A Market Game with Infinitely Many Players (Mathematical Economics: Game Theory)

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A Market Game with Infinitely Many Players

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1 Introduction

A market game that derive from an exchange economy in which the finite number of traders have continuous concave monetary utility functions was studied fully in [4] and a market game with infinitely many traders described with a non-atomic measure space was extensively investigated in [1]. The non-atomic measure space played a crucial role to remove the concavity of utility functions from the assumption in [4]. In this paper, we shall study a market game with infinite traders described with a general measure space preserving the concavity assumption for utilities. It will be shown that such a market game has properties parallel to those of an exact game studied in [3] and each member of the core of a market game has an outcome density with respect to the measure.

Let $(\Omega, \mathcal{F})$ be a measurable space. A game $v$ is a nonnegative real valued function, defined on the $\sigma$-field $\mathcal{F}$, which maps the empty set to zero. An outcome of a game $v$ is a finitely additive real valued function $\alpha$ on $\mathcal{F}$ such that $\alpha(\Omega) = v(\Omega)$. For an outcome $\alpha$ of $v$, an integrable function $f$ satisfying $\int_S f \, d\mu = \alpha(S)$ for all $S \in \mathcal{F}$ is said to be an outcome density of $\alpha$ with respect to $\mu$. An outcome indicates outcomes to each coalition while an outcome density designates outcomes to every players. The core of $v$ is the set of outcomes $\alpha$ satisfying $\alpha(S) \geq v(S)$ for all $S \in \mathcal{F}$.

To every game $v$ we associate an extended real number $|v|$ defined by

$$|v| = \sup \left\{ \sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_\Omega \right\} ,$$

(1)

where $n = 1, 2, \ldots, S_i \in \mathcal{F}$, $\lambda_i$ is a real number. The notation $\chi_A$ denotes the characteristic function of a subset $A$ of $\Omega$. For a game $v$ with $|v| < \infty$, \ldots
we define two games $\bar{v}$ and $\hat{v}$ by

$$\bar{v}(S) = \sup \left\{ \sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_S \right\}, \quad S \in \mathcal{F},$$  

(2)

$$\hat{v}(S) = \min \{ \alpha(S) : \alpha \text{ is additive, } \alpha \geq v, \alpha(\Omega) = |v| \}, \quad S \in \mathcal{F},$$  

(3)

following [3]. A game $v$ is said to be balanced if $v(\Omega) = |v|$, totally balanced if $v = \bar{v}$ and exact if $v = \hat{v}$, respectively. It is proved in [3] that the core of a game is nonempty if and only if it is balanced, every exact game is totally balanced, and every totally balanced game is balanced.

A game $v$ is said to be monotone if $S \subseteq T$ implies $v(S) \leq v(T)$. A game $v$ is said to be inner continuous at $S \in \mathcal{F}$ if it follows that $\lim_{n \to \infty} v(S_n) = v(S)$ for any nondecreasing sequence $\{S_n\}$ of measurable sets such that $\bigcup_{n=1}^{\infty} S_n = S$. Similarly, a game $v$ is said to be outer continuous at $S \in \mathcal{F}$ if it follows that $\lim_{n \to \infty} v(S_n) = v(S)$ for any nonincreasing sequence $\{S_n\}$ of measurable sets such that $\bigcap_{n=1}^{\infty} S_n = S$. A game $v$ is continuous at $S \in \mathcal{F}$ if it is both inner and outer continuous at $S$.

2 Market Games

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space throughout this paper. We denote utilities of players by a Carathéodory type function $u$ defined on $\Omega \times R_{+}^{l}$ to $R_{+}$, where $R_{+}^{l}$ denotes the nonnegative orthant of the $l$-dimensional Euclidean space $R_{+}$, and $R_{+}$ is the set of nonnegative real numbers. The nonnegative number $u(\omega, x)$ designates the density of the utility of a player $\omega$ getting goods $x$. We always use the ordinary coordinatewise order when having concern with an order in $R_{+}^{l}$. We suppose that the function $u : \Omega \times R_{+}^{l} \to R_{+}$ satisfies the conditions:

1. The function $\omega \mapsto u(\omega, x)$ is measurable for all $x \in R_{+}^{l}$;

2. The function $x \mapsto u(\omega, x)$ is continuous, concave, nondecreasing, and $u(\omega, 0) = 0$, for almost all $\omega$ in $\Omega$;

3. $\sigma \equiv \sup\{u(\omega, x) : (\omega, x) \in \Omega \times B_{+} \} < \infty$, where $B_{+} = \{x \in R_{+}^{l} : ||x|| \leq 1\}$, and $||x||$ denotes the Euclidean norm of $x \in R_{+}^{l}$.

For any measurable set $S \in \mathcal{F}$, the set of integrable functions on $S$ to $R_{+}^{l}$ is denoted by $L_{1}(S, R_{+}^{l})$. We take an element $e$ of $L_{1}(S, R_{+}^{l})$ as the
density of initial endowments for the players. For any $S \in \mathcal{F}$, define

$$v(S) \equiv \sup \left\{ \int_S u(\omega, x(\omega)) d\mu(w) : x \in L_1(S, R_+^l), \int_S x d\mu = \int_S e d\mu \right\}.$$  \hspace{1cm} (4)

The set function $v$ defined above is called a *market game* derived from the market $(\Omega, \mathcal{F}, \mu, u, e)$.

We shall confirm that the market game $v$ is actually a game in the rest of this section. It is well known that the function $\omega \mapsto u(\omega, x(\omega))$ is measurable for any $x \in L_1(S, R_+^l)$. Moreover we need to show that the mapping $\omega \mapsto u(\omega, x(\omega))$ is integrable in order to define $v(S)$ as a real number.

**Lemma 1** If $x \in L_1(S, R_+^l)$, then $u(\cdot, x(\cdot)) \in L_1(S, R_+)$ for any $S \in \mathcal{F}$ and the map $x \mapsto u(\cdot, x(\cdot))$ is continuous with respect to the norm topologies of $L_1(S, R_+^l)$ and $L_1(S, R_+)$.  

**Proof** Let $x \in L_1(S, R_+^l)$. Since $u(\omega, \cdot)$ is concave, for any $x \in R_+^l$ with $\|x\| > 1$, we have the inequality

$$\frac{u(\omega, x) - u(\omega, x/\|x\|)}{\|x - x/\|x\||} \leq \frac{u(\omega, x/\|x\|) - u(\omega, 0)}{\|x/\|x\||},$$  \hspace{1cm} (5)

and hence we have $u(\omega, x) \leq \|x\| \sigma$. It is obvious from the definition of $\sigma$ that $u(\omega, x) \leq \sigma$ for all $x$ with $\|x\| \leq 1$. Thus we have $u(\omega, x) \leq \sigma(1 + \|x\|)$ for any $x \in R_+^l$ and this leads to the inequalities

$$\int_S u(\omega, x(\omega)) d\mu \leq \int_S \sigma(1 + \|x(\omega)\|) d\mu = \sigma \left( \mu(S) + \int_S \|x(\omega)\| d\mu \right) < \infty.$$  \hspace{1cm} (6)

Thus it follows that $u(\cdot, x(\cdot)) \in L_1(S, R_+)$. The second part of the assertion is verified in Theorem 2.1 of [2]. Although Theorem 2.1 of [2] is proved under the hypotheses that $\Omega$ is a measurable set in $R^l$ and the second argument $x$ of the function $u$ runs over $R$, the proof of Theorem 2.1 of [2] is valid even in our setting. Thus the map $x \mapsto u(\cdot, x(\cdot))$ is norm continuous. Q.E.D.

**Remark 1** The assumption of the finiteness of $\sigma$ is necessary to prove Lemma 1. The following example violates the assumption and shows that $u$ does not necessarily convey an integrable function to an integrable function.
Example 1 Let \( l = 1 \) and \( \Omega = (0, 1) \). Define \( u : (0, 1) \times R_+ \to R_+ \) by \( u(\omega, x) = \sqrt{x}/\omega \). Then, for the function \( x(\omega) = 1 \) for all \( \omega \in (0, 1) \), it follows \( u(\omega, x(\omega)) = 1/\omega \), and obviously it is not integrable.

Lemma 2 A market game \( v \) is actually a game and is monotone.

**Proof** It is obvious \( v(\emptyset) = 0 \). The finiteness of \( v(S) \) follows since the inequalities

\[
\int_S u(\omega, x(\omega)) \, d\mu(\omega) \leq \sigma \int_S (1 + ||x||) \, d\mu \\
\leq \sigma \left( \mu(S) + \sum_{i=1}^l \int_S x^i \, d\mu \right) = \sigma \left( \mu(S) + \sum_{i=1}^l \int_S e^i \, d\mu \right)
\]

hold if

\[
\int_S x \, d\mu = \int_S e \, d\mu,
\]

where \( x^i \) and \( e^i \) are the \( i \)-th coordinate functions of \( x \) and \( e \), respectively. Moreover \( v \) is monotone because the function \( x \mapsto u(\omega, x) \) is nondecreasing for almost all \( \omega \in \Omega \). Q.E.D.

**Remark 2** The supremum in the definition of a market game cannot be replaced by maximum in general as the following example shows.

Example 2 [[1], pp. 204] Let \( l = 1 \), \( \Omega = [0, 1] \) and \( \mu \) be the Lebesgue measure. Define \( u : [0, 1] \times R_+ \to R_+ \) by \( u(\omega, x) = \omega x \) and let \( e(\omega) = 1 \) for all \( \omega \in \Omega \). Then \( v([0, 1]) = 1 \) but, for any \( x \in L_1([0, 1], R_+) \) with \( \int_0^1 x \, d\mu = 1, \int_0^1 \omega x(\omega) \, d\mu(\omega) \) never reaches 1.

### 3 Cores of Market Games

We shall investigate properties of the cores of the market games in this section. We start with a lemma on concave functions.

**Lemma 3** If \( f : R_+^l \to R \) is concave and \( f(0) = 0 \), then for any \( x_1, \ldots, x_n \in R_+^l \) and \( \lambda_1, \ldots, \lambda_n \geq 0 \) with \( \sum_{i=1}^n \lambda_i \leq 1 \), it follows that

\[
\sum_{i=1}^n \lambda_i f(x_i) \leq f(\sum_{i=1}^n \lambda_i x_i).
\]
Proof We can assume that $\lambda = \sum_{i=1}^{n} \lambda_i > 0$ without loss of generality. It follows that

\[
\sum_{i=1}^{n} \lambda_i f(x_i) = \lambda \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} f(x_i)
\]

(10)

\[
\leq \lambda f\left(\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} x_i\right)
\]

(11)

\[
= (1 - \lambda)f(0) + \lambda f\left(\frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i x_i\right)
\]

(12)

\[
\leq f\left(\sum_{i=1}^{n} \lambda_i x_i\right).
\]

(13)

Q.E.D.

Let $S'$ and $S$ be measurable sets with $S' \subset S$. For any $x \in L_1(S', R_+^l)$, define an extension $\overline{x} \in L_1(S, R_+^1)$ of $x$ to $S$ by

\[
\overline{x}(\omega) = \begin{cases} 
 x(\omega), & \text{if } \omega \in S'; \\
 0, & \text{if } \omega \in S \setminus S'.
\end{cases}
\]

(14)

Proposition 1 A market game $v$ is totally balanced.

Proof Take any $S \in \mathcal{F}$ and $S_i \in \mathcal{F}$ and $\lambda_i > 0$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} \lambda_i x_{S_i} \leq x_S$. We can assume that $\mu(S) > 0$ without loss of generality.

Let $\epsilon$ be an arbitrary positive number. Take $x_i \in L_1(S_i, R_+^l)$ such that

\[
\int_{S_i} x_i \, d\mu = \int_{S_i} e \, d\mu \quad \text{and} \quad v(S_i) - \frac{\epsilon}{n} < \int_{S_i} u(\omega, x_i(\omega)) \, d\mu(\omega),
\]

(15)

and define $y \in L_1(S, R_+^l)$ by

\[
y = \sum_{i=1}^{n} \lambda_i \overline{x}_i.
\]

(16)
Then we have the following:

\[ \int_S y \, d\mu = \sum_{i=1}^{n} \lambda_i \int_S \overline{x}_i \, d\mu \tag{17} \]

\[ = \sum_{i=1}^{n} \lambda_i \int_{S_i} e \, d\mu \tag{18} \]

\[ = \int_S e \sum_{i=1}^{n} \lambda_i \chi_{S_i} \, d\mu \tag{19} \]

\[ \leq \int_S e \, d\mu. \tag{20} \]

Define \( y' \in L_1(S, R^+_{\mu}) \) by

\[ y' = y + \frac{1}{\mu(S)} \left( \int_S e \, d\mu - \int_S y \, d\mu \right). \tag{21} \]

Then it is easily seen that \( \int_S y' \, d\mu = \int_S e \, d\mu. \)

On the other hand, let \( A \) be the family of all nonempty subsets \( A \) of \( \{1, \ldots, n\} \) such that \( T_A \equiv \bigcap_{i \in A} S_i \cap \bigcap_{j \in A^c} (S \setminus S_j) \neq \emptyset \). Then it is easily seen that \( S_i = \bigcup_{A \ni i} T_A \) for \( i = 1, \ldots, n \) and \( \{T_A : A \in A\} \) is a partition of \( \bigcup_{i=1}^{n} S_i \), and \( \sum_{i \in A} \lambda_i \leq 1 \) for all \( A \in A \). For any \( i \) and \( A \) with \( i \in A \in A \), define \( x_i^A = x_i|_{T_A} \), the restriction of \( x_i \) to \( T_A \). Then we have

\[ \overline{x}_i = \sum_{A \ni i} \overline{x}_i^A \quad \text{and} \quad y = \sum_{A \in A} \sum_{i \in A} \lambda_i \overline{x}_i^A. \tag{22} \]
Thus we have

\[\sum_{i=1}^{n} \lambda_i v(S_i) - \epsilon < \sum_{i=1}^{n} \lambda_i \int_{S_i} u(\omega, x_i(\omega)) \, d\mu(\omega)\]

(23)

\[= \sum_{i=1}^{n} \sum_{A \ni i} \lambda_i \int_{T_A} u(\omega, x^A_i(\omega)) \, d\mu(\omega)\]

(24)

\[= \sum_{A \in A} \sum_{i \in A} \lambda_i \int_{T_A} u(\omega, x^A_i(\omega)) \, d\mu(\omega)\]

(25)

\[= \sum_{A \in A} \int_{T_A} \sum_{i \in A} \lambda_i u(\omega, x^A_i(\omega)) \, d\mu(\omega)\]

(26)

\[\leq \sum_{A \in A} \int_{T_A} u(\omega, \sum_{i \in A} \lambda_i x^A_i(\omega)) \, d\mu(\omega)\] by Lemma 3

(27)

\[= \int_{S} u(\omega, \sum_{A \in A} \sum_{i \in A} \lambda_i x^A_i(\omega)) \, d\mu(\omega)\] by \(u(\omega, 0) = 0\)

(28)

\[= \int_{S} u(\omega, y(\omega)) \, d\mu(\omega)\]

(29)

\[\leq \int_{S} u(\omega, y'(\omega)) \, d\mu(\omega)\] by monotonicity of \(u(\omega, \cdot)\)

(30)

\[\leq v(S).\]

(31)

Therefore, we have

\[\sum_{i=1}^{n} \lambda_i v(S_i) \leq v(S).\]

(32)

Thus \(\overline{v}(S) \leq v(S)\) and the reverse inequality is obvious. Hence we have \(\overline{v} = v\). Q.E.D.

A market game has a continuity property by nature.

\textbf{Proposition 2} A market game \(v\) is inner continuous at any \(S\) in \(\mathcal{F}\).

\textbf{Proof} Let \(\{S_n\}\) be a sequence of measurable sets with \(\bigcup_{n=1}^{\infty} S_n = S\) and \(\epsilon\) an arbitrary positive number. Then, there is \(x \in L_1(S, R^+_1)\) such that

\[v(S) - \epsilon < \int_{S} u(\omega, x(\omega)) \, d\mu(\omega) \quad \text{and} \quad \int_{S} x \, d\mu = \int_{S} e \, d\mu.\]

(33)
Let $x_n$ be the restriction $x|_{S_n}$ and define a sequence $\{y_n\}$ of functions in $L_1(S_n, R_+^l)$ by

$$y_n^i = \begin{cases} \frac{\int_{S_n} e^i d\mu}{\int_{S_n} x_n^i d\mu} x_n^i, & \text{if } \int_{S_n} x_n^i d\mu > \int_{S_n} e^i d\mu; \\ x_n^i + \frac{1}{\mu(S_n)} \left( \int_{S_n} e^i d\mu - \int_{S_n} x_n^i d\mu \right), & \text{if } \int_{S_n} x_n^i d\mu \leq \int_{S_n} e^i d\mu, \end{cases}$$

for $i = 1, \ldots, l$. It is obvious that

$$\int_{S_n} y_n d\mu = \int_{S_n} e d\mu.$$ 

On the other hand, since

$$\lim_{n \to \infty} \int_{S_n} |y_n^i - x_n^i| d\mu = \lim_{n \to \infty} \left| \int_{S_n} e^i d\mu - \int_{S_n} x_n^i d\mu \right| = 0,$$

for $i = 1, \ldots, l$, we have

$$\lim_{n \to \infty} \int_S \|\bar{y}_n - x\| d\mu = \lim_{n \to \infty} \int_{S_n} \|y_n - x\| d\mu + \lim_{n \to \infty} \int_{S \setminus S_n} \|x\| d\mu = 0,$$

and hence $\bar{y}_n$ converges to $x$ with respect to the norm topology of $L_1(S, R_+^l)$. Therefore, by Lemma 1, it follows that

$$\lim_{n \to \infty} \int_{S_n} u(\omega, y_n(\omega)) d\mu(\omega) = \lim_{n \to \infty} \int_S u(\omega, \bar{y}_n(\omega)) d\mu(\omega) = \int_S u(\omega, x(\omega)) d\mu(\omega)$$

and hence, for sufficiently large $n$,

$$v(S) - \epsilon < \int_{S_n} u(\omega, y_n(\omega)) d\mu(\omega) \leq v(S_n).$$

Thus we have $\lim_{n \to \infty} v(S_n) = v(S)$. Q.E.D.

Remark 3 Every exact game which is continuous at $\Omega$, equivalently inner continuous at $\Omega$, is continuous at every $S \in \mathcal{F}$ according to [3]. A market game, however, is not necessarily continuous at each $S \in \mathcal{F}$. Consider again the market game in Example 2. The game is not outer continuous at each $S \in \mathcal{F}$ with $0 < \mu(S) < \mu(\Omega)$ according to [1].

Now we have reached our main theorem combining Proposition 1 and Proposition 2.
Theorem 1 A market game $v$ has a nonempty core, and every element $\alpha$ of the core is countably additive and has a unique outcome density $f \in L_1(\Omega, R_+)$, and hence it follows that

$$\alpha(S) = \int_S f \, d\mu, \quad S \in \mathcal{F}. \quad (40)$$

Proof The core is nonempty by Proposition 1. Each element $\alpha$ of the core is continuous at $\Omega$ by Proposition 2, and hence $\alpha$ is countably additive. To prove existence of an outcome density for $\alpha$, it is sufficient to show that $\alpha$ is absolutely continuous with respect to $\mu$ by virtue of the Radon-Nikodym theorem. If $\mu(S) = 0$, then $v(S^c) = v(\Omega)$ by the definition of the game $v$, and hence we have $\alpha(S^c) \geq v(S^c) = v(\Omega) = \alpha(\Omega)$, that is, $\alpha(S) = 0$. Q.E.D.

Remark 4 Similar to the assertion of Theorem 1, an exact game which is continuous at $\Omega$ has a nonempty core and every member of the core is countably additive. Moreover, there is a measure $\lambda$ on $\mathcal{F}$ such that every member of the core is absolutely continuous with respect to $\lambda$ according to [3]. The following example shows that there is a market game which is not exact, and hence Theorem 1 is independent of the results of [3].

Example 3 [[1], pp. 192] Let $l = 1$, $\Omega = [0, 1]$ and $\mu$ be the Lebesgue measure. Define $u : [0, 1] \times R_+ \to R_+$ by

$$u(\omega, x) = \sqrt{x + \omega} - \sqrt{\omega} \quad \text{and} \quad e(\omega) = \frac{1}{32} \quad \text{for all } \omega \in [0, 1]. \quad (41)$$

According to [1], the core of the market game has only one member $\alpha$ and the outcome density $f$ of $\alpha$ is given by

$$f(\omega) = \begin{cases} \frac{1}{32}, & \text{if } \omega \in [0, \frac{1}{4}]; \\ (\frac{1}{4} - \sqrt{\omega})^2 + \frac{1}{32}, & \text{if } \omega \in [\frac{1}{4}, 1]. \end{cases} \quad (42)$$

Thus it follows $\alpha([\frac{1}{2}, 1]) = \frac{1}{64}$, and hence $\hat{v}([\frac{1}{2}, 1]) = \frac{1}{64}$. On the other hand, we have

$$\sqrt{x + \omega} - \sqrt{\omega} \leq \sqrt{x + \frac{1}{2}} - \sqrt{\frac{1}{2}} \leq \sqrt{\frac{1}{2}}x \quad \text{for } 1/2 \leq \omega \leq 1 \quad \text{and} \quad x \geq 0. \quad (43)$$

Thus, if $x \in L_1([0, 1], R_+)$ satisfies

$$\int_{\frac{1}{2}}^{1} x \, d\mu = \int_{\frac{1}{2}}^{1} e \, d\mu = \frac{1}{64}, \quad (44)$$
\[
\int_{\frac{1}{2}}^{1} u(\omega, x(\omega)) \, d\mu(\omega) \leq \int_{\frac{1}{2}}^{1} \sqrt{\frac{1}{2}} \, x \, d\mu = \frac{1}{64\sqrt{2}} < \frac{1}{64}, \tag{45}
\]

Therefore we have \( v([\frac{1}{2}, 1]) < \hat{v}([\frac{1}{2}, 1]) \) and \( v \) is not exact.

4 Concluding Remark

We have shown that every member of the core of a market game is countably additive and hence has an outcome density, and an exact game which is continuous at \( \Omega \) has these properties as written in Remark 4. If we proved that every totally balanced game that is continuous at \( \Omega \) is a game derived from a market in our sense, then we could deduce from Theorem 1 that every totally balanced game that is continuous at \( \Omega \) has a nonempty core whose members are all countably additive and have outcome densities. This problem is the infinite version of the problem solved in [4], but it is still open.

References


