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Monte Carlo Method for pricing of Bermudan type derivatives

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1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, \infty)}, P)$ be a filtered space with the usual condition, and $\{B_t\}_{t \in [0, \infty)}$ be a $d$-dimensional Brownian motion. Let $T > 0$, and let $\sigma : [0, T] \times \mathbb{R}^D \to \mathbb{R}^D \times \mathbb{R}^d$ and $b : [0, T] \times \mathbb{R}^D \to \mathbb{R}^D$ be continuous functions. For each $s \in [0, T]$ and $x \in \mathbb{R}^D$, let $X(t; s, x), t \in [s, T]$ be a solution of the following SDE.

$$X(t; s, x) = x + \int_s^t \sigma(r, X(r; s, x)) dB_r + \int_s^t b(r, X(r; t, x)) dr, \quad t \in [s, T]. \quad (1)$$

We assume that the above SDE 1 has a path-wise unique solution for every $(s, x) \in [0, T] \times \mathbb{R}^D$.

Let $\tilde{S}_{s}^{t}, \ 0 \leq s \leq t \leq T$, be the set of $\mathcal{F}_t$-stopping times $\tau$ with $s \leq \tau \leq t$. Let $g : [0, T] \times \mathbb{R}^D \to \mathbb{R}$ be a continuous function with suitable conditions. Then, concerning the pricing of American derivatives, we are interested in computing the following value function,

$$u(s, x) = \sup \{E[g(\tau, X(\tau; s, x))]; \tau \in \tilde{S}_{s}^{T}, (s, x) \in [0, T] \times \mathbb{R}^D. \$$

There are several attempts to compute the value function $u$ numerically. However, it seems that there are not so good method if $D$ is not small. Let $N \geq 2$ and let $T_n, n = 0, 1, \ldots, N$, be positive numbers such that $0 = T_0 < T_1 < \ldots < T_N = T$. Let $S_n, n = 0, 1, \ldots, N$, be the set of $\mathcal{F}_t$-stopping times taking value in $\{T_n, T_{n+1}, \ldots, T_N\}$. Concerning the pricing of Bermudan type derivatives, we are interested in computing the following value functions.

$$v_n(x) = \sup \{E[g(\tau, X(\tau; s, x))]; \tau \in S_n\}, \quad n = 0, 1, \ldots, N.$$

Let us define a probability measure $p_n(x, \cdot)$ over $\mathbb{R}^D$ for each $n = 0, 1, \ldots, N - 1$, and $x \in \mathbb{R}^D$ by

$$p_n(x, A) = P(X(T_{n+1}; T_n, x) \in A), \quad \text{for a Borel set } A \text{ in } \mathbb{R}^D,$$

and define a operator $P_n, n = 0, 1, \ldots, N - 1, \text{ by}$

$$P_n f(x) = \int_{\mathbb{R}^D} f(y)p_n(x, dy) = E[f(X(T_{n+1}; T_n, x))]$$
for a measurable function $f$ on $\mathbb{R}^D$. Then $v_n$, $n = N, N - 1, \ldots, 0$, are given inductively by the following.

$$v_N(x) = g(T_N, x),$$

$$v_{n-1}(x) = (P_{n-1}v_n)(x) \vee g(T_{n-1}, x).$$

So the value function $v_0(x)$ is easily given mathematically. However, if $D$ is not small, it is not easy to memorize a function on $\mathbb{R}^D$, and so it is not easy to compute $v_0(x)$.

Several people suggest a Monte-Carlo method to compute the value function. In this paper, we discuss the method given by [?]. We assume the following assumption (A).

(A1) $D_n$, $n = 0, 1, \ldots, N - 1$, are measurable sets in $\mathbb{R}^N$ such that $(P_nv_{n+1})(x) \geq g(T_{n}, x)$ for any $x \in \mathbb{R}^D \setminus D_n$.

Remark 1 (1) $D_n = \mathbb{R}^D$ satisfies the assumption (A1).

(2) If $g(t, x) \geq 0$, for any $(t, x) \in [0, T] \times \mathbb{R}$, then $D_n = \{x \in \mathbb{R}^D; g(T_n, x) > 0\}$ satisfies the assumption (A1).

Now let $L_n \geq 1$, $n = 0, 1, \ldots, N - 1$, and $X_{n, \ell} = \{X_{n, \ell}(m)\}_{m=0}^{N}$, $\ell = 1, \ldots, L_n$, $n = 0, 1, \ldots, N - 1$, are identically independent random vectors whose distribution is the same as the distribution of $\{X(T_m; 0, x)\}_{m=0}^{N}$. Let $K_n \geq 1$, $n = 0, 1, \ldots, N - 1$, and $\psi_{n,k}$, $k = 1, \ldots, K_n$, $n = 0, 1, \ldots, N - 1$, are functions on $\mathbb{R}^D$. Then we define functions $H_n$, $n = N, N - 1, \ldots, 1, 0$, on $\mathbb{R}^D$ inductively by the following.

$$H_N(x) = 1.$$  

When $H_{n+1} = \{H_m\}_{m=n+1}^{N}$, are given we let

$$\sigma_{n, \ell} = \min\{m \geq n + 1; H_m(X_{n, \ell}(m)) > 0\}, \quad \ell = 1, \ldots, L_n.$$  

Then we let $\tilde{v}_{0, n, k}$ be the minimizing point of the function

$$F_n(\{a_k\}_{k=1}^{K_n}) = \frac{1}{L_n} \sum_{\ell=1}^{L_n} |g(T_{\sigma_{\ell}, \ell}, X_{n, \ell}(\sigma_{n, \ell})) - \sum_{k=1}^{K_n} a_{\ell} \psi_{n,k}(X_{n, \ell}(\sigma_{n, \ell}))|^2 1_{D_n}(X_{n, \ell}(\sigma_{n, \ell})).$$  

Finally we define $H_n$ by

$$H_n(x) = \begin{cases}  g(T_n, x) - \sum_{k=1}^{K_n} \tilde{a}_{n,k} \psi_{n,k}(x), & x \in D_n \\ -1, & x \in \mathbb{R}^D \setminus D_n. \end{cases}$$  

Then we let

$$\tilde{v}_0 = \frac{1}{L_0} \sum_{\ell=1}^{L_0} g(\sigma_{0, \ell}, X_{0, \ell}(\sigma_{0, \ell}))$$  

and

$$\tilde{\sigma} = \min\{T_n; H_n(X(T_n; 0, x)) > 0\}.$$  

We think that $\tilde{v}_0$ is an approximation of the value function $v_0(x)$ and the stopping time $\tilde{\sigma}$ as a candidate of the optimal stopping time.
2 Preliminary Results

Let \( W_n = \mathbb{R}^{(N+1-n)D} \), \( n = 0, 1, \ldots, N \), and let \( P^{(n)}_n \), \( x \in \mathbb{R}^D \) be the distribution of \( \{X(T_m; T_n, x)\}_{m=n}^N \) on \( W_n \). Then \( P^{(n)}_n \), \( n = 0, 1, \ldots, N \), \( x \in \mathbb{R}^D \), is a Markov chain on \( \mathbb{R}^D \).

For any measurable function \( h \) on \( \mathbb{R}^D \) and \( n, m = 0, 1, \ldots, N \) with \( n \leq m \), let \( \tau_{m}(\cdot; h) : W_n \to \{m, N\} \) be
\[
\tau_{m}(w; h) = \begin{cases} 
  m, & h(w(m)) > 0, \\
  N, & h(w(m)) \leq 0.
\end{cases}
\]

Lemma 2 Let \( h_n : \mathbb{R}^N \to \mathbb{R} \), \( n = 0, 1, \ldots, N \), be given, and assume that \( h_n(x) \leq 0 \), \( x \in \mathbb{R}^N \setminus D_n \), and that \( h_N(x) = 1 \). Let \( \sigma_n : W_n \to \{n, n+1, \ldots, N\} \) be given by
\[
\sigma_n(w) = \sigma_n(w; \{h_m\}_{m=n}^{N-1}) = \bigwedge_{m=n}^{N-1} \tau_m(w; h_m), \quad w \in W_n.
\]

Moreover, let \( u_n : \mathbb{R}^D \to \mathbb{R} \) be given by
\[
u_n(x) = u_n(x; \{h_m\}_{m=n}^{N}) = E^{P^{(n)}_n}[g(T_{\sigma_n}, w(\sigma_n))], \quad x \in \mathbb{R}^D,
\]

Then we have the following.
(1) \(|u_n(x) - v_n(x)| \leq |P_n(u_{n+1} - v_{n+1})(x)| + 1_{D_n}(x)|P_nu_{n+1}(x) - (g(T_n, x) - h_n(x))|
for any \( n = 0, 1, \ldots, N - 1 \), and \( x \in \mathbb{R}^D \).
(2) \(|u_n(x) - v_n(x)| \leq |P_n(u_{n+1} - v_{n+1})(x)| + 1_{D_n}(x)|\text{sgn}(P_nu_{n+1}(x) - g(T_n, x))\text{sgn}(h_n(x))|P_nu_{n+1}(x) - g(n, x)|.

Here
\[
\text{sgn}(a) = \begin{cases} 
  1, & a > 0, \\
  0, & a = 0, \\
  -1, & a < 0.
\end{cases}
\]

Proof. Note that \( u_n(x) \leq v_n(x) \), for all \( n = 0, 1, \ldots, N - 1 \), and \( x \in \mathbb{R}^D \). Let \( \tilde{u}_n(x) = g(T_n, x) - h_n(x) \), \( x \in \mathbb{R}^D \).

Let \( n = 0, 1, \ldots, N - 1 \), and \( x \in \mathbb{R}^D \), and fix them for a while.

Case 1. Suppose that \( h_n(x) > 0 \). Then we see that \( x \in D_n \) and \( g(T_n, x) > \tilde{u}_n(x) \). So we have
\[
u_n(x) = g(T_n, x) + (P_nv_{n+1}(x) - g(T_n, x)) \vee 0 \leq g(T_n, x) + |P_nv_{n+1}(x) - \tilde{u}_n(x)|.
\]
This implies
\[
g(T_n, x) \geq v_n(x) - |P_n(v_{n+1} - u_{n+1})(x)| - |P_nu_{n+1}(x) - \tilde{u}_n(x)|.
\]

Case 2. Suppose that \( h_n(x) \leq 0 \), and \( x \in D_n \). Then we see that \( g(T_n, x) \leq \tilde{u}_n(x) \). So we see that
\[
u_n(x) \leq P_nv_{n+1}(x) \vee \tilde{u}_n(x) \leq P_nu_{n+1}(x) + |P_n(v_{n+1} - u_{n+1})(x)| + |P_nu_{n+1}(x) - \tilde{u}_n(x)|.
\]
Case 3. Suppose that $h_n(x) \leq 0$, and $x \in \mathbb{R}^D \setminus D_n$. Then we see that $g(T_n, x) \leq (P_n v_{n+1})(x)$. So we have

$$v_n(x) = P_n v_{n+1}(x) \leq P_n u_{n+1}(x) + |P_n(u_{n+1} - v_{n+1})(x)|.$$

So we see that for any $n = 0, 1, \ldots, N - 1$, we have

$$u_n = I_{\{h_\in 0\}} g(T_n, \cdot) + I_{\{h_\leq 0\}} (P_n u_{n+1})$$

$$\geq I_{\{h_\in 0\}} (v_n - |P_n(u_{n+1} - u_{n+1})| - |P_n u_{n+1} - \tilde{u}_n|)$$

$$+ I_{\{h_\leq 0\}} I_{D_n} (v_n - |P_n(u_{n+1} - u_{n+1})| - |P_n u_{n+1} - \tilde{u}_n|)$$

Thus we see that

$$0 \leq v_n - u_n \leq |P_n(u_{n+1} - u_{n+1})| + |1_D P_n u_{n+1} - \tilde{u}_n|.$$

This implies the assertion (1).

Now let us prove the assertion (2). Let $\xi$ be a positive measurable function on $\mathbb{R}^D$. Since $\tau_n(w; \xi h_n) = \tau_n(w; h_n)$, we see from the assertion (1) that

$$|u_n(x) - v_n(x)| \leq |P_n(u_{n+1} - u_{n+1})(x)| + 1_D(x) |P_n u_{n+1}(x) - g(T_n, x) + \xi(x) h_n(x)|.$$

Noting that

$$\inf\{a + tb; t > 0\} = I_{\{1\}} (sgn(a) sgn(b)) |a|, \quad a, b \in \mathbb{R},$$

we have the assertion (2).

This completes the proof.

Let $\nu_0$ be a probability measure on $\mathbb{R}^D$ and define probability measures $\nu_n$, $n = 1, \ldots, N$, inductively by

$$\nu_{n+1}(dx) = \int_{\mathbb{R}^D} p_n(y; dx) \nu_n(dy), \quad n = 0, 1, \ldots, N - 1.$$

Then we have the following as an easy consequence of Lemma 2.

**Corollary 3** Let $h_n$ and $u_n$ be the same as the previous lemma. Then we have the following.

$$\left( \int_{\mathbb{R}^D} |u_n(x) - v_n(x)|^2 \nu_n(dx) \right)^{1/2}$$

$$\leq |(\int_{\mathbb{R}^D} |u_{n+1} - v_{n+1}(x)|^2 \nu_{n+1}(dx))^{1/2} + \int_{D_n} |P_n u_{n+1}(x) - g(T_n, x) - h_n(x)|^2 \right)^{1/2}$$

for any $n = 0, 1, \ldots, N - 1$. 

3 Main Result

Let $\nu_0$ be a probability measure over $\mathbb{R}^D$. Let $L_n \geq 1$, $n = 0, 1, \ldots, N - 1$, and $X_{n,\ell}(m) = \{X_{n,\ell}(m)\}_{m=0}^{N}$, $\ell = 1, \ldots, L_n$, $n = 0, 1, \ldots, N - 1$, are identically independent random vectors defined on the probability measure $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ whose distribution is $P^{(0)}_{\eta} = \int_{\mathbb{R}^D} P^{(0)}_{\eta}(dx)$. Let $K_n \geq 1$, $n = 0, 1, \ldots, N - 1$, and $\psi_{n,k}$, $k = 1, \ldots, K_n$, $n = 0, 1, \ldots, N - 1$, are functions on $\mathbb{R}^D$.

Then we define functions $H_n : \mathbb{R}^D \rightarrow \mathbb{R}$, $n = N, N - 1, \ldots, 0$, on $\mathbb{R}^D$ inductively by the following procedure.

$H_N(x) = 1$.

When $\tilde{H}_{n+1} = \{H_m\}_{m=n+1}^{N}$, are given we let

$\sigma_{n,\ell} = \min\{m \geq n+1; H_m(X_{n,\ell}(m)) > 0\}$, $\ell = 1, \ldots, L_n$.

Then we let $\tilde{a}_n = \{\tilde{a}_{n,k}\}_{k=1}^{K_n}$ be the minimizing point of the function

$$F_n(\{a_k\}_{k=1}^{K_n}) = \frac{1}{L_n} \sum_{\ell=1}^{L_n} \left| g(T_{\sigma_{n,\ell}}, X_{n,\ell}(\sigma_{n,\ell})) - \sum_{k=1}^{K_n} a_n \psi_{n,k}(X_{n,\ell}(\sigma_{n,\ell})) \right|^2 1_{D_n}(X_{n,\ell}(\sigma_{n,\ell})).$$

Finally we define $H_n$ by

$$H_n(x) = \begin{cases} g(T_n, x) - \sum_{k=1}^{K_n} \tilde{a}_{n,k} \psi_{n,k}(x), & x \in D_n \\ -1, & x \in \mathbb{R}^D \setminus D_n. \end{cases}$$

Let $U_n(x) = u_n(\cdot; \{H_m\}_{m=n+1}^{N})(x)$. Here $u_n$ is as in Lemma 2. Let $\bar{a}_n = \{\bar{a}_{n,k}\}_{k=1}^{K_n}$ be the minimizing point of the function

$$\overline{F}_n(\{a_k\}_{k=1}^{K_n}) = \int_{D_n} \left| (P_n U_{n+1})(x) - \sum_{k=1}^{K_n} a_n \psi_{n,k}(x) \right|^2 \nu_n(dx).$$

We assume the following.

(A2) $\psi_{n,k}$, $k = 1, \ldots, K_n$, is linear ly independent in $L^2(D_n; d\nu_n)$, $n = 0, 1, \ldots, N - 1$, where $\nu_n$ is the probability law of $w(n)$ under $P^{(0)}_{\eta}(dw)$.

(A3) $\int_{D_n} \psi_{n,k}(x)^4 \nu_n(dx) < \infty$ $k = 1, \ldots, K_n$, $n = 0, 1, \ldots, N - 1$. and

$$\int_{\mathbb{R}^D} E^{P^{(0)}} \left[ \left( \sum_{m=1}^{N} g(T_m, w(T_m))^4 \right)^4 \nu_0(dx) \right] < \infty, \quad n = 0, 1, \ldots, N.$$

Then we have the following.

Theorem 4 (1) There is a constant $C > 0$ such that

$$E^{\tilde{P}}[1 \wedge \left( \sum_{k=1}^{K_n} |\tilde{a}_{n,k} - \bar{a}_{n,k}|^2 \right)] \leq \frac{C}{L_n}.$$
(2) \( (\int_{\mathbb{R}^D} |U_n(x) - v_n(x)|^2 \nu_n(dx))^{1/2} \)

\[
\leq (\int_{\mathbb{R}^D} |U_{n+1}(x) - v_{n+1}(x)|^2 \nu_{n+1}(dx))^{1/2} + \left( \int_{\mathbb{R}^D} \left( \sum_{k=1}^{K_n} (\tilde{a}_{n,k} - \overline{a}_{n,k}) \psi_{n,k}(x)^2 \nu_n(dx) \right)^2 \right)^{1/2}
\]

\[+ \inf \{ (\int_{D_n} |(P_n U_{n+1})(x) - \sum_{k=1}^{K_n} a_k \psi_{n,k}(x)|^2 \nu_n(dx))^{1/2}; a_k \in \mathbb{R}, k = 1, \ldots, K_n \} \]

Proof. Let \( \mathcal{I}_n, n = 0, 1, \ldots, N - 1 \), be the \( \sigma \)-algebra generated by \( \vec{X}_{n, \ell}, \ell = 1, \ldots, L_n \), and let \( \mathcal{B}_n, n = 0, 1, \ldots, N - 1 \), be the \( \sigma \)-algebra generated by \( \cup_{m=m+1}^{N} \mathcal{I}_m \). Inductively, we see that \( H_n \) is \( \mathcal{B}_n \)-measurable, \( n = N - 1, N - 2, \ldots, 0 \). Also, we have

\[
F_n(\{a_k\}_{k=1}^{K_n}) = \sum_{k,k'=1}^{K_n} C_{n,k,k'}^{(2)} a_k a_{k'} + 2 \sum_{k=1}^{K_n} c_{n,k}^{(1)} a_k + C_n^{(0)},
\]

where

\[
C_{n,k,k'}^{(2)} = \frac{1}{L_n} \sum_{\ell=1}^{L_n} \left( 1_{D_n} \psi_{n,k} \psi_{n,k'} \right)(X_{n, \ell})
\]

\[
c_{n,k}^{(1)} = \frac{1}{L_n} \sum_{\ell=1}^{L_n} \left( 1_{D_n} \psi_{n,k} \right)(T_{\sigma_{n, \ell}} X_{n, \ell}(\sigma_{n, \ell})).
\]

Note that \( \sigma_{n, \ell} = \sigma_{n+1}(\vec{X}_{n, \ell}(\cdot); \{H_m\}_{m=m+1}^{N}) \). Therefore we have

\[
\tilde{C}_{n,k,k'}^{(2)} = E^\overline{P}[C_{n,k,k'}^{(2)}|\mathcal{B}_{n+1}] = \int_{D_n} \psi_{n,k}(x) \psi_{n,k'}(x) \nu_n(dx),
\]

and

\[
\tilde{c}_{n,k}^{(1)} = E^\overline{P}[c_{n,k}^{(1)}|\mathcal{B}_{n+1}] = \int_{D_n} \psi_{n,k}(x) (P_n U_{n+1})(x) \nu_n(dx).
\]

Let \( R_{n,k,k'}^{(2)} = C_{n,k,k'}^{(2)} - \tilde{C}_{n,k,k'}^{(2)} \), and \( c_{n,k}^{(1)} = c_{n,k}^{(1)} - \tilde{c}_{n,k}^{(1)} \). Let \( C_n^{(2)} = \{C_{n,k,k'}^{(2)}\}_{k,k'=1}^{D}, \tilde{C}_n^{(2)} = \{\tilde{C}_{n,k,k'}^{(2)}\}_{k,k'=1}^{D}, \) and \( R_n^{(2)} = \{R_{n,k,k'}^{(2)}\}_{k,k'=1}^{D} \) be \( D \times D \) random matrices, and let Let \( C_n^{(1)} = \{c_{n,k}^{(1)}\}_{k=1}^{D}, \tilde{C}_n^{(1)} = \{\tilde{c}_{n,k}^{(1)}\}_{k=1}^{D}, \) and \( R_n^{(1)} = \{r_{n,k}^{(1)}\}_{k=1}^{D} \) be \( D \)-dimensional random vectors. Then we see that

\[
\tilde{a}_n = C_n^{(2)-1} c_n^{(1)}, \quad \overline{a}_n = \tilde{C}_n^{(2)-1} \tilde{c}_n^{(1)}, \quad n = 0, \ldots, N - 1.
\]

Also, we see that

\[
E^\overline{P}[(R_{n,k,k'}^{(2)})^2]
\]

\[
= \frac{1}{L_n} E[Var[1_{D_n}(X_{n,1}(n)) \psi_{n,k}(X_{n,1}(n)) \psi_{n,k'}(X_{n,1}(n))|\mathcal{B}_{n+1}]]
\]

\[
\leq \frac{1}{L_n} \int_{D_n} \psi_{n,k}(x)^2 \psi_{n,k'}(x)^2 \nu_n(dx)
\]
\[ E^\tilde{P}[\{r_{n,k}^{(1)}\}^2] = \frac{1}{L_n} E[Var[1_{D_n}(X_{n,1}(n))\psi_{n,k}(X_{n,1}(n))g(\sigma_{n,1}, X_{n,1}(\sigma_{n,1}))|B_{n+1}]] \]
\[ \leq \frac{1}{L_n} \int_{D_n} \psi_{n,k}(x)^2 E^P\left[ g(T_{\sigma_{n+1}(w;\{H_m\}_{m=n+1}^{N})}w(\sigma_{n+1}(w;\{H_m\}_{m=n+1}^{N}))\right] \nu_n(dx). \]

If \( ||\bar{C}_n^{(2)-1}R_n^{(2)}|| \leq 1/2 \), we have
\[ ||(\bar{C}_n^{(2)} + R_n^{(2)})^{-1}\bar{C}_n^{(2)-1}|| = ||(I + \bar{C}_n^{(2)-1}R_n^{(2)})^{-1} - I)\bar{C}_n^{(2)-1}|| \leq 2 ||\bar{C}_n^{(2)-1}|| ||R_n^{(2)}||. \]

Here \( ||\cdot|| \) is the operator norm of a matrix. So if \( ||\bar{C}_n^{(2)-1}|| ||R_n^{(2)}|| \leq 1/2 \) and \(|\bar{c}_n^{(1)}| \leq 1\), we have
\[ |	ilde{a}_n - \bar{a}_n| = |((\bar{C}_n^{(2)} + R_n^{(2)})^{-1} - \bar{C}_n^{(2)-1})(\bar{c}_n^{(1)} + r_n^{(1)}) + \bar{C}_n^{(2)-1}r_n^{(1)}| \]
\[ \leq 2 ||\bar{C}_n^{(2)-1}|| ||R_n^{(2)}|| (|\bar{c}_n^{(1)}| + 1) ||\bar{C}_n^{(2)-1}|| ||r_n^{(1)}|| \]

So we have
\[ E^\tilde{P}[||\tilde{a}_n - \bar{a}_n||^2 \wedge 1] \]
\[ \leq E^\tilde{P}[||\tilde{a}_n - \bar{a}_n||^2, ||\bar{C}_n^{(2)-1}|| ||R_n^{(2)}|| \leq 1/2, |\bar{c}_n^{(1)}| \leq 1] \]
\[ + \tilde{P}(||\bar{C}_n^{(2)-1}|| ||R_n^{(2)}|| > 1/2) + \tilde{P}(|\bar{c}_n^{(1)}| > 1) \]
\[ \leq (4(|\bar{c}_n^{(1)}| + 1)^2 ||\bar{C}_n^{(2)-1}||^4 + 4 ||\bar{C}_n^{(2)-1}||^{-2}) E^\tilde{P}[||R_n^{(2)}||^2] + (||\bar{C}_n^{(2)-1}||^2 + 1) E^\tilde{P}[|r_n^{(1)}|^2]. \]

Also we have
\[ |\bar{c}_n^{(1)}| \leq (\int_{D_n}(\sum_{k=1}^{K_n}\psi_{n,k}(x)^2)\nu_n(dx))^{1/2}(\int_{\mathbb{R}^D} E^{P^{(0)}}\left[ (\sum_{m=1}^{N}g(T_m, w(T_m)))^2\right] \nu_0(dx))^{1/2}, \]
\[ E^\tilde{P}[||R_n^{(2)}||^2] \leq \frac{1}{L_n} \int_{D_n} (\sum_{k=1}^{K_n}\psi_{n,k}(x)^2)^2 \nu_n(dx), \]

and
\[ E^\tilde{P}[|r_n^{(1)}|^2] \leq \frac{1}{L_n}(\int_{D_n}(\sum_{k=1}^{K_n}\psi_{n,k}(x)^2)^2 \nu_n(dx))^{1/2}(\int_{\mathbb{R}^D} E^{P^{(0)}}\left[ (\sum_{m=1}^{N}g(T_m, w(T_m)))^4\right] \nu_0(dx))^{1/2}. \]

This implies the assertion (1).

The assertion (2) is an easy consequence of Lemma 2.

Let \( V_n = \sum_{k=1}^{K_n} R_k \psi_{n,k} \subset L^2(\mathbb{R}^D; d\nu_n), n = 0, 1, \ldots, N - 1 \). Then it is easy to see that \( U_n \)'s are determined by \( \bar{X}_{n,\ell}, \ell = 1, \ldots, L_n, n = 0, \ldots, N \) and \( V_n \)'s and are independent of a choice of bases \( \{\psi_{n,k}\}_{k=1}^{K_n} \). Let
\[ d_n = \inf\{(\int_{\mathbb{R}^D}(\sum_{k=1}^{K_n}\psi_{n,k}(x)^2)\nu_n(dx))^{1/2}; \{\psi_{n,k}\}_{k=1}^{K_n} \text{ is an orhogonal basis of } V_n\}, \]
and
\[ c_0 = (\sum_{k=1}^{K_n} \int_{\mathbb{R}^D} E^{P^{(0)}}\left[ (\sum_{m=1}^{N}g(T_m, w(T_m)))^4\right] \nu_0(dx))^{1/4}. \]

Then we have the following from the proof of Theorem 4.
Corollary 5 \[ E[(\int_{\mathbb{R}^D}|U_n(x) - v_n(x)|^2 \nu_n(dx)) \wedge 1]^{1/2} \]

\[ \leq E[(\int_{\mathbb{R}^D}|U_{n+1}(x) - v_{n+1}(x)|^2 \nu_{n+1}(dx)) \wedge 1]^{1/2} + 4(L_n)^{-1/2} d_n (K_n c_0^{1/2} + 1) \]

\[ + E[\inf\{(\int_{D_n}|(P_n U_{n+1})(x) - \psi(x)|^2 \nu_n(dx)) \psi \in V_n\} \wedge 1]^{1/2}. \]

References
