g(\tau, X(\tau; s, x))]; \tau \in \tilde{S}_s^T\}, (s, x) \in [0, T] \times \mathbb{R}^D.
\]

There are several attempts to compute the value function \(u\) numerically. However, it seems that there are not so good method if \(D\) is not small. Let \(N \geq 2\) and let \(T_n, n = 0, 1, \ldots, N\), be positive numbers such that \(0 = T_0 < T_1 < \ldots < T_N = T\). Let \(S_n, n = 0, 1, \ldots, N\), be the set of \(\mathcal{F}_t\)-stopping times taking value in \(\{T_0, T_1, \ldots, T_N\}\). Concerning the pricing of Bermuda type derivatives, we are interested in computing the following value functions.

\[
v_n(x) = \sup \{E[g(\tau, X(\tau; s, x))]; \tau \in S_n\}, \quad n = 0, 1, \ldots, N.
\]

Let us define a probability measure \(p_n(x, \cdot)\) over \(\mathbb{R}^D\) for each \(n = 0, 1, \ldots, N - 1\), and \(x \in \mathbb{R}^D\) by

\[
p_n(x, A) = P(X(T_{n+1}; T_n, x) \in A), \quad \text{for a Borel set } A \text{ in } \mathbb{R}^D,
\]

and define a operator \(P_n, n = 0, 1, \ldots, N - 1\), by

\[
P_n f(x) = \int_{\mathbb{R}^D} f(y)p_n(x, dy) = E[f(X(T_{n+1}; T_n, x))].
\]
for a measurable function $f$ on $\mathbb{R}^D$. Then $v_n$, $n = N, N - 1, \ldots, 0$, are given inductively by the following.

$$v_N(x) = g(T_N, x),$$

$$v_{n-1}(x) = (P_{n-1}v_n)(x) \vee g(T_{n-1}, x).$$

So the value function $v_0(x)$ is easily given mathematically. However, if $D$ is not small, it is not easy to memorize a function on $\mathbb{R}^D$, and so it is not easy to compute $v_0(x)$.

Several people suggest a Monte-Carlo method to compute the value function. In this paper, we discuss the method given by [7]. We assume the following assumption (A).

(A1) $D_n$, $n = 0, 1, \ldots, N - 1$, are measurable sets in $\mathbb{R}^N$ such that $(P_n v_{n+1})(x) \geq g(T_1, x)$ for any $x \in \mathbb{R}^D \setminus D_1$.

Remark 1 (1) $D_0 = \mathbb{R}^D$ satisfies the assumption (A1).

(2) If $g(t, x) \geq 0$, for any $(t, x) \in [0, T] \times \mathbb{R}$, then $D_n = \{x \in \mathbb{R}^D; g(T_n, x) > 0\}$ satisfies the assumption (A1).

Now let $L_n \geq 1$, $n = 0, 1, \ldots, N - 1$, and $\bar{X}_{n, \ell} = \{X_{n, \ell}(m)\}_{m=0}^{N}$, $\ell = 1, \ldots, L_n$, $n = 0, 1, \ldots, N - 1$, are identically independent random vectors whose distribution is the same as the distribution of $\{X(T_m; 0, x)\}_{m=0}^{N}$. Let $K_n \geq 1$, $n = 0, 1, \ldots, N - 1$, and $\psi_{n,k}$, $k = 1, \ldots, K_n$, $n = 0, 1, \ldots, N - 1$, are functions on $\mathbb{R}^D$. Then we define functions $H_n$, $n = N, N - 1, \ldots, 1, 0$, on $\mathbb{R}^D$ inductively by the following.

$$H_N(x) = 1.$$

When $\bar{H}_{n+1} = \{H_m\}_{m=n+1}^{N}$, are given we let

$$\sigma_{n, \ell} = \min\{m \geq n + 1; H_m(X_{n, \ell}(m)) > 0\}, \quad \ell = 1, \ldots, L_n.$$

Then we let $\{\bar{a}_{n,k}\}_{k=1}^{K_n}$ be the minimizing point of the function

$$F_n(\bar{a}_k)_{k=1}^{K_n} = \frac{1}{L_n} \sum_{\ell=1}^{L_n} |g(T_{\sigma_{n, \ell}}, X_{n, \ell}(\sigma_{n, \ell})) - \sum_{k=1}^{K_n} a_n \psi_{n,k}(X_{n, \ell}(n))|^2 1_D(X_{n, \ell}(n)).$$

Finally we define $H_n$ by

$$H_n(x) = \begin{cases} g(T_n, x) - \sum_{k=1}^{K_n} \bar{a}_{n,k} \psi_{n,k}(x), & x \in D_n \\ -1, & x \in \mathbb{R}^D \setminus D_n. \end{cases}$$

Then we let

$$\bar{v}_0 = \frac{1}{L_0} \sum_{\ell=1}^{L_0} g(\sigma_{0, \ell}, X_{0, \ell}(\sigma_{0, \ell}))$$

and

$$\bar{\sigma} = \min\{T_n; H_n(X(T_n; 0, x)) > 0\}.$$

We think that $\bar{v}_0$ is an approximation of the value function $v_0(x)$ and the stopping time $\bar{\sigma}$ as a candidate of the optimal stopping time.
2 Preliminary Results

Let \( W_n = \mathbb{R}^{(N+1-n)D} \), \( n = 0, 1, \ldots, N \), and let \( P_n^{(n)} \), \( x \in \mathbb{R}^D \) be the distribution of \( \{X(T_m; T_n, x)\}_{m=n}^N \) on \( W_n \). Then \( P_n^{(n)} \), \( n = 0, 1, \ldots, N \), \( x \in \mathbb{R}^D \), is a Markov chain on \( \mathbb{R}^D \).

For any measurable function \( h \) on \( \mathbb{R}^D \) and \( n, m = 0, 1, \ldots, N \) with \( n \leq m \), let \( \tau_m(\cdot; h) : W_n \rightarrow \{m, N\} \) by

\[
\tau_m(w; h) = \begin{cases} 
  m, & h(w(m)) > 0, \\
  N, & h(w(m)) \leq 0.
\end{cases}
\]

**Lemma 2** Let \( h_n : \mathbb{R}^N \rightarrow \mathbb{R} \), \( n = 0, 1, \ldots, N \), be given, and assume that \( h_n(x) \leq 0 \), \( x \in \mathbb{R}^N \setminus D_n \), and that \( h_N(x) = 1 \). Let \( \sigma_n : W_n \rightarrow \{n, n+1, \ldots, N\} \) be given by

\[
\sigma_n(w) = \sigma_n(w; \{h_m\}_{m=n}^{N-1}) = \bigwedge_{m=n}^{N-1} \tau_m(w; h_m), \quad w \in W_n.
\]

Moreover, let \( u_n : \mathbb{R}^D \rightarrow \mathbb{R} \) be given by

\[
u_n(x) = u_n(x; \{h_m\}_{m=n}^N) = E^{P_n^{(n)}}[g(T_{\sigma_n}, w(\sigma_n))], \quad x \in \mathbb{R}^D,
\]

Then we have the following.

(1) \( |u_n(x) - v_n(x)| \leq |P_n(u_n+1 - v_{n+1})(x)| + 1_{D_n}(x)|P_n u_n+1(x) - (g(T_n, x) - h_n(x))| \)

for any \( n = 0, 1, \ldots, N - 1 \), and \( x \in \mathbb{R}^D \).

(2) \( |u_n(x) - v_n(x)| \leq |P_n(u_{n+1}-v_{n+1})(x)| + 1_{D_n}(x)1_{\{1\}}(\text{sgn}(P_n u_n(x) - g(T_n, x)) \text{sgn}(h_{n}(x)))|P_n u_n(x) - g(n, x)| \).

Here

\[
\text{sgn}(a) = \begin{cases} 
  1, & a > 0, \\
  0, & a = 0, \\
  -1, & a < 0.
\end{cases}
\]

**Proof.** Note that \( u_n(x) \leq v_n(x) \), for all \( n = 0, 1, \ldots, N - 1 \), and \( x \in \mathbb{R}^D \). Let \( \tilde{u}_n(x) = g(T_n, x) - h_n(x), x \in \mathbb{R}^D \).

Let \( n = 0, 1, \ldots, N - 1 \), and \( x \in \mathbb{R}^D \), and fix them for a while.

Case 1. Suppose that \( h_n(x) > 0 \).

Then we see that \( x \in D_n \) and \( g(T_n, x) > \tilde{u}_n(x) \). So we have

\[

v_n(x) = g(T_n, x) + (P_n v_{n+1}(x) - g(T_n, x)) \vee 0 \leq g(T_n, x) + |P_n v_{n+1}(x) - \tilde{u}_n(x)|.
\]

This implies

\[
g(T_n, x) \geq v_n(x) - |P_n(v_{n+1} - u_{n+1})(x)| - |P_n u_{n+1}(x) - \tilde{u}_n(x)|.
\]

Case 2. Suppose that \( h_n(x) \leq 0 \), and \( x \in D_n \).

Then we see that \( g(T_n, x) \leq \tilde{u}_n(x) \). So we see that

\[
v_n(x) \leq P_n v_{n+1}(x) \vee \tilde{u}_n(x) \leq P_n u_{n+1}(x) + |P_n(v_{n+1} - u_{n+1})(x)| + |P_n u_{n+1}(x) - \tilde{u}_n(x)|.
\]
Case 3. Suppose that $h_n(x) \leq 0$, and $x \in \mathbb{R}^D \setminus D_n$. Then we see that $g(T_n, x) \leq (P_n v_{n+1})(x)$. So we have

$$v_n(x) = P_n v_{n+1}(x) \leq P_n v_{n+1}(x) + |P_n(u_{n+1} - v_{n+1})(x)|.$$

So we see that for any $n = 0, 1, \ldots, N - 1$,

$$u_n = 1_{\{h_n > 0\}} g(T_n, \cdot) + 1_{\{h_n \leq 0\}} (P_n u_{n+1}) \geq 1_{\{h_n > 0\}} (v_n - |P_n(u_{n+1} - v_{n+1})| - |P_n u_{n+1} - \bar{u}_n|) + 1_{\{h_n \leq 0\}} 1_{D_n}(v_n - |P_n(u_{n+1} - v_{n+1})| - |P_n u_{n+1} - \bar{u}_n|).$$

Thus we see that

$$0 \leq v_n - u_n \leq |P_n(u_{n+1} - v_{n+1})| + |1_{\mathcal{D}} P_n u_{n+1} - \bar{u}_n|.$$

This implies the assertion (1).

Now let us prove the assertion (2). Let $\xi$ is a positive measurable function on $\mathbb{R}^D$. Since $\tau_n(w; \xi h_n) = \tau_n(w; h_n)$, we see from the assertion (1) that

$$|u_n(x) - v_n(x)| \leq |P_n(u_{n+1} - v_{n+1})(x)| + 1_{D_n}(x)|P_n u_{n+1}(x) - g(T_n, x) + \xi(x) h_n(x)|.$$

Noting that

$$\inf\{a + tb; t > 0\} = 1_{(1)}(\text{sgn}(a) \text{sgn}(b))|a|, \quad a, b \in \mathbb{R},$$

we have the assertion (2). This completes the proof.

Let $\nu_0$ be a probability measure on $\mathbb{R}^D$ and define probability measures $\nu_n$, $n = 1, \ldots, N$, inductively by

$$\nu_{n+1}(dx) = \int_{\mathbb{R}^D} p_n(y; dx) \nu_n(dy), \quad n = 0, 1, \ldots, N - 1.$$

Then we have the following as an easy consequence of Lemma 2.

**Corollary 3** Let $h_n$ and $u_n$ be the same as the previous lemma. Then we have the following.

$$(\int_{\mathbb{R}^D} |u_n(x) - v_n(x)|^2 \nu_n(dx))^{1/2} \leq |(\int_{\mathbb{R}^D} |u_{n+1} - v_{n+1}(x)|^2 \nu_{n+1}(dx))^{1/2} + \int_{D_n} |P_n u_{n+1}(x) - (g(T_n, x) - h_n(x))|^2|^{1/2}$$

for any $n = 0, 1, \ldots, N - 1$. 
3 Main Result

Let \( \nu_0 \) be a probability measure over \( \mathbb{R}^D \). Let \( L_n \geq 1, n = 0, 1, \ldots, N - 1 \), and \( \tilde{X}_{n,\ell} = \{X_{n,\ell}(m)\}_{m=0}^{N}, \ell = 1, \ldots, L_n, n = 0, 1, \ldots, N - 1 \), are identically independent random vectors defined on the probability measure \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \) whose distribution is \( P_{\nu_0}^{(0)} = \int_{\mathbb{R}^D} P_{\tilde{\mathbb{P}}_0}^{(0)} \nu_0(dx) \). Let \( K_n \geq 1, n = 0, 1, \ldots, N - 1 \), and \( \psi_{n,k}, k = 1, \ldots, K_n, n = 0, 1, \ldots, N - 1 \), are functions on \( \mathbb{R}^D \).

Then we define functions \( H_n : \mathbb{R}^D \rightarrow \mathbb{R}, n = N, N - 1, \ldots, 1, 0 \), on \( \mathbb{R}^D \) inductively by the following procedure.

\[
H_N(x) = 1.
\]

When \( \vec{H}_{n+1} = \{H_m\}_{m=n+1}^{N} \), are given we let

\[
\sigma_{n,\ell} = \min\{m \geq n + 1; H_m(X_{n,\ell}(m)) > 0\}, \quad \ell = 1, \ldots, L_n.
\]

Then we let \( \tilde{a}_n = \{\tilde{a}_{n,k}\}_{k=1}^{K_n} \) be the minimizing point of the function

\[
F_n(\{a_k\}_{k=1}^{K_n}) = \frac{1}{L_n} \sum_{k=1}^{K_n} |g(T_{\sigma_{n,\ell}}, X_{n,\ell}(\sigma_{n,\ell})) - \sum_{k=1}^{K_n} a_n \psi_{n,k}(X_{n,\ell}(n))|^2 1_{D_n}(X_{n,\ell}(n)).
\]

Finally we define \( H_n \) by

\[
H_n(x) = \begin{cases}  
g(T_n, x) - \sum_{k=1}^{K_n} \tilde{a}_{n,k} \psi_{n,k}(x), & x \in D_n \\
-1, & x \in \mathbb{R}^D \setminus D_n. \end{cases}
\]

Let \( U_n(x) = u_n(\cdot; \{H_m\}_{m=n}^{N})(x) \). Here \( u_n \) is as in Lemma 2. Let \( \tilde{a}_n = \{\tilde{a}_{n,k}\}_{k=1}^{K_n} \) be the minimizing point of the function

\[
\tilde{F}_n(\{a_k\}_{k=1}^{K_n}) = \int_{D_n} |(P_n U_{n+1})(x) - \sum_{k=1}^{K_n} a_n \psi_{n,k}(x))|^2 \nu_n(dx).
\]

We assume the following.

(A2) \( \psi_{n,k}, k = 1, \ldots, K_n, \) is linearly independent in \( L^2(D_n; d\nu_n), n = 0, 1, \ldots, N - 1 \), where \( \nu_n \) is the probability law of \( w(n) \) under \( P_{\nu_0}^{(0)}(dw) \).

(A3) \( \int_{D_n} \psi_{n,k}(x)^4 \nu_n(dx) < \infty k = 1, \ldots, K_n, n = 0, 1, \ldots, N - 1 \). and

\[
\int_{\mathbb{R}^D} E_{\tilde{\mathbb{P}}_0}^{P_{\nu_0}^{(0)}} \left[ (\sum_{m=1}^{N} g(T_n, w(T_n)))^4 \nu_0(dx) \right] < \infty, \quad n = 0, 1, \ldots, N.
\]

Then we have the following.

Theorem 4 (1) There is a constant \( C > 0 \) such that

\[
E_{\tilde{\mathbb{P}}} \left[ 1 \wedge \left( \sum_{k=1}^{K_n} |\tilde{a}_{n,k} - \tilde{a}_{n,k}|^2 \right) \right] \leq \frac{C}{L_n}.
\]
(2) \[
\left( \int_{\mathbb{R}^D} |U_n(x) - v_n(x)|^2 \nu_n(dx) \right)^{1/2} \leq \left( \int_{\mathbb{R}^D} |U_{n+1}(x) - v_{n+1}(x)|^2 \nu_{n+1}(dx) \right)^{1/2} + \inf \left\{ \left( \int_{\mathbb{R}^D} \left| \left( P_n U_{n+1}(x) - \sum_{k=1}^{K_n} a_k \psi_{n,k}(x) \right) \right|^2 \nu_n(dx) \right)^{1/2} ; \quad a_k \in \mathbb{R}, \quad k = 1, \ldots, K_n \right\}
\]

Proof. Let \( \mathcal{I}_n, n = 0, 1, \ldots, N-1 \), be the \( \sigma \)-algebra generated by \( X_{n,\ell}, \ell = 1, \ldots, L_n \), and let \( \mathcal{B}_n, n = 0, 1, \ldots, N-1 \), be the \( \sigma \)-algebra generated by \( \cup_{m=1}^{N-1} \mathcal{I}_m \). Inductively, we see that \( H_n \) is \( \mathcal{B}_n \)-measurable, \( n = N-1, N-2, \ldots, 0 \). Also, we have

\[
F_n(\{a_k\}_{k=1}^{K_n}) = \sum_{k,k'=1}^{K_n} C^{(2)}_{n,k,k'} a_k a_{k'} - 2 \sum_{k=1}^{K_n} c^{(1)}_{n,k} a_k + C^{(0)}_n,
\]
where

\[
C^{(2)}_{n,k,k'} = \frac{1}{L_n} \sum_{\ell=1}^{L_n} (1_{D_n} \psi_{n,k} \psi_{n,k'})(X_{n,\ell})
\]
\[
c^{(1)}_{n,k} = \frac{1}{L_n} \sum_{\ell=1}^{L_n} (1_{D_n} \psi_{n,k})(X_{n,\ell}) g(T_{\sigma_n,\ell}, X_{n,\ell}(\sigma_n)).
\]

Note that \( \sigma_n = \sigma_{n+1}(\vec{X}_{n,\ell}(\cdot); \{H_m\}_{m=0}^{N-1}) \). Therefore we have

\[
\overline{C}^{(2)}_{n,k,k'} = E^\overline{P}[C^{(2)}_{n,k,k'}|\mathcal{B}_{n+1}] = \int_{D_n} \psi_{n,k}(x) \psi_{n,k'}(x) \nu_n(dx),
\]
and

\[
\overline{c}^{(1)}_{n,k} = E^\overline{P}[c^{(1)}_{n,k}|\mathcal{B}_{n+1}] = \int_{D_n} \psi_{n,k}(x) (P_n U_{n+1})(x) \nu_n(dx).
\]

Let \( R^{(2)}_{n,k,k'} = C^{(2)}_{n,k,k'} - \overline{C}^{(2)}_{n,k,k'} \), and \( r^{(1)}_{n,k} = c^{(1)}_{n,k} - \overline{r}^{(1)}_{n,k} \). Let \( C^{(2)}_n = \{C^{(2)}_{n,k,k'}\}_{k,k'=1}^{D,n} \), \( \overline{C}^{(2)}_n = \{\overline{C}^{(2)}_{n,k,k'}\}_{k,k'=1}^{D,n} \), and \( R^{(2)}_n = \{R^{(2)}_{n,k,k'}\}_{k,k'=1}^{D,n} \) be \( D \times D \) random matrices, and let Let \( c^{(1)}_n = \{c^{(1)}_{n,k}\}_{k=1}^{D,n} \), \( \overline{c}^{(1)}_n = \{\overline{c}^{(1)}_{n,k}\}_{k=1}^{D,n} \), and \( r^{(1)}_n = \{r^{(1)}_{n,k}\}_{k=1}^{D,n} \) be \( D \)-dimensional random vectors. Then we see that

\[
\tilde{a}_n = C^{(2)-1}_n c^{(1)}_n, \quad \overline{a}_n = \overline{C}^{(2)-1}_n \overline{c}^{(1)}_n, \quad n = 0, \ldots, N-1.
\]

Also, we see that

\[
E^\overline{P}[(R^{(2)}_{n,k,k'})^2]
\]
\[
= \frac{1}{L_n} E[\text{Var}[1_{D_n}(X_{n,1}(n)) \psi_{n,k}(X_{n,1}(n)) \psi_{n,k'}(X_{n,1}(n))]|\mathcal{B}_{n+1}]
\]
\[
\leq \frac{1}{L_n} \int_{D_n} \psi_{n,k}(x)^2 \psi_{n,k'}(x)^2 \nu_n(dx)
\]
\[ E^\tilde{P}(\rho_{n, k}^{(1)})^2 = \frac{1}{L_n} E[\text{Var}[\psi_{n, k}(X_{n, 1}(\sigma_{n, 1}))g(\sigma_{n, 1}, X_{n, 1}(\sigma_{n, 1}))|B_{n+1}]] \]

\[ \leq \frac{1}{L_n} \int_{D_n} \psi_{n, k}(x)^2 E^{\tilde{P}}[g(T_{\sigma_{n+1}}(w; H_{m, n+1}^N))w(\sigma_{n+1} + \{H_{m,m+1}^N\})]|\nu_n(dx). \]

If \( \|C_n^{(2)} - R_n^{(2)}\| \leq 1/2 \), we have

\[ \|C_n^{(2)} + R_n^{(2)} - C_n^{(2)}\| = \|(I + C_n^{(2)} - R_n^{(2)})^{-1} - I\) \( C_n^{(2)} - R_n^{(2)}\| \leq 2 \|C_n^{(2)}\| \|R_n^{(2)}\| \leq 1/2 \].

Here \( \| \cdot \| \) is the operator norm of a matrix. So if \( \|C_n^{(2)}\| \|R_n^{(2)}\| \leq 1/2 \) and \( \|c_n^{(1)}\| \leq 1 \), we have

\[ |\tilde{a}_n - \overline{a}_n| = |((C_n^{(2)} + R_n^{(2)})^{-1} - C_n^{(2)} - R_n^{(1)}) + C_n^{(2)} - R_n^{(1)}| \]

\[ \leq 2 \|C_n^{(2)}\|^2 \|R_n^{(2)}\| (\|c_n^{(1)}\| + 1) \|C_n^{(2)}\| \|R_n^{(1)}\| \]

So we have

\[ E^\tilde{P}[|\tilde{a}_n - \overline{a}_n|^2 \wedge 1] \]

\[ \leq E^\tilde{P}[|\tilde{a}_n - \overline{a}_n|^2, \|C_n^{(2)}\| \|R_n^{(2)}\| \leq 1/2, \|c_n^{(1)}\| \leq 1] \]

\[ + \tilde{P}(\|C_n^{(2)}\| \|R_n^{(2)}\| > 1/2) + \tilde{P}(\|c_n^{(1)}\| > 1) \]

\[ \leq 4(\|c_n^{(1)}\| + 1)^2 \|C_n^{(2)}\|^4 + 4 \|C_n^{(2)}\|^{-2} E^\tilde{P}[\|R_n^{(2)}\|^2] + (\|C_n^{(2)}\|^2 + 1)E^\tilde{P}[|r_n^{(1)}|^2]. \]

Also we have

\[ \|c_n^{(1)}\| \leq \left( \int_{D_n} \sum_{k=1}^{K_n} \psi_{n,k}(x)^2 \nu_n(dx) \right)^{1/2} \left( \int_{\mathbb{R}^D} E^{\tilde{P}}[\sum_{m=1}^{N} g(T_m, w(T_m))] \nu_0(dx) \right)^{1/2}, \]

\[ E^\tilde{P}[\|R_n^{(2)}\|^2] \leq \frac{1}{L_n} \int_{D_n} \sum_{k=1}^{K_n} \psi_{n,k}(x)^2 \nu_n(dx) , \]

and

\[ E^\tilde{P}[|r_n^{(1)}|^2] \leq \frac{1}{L_n} \left( \int_{\mathbb{R}^D} \sum_{k=1}^{K_n} \psi_{n,k}(x)^2 \nu_n(dx) \right)^{1/2} \left( \int_{\mathbb{R}^D} E^{\tilde{P}}[\sum_{m=1}^{N} g(T_m, w(T_m))] \nu_0(dx) \right)^{1/2}. \]

This implies the assertion (1).

The assertion (2) is an easy consequence of Lemma 2.

Let \( V_n = \sum_{k=1}^{K_n} R_k \psi_{n,k} \subset L^2(\mathbb{R}^D, \nu_0), \) \( n = 0, 1, \ldots, N - 1 \). Then it is easy to see that \( U_n \)'s are determined by \( \bar{X}_{n, \ell}, \ell = 1, \ldots, L_n, \) \( n = 0, \ldots, N \) and \( V_n \)'s and are independent of a choice of bases \( \{\psi_{n,k}\}_{k=1}^{K_n} \). Let

\[ d_n = \inf \{\int_{\mathbb{R}^D} \sum_{k=1}^{K_n} \psi_{n,k}(x)^2 \nu_n(dx) \}^{1/2}; \{\psi_{k}\}_{k=1}^{K_n} \text{ is a orthogonal basis of } V_n, \]

and

\[ c_0 = \left( \int_{\mathbb{R}^D} E^{\tilde{P}}[\sum_{m=1}^{N} g(T_m, w(T_m))] \nu_0(dx) \right)^{1/4}. \]

Then we have the following from the proof of Theorem 4.
Corollary 5 \[ E[(\int_{\mathbb{R}^D} |U_n(x) - v_n(x)|^2 \nu_n(dx)) \wedge 1]^{1/2} \leq E[(\int_{\mathbb{R}^D} |U_{n+1}(x) - v_{n+1}(x)|^2 \nu_{n+1}(dx)) \wedge 1]^{1/2} + 4(L_n)^{-1/2}d_n(K_n^{1/2}c_0^{1/2} + 1) \]

\[ + E[\inf\{(\int_{D_n} |(P_nU_{n+1})(x) - \psi(x)|^2 \nu_n(dx)) \psi \in V_n\} \wedge 1]^{1/2}. \]

References
