数理解析研究所講究録 1264 巻 2002 年 253-262

A Market Game with Infinitely Many Players

Hidetoshi Komiya (小宮 英敏)
Faculty of Business and Commerce
Keio University (慶応大学商学部)

1 Introduction

A market game that derive from an exchange economy in which the finite number of traders have continuous concave monetary utility functions was studied fully in [4] and a market game with infinitely many traders described with a non-atomic measure space was extensively investigated in [1]. The non-atomic measure space played a crucial role to remove the concavity of utility functions from the assumption in [4]. In this paper, we shall study a market game with infinite traders described with a general measure space preserving the concavity assumption for utilities. It will be shown that such a market game has properties parallel to those of an exact game studied in [3] and each member of the core of a market game has an outcome density with respect to the measure.

Let $(\Omega, \mathscr{F})$ be a measurable space. A *game* v is a nonnegative real valued function, defined on the σ-field $\mathscr{F}$, which maps the empty set to zero. An *outcome* of a game v is a finitely additive real valued function α on $\mathscr{F}$ scuh that $\alpha(\Omega) = v(\Omega)$. For an outcome α of v, an integrable function f satisfying $\int_S f\, d\mu = \alpha(S)$ for all $S \in \mathscr{F}$ is said to be an *outcome density* of α with respect to μ. An outcome indicates outcomes to each coalitions while an outcome density designates outcomes to every players. The *core* of v is the set of outcomes α satisfying $\alpha(S) \geq v(S)$ for all $S \in \mathscr{F}$.

To every game v we associate an extended real number $|v|$ defined by

$$|v| = \sup \left\{ \sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_\Omega \right\}, \tag{1}$$

where $n = 1, 2, \ldots, S_i \in \mathscr{F}$, λ_i is a real number. The notation χ_A denotes the characteristic function of a subset A of Ω. For a game v with $|v| < \infty$, we define two games $\overline{v}$ and $\hat{v}$ by

$$\overline{v}(S) = \sup\left\{\sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_S\right\}, \quad S \in \mathscr{F}, \tag{2}$$

$$\hat{v}(S) = \min\{\alpha(S) : \alpha \text{ is additive, } \alpha \geq v,\ \alpha(\Omega) = |v|\}, \quad S \in \mathscr{F}, \tag{3}$$

following [3]. A game v is said to be *balanced* if $v(\Omega) = |v|$, *totally balanced* if $v = \overline{v}$ and *exact* if $v = \hat{v}$, respectively. It is proved in [3] that the core of a game is nonempty if and only if it is balanced, every exact game is totally balanced, and every totally balanced game is balanced.

A game v is said to be *monotone* if $S \subset T$ implies $v(S) \leq v(T)$. A game v is said to be *inner continuous* at $S \in \mathscr{F}$ if it follows that $\lim_{n\to\infty} v(S_n) = v(S)$ for any nondecreasing sequence $\{S_n\}$ of measurable sets such that $\bigcup_{n=1}^{\infty} S_n = S$. Similarly, a game v is said to be *outer continuous* at $S \in \mathscr{F}$ if it follows that $\lim_{n\to\infty} v(S_n) = v(S)$ for any nonincreasing sequence $\{S_n\}$ of measurable sets such that $\bigcap_{n=1}^{\infty} S_n = S$. A game v is *continuous* at $S \in \mathscr{F}$ if it is both inner and outer continuous at S.

2 Market Games

Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space throughout this paper. We denote utilities of players by a Carathéodory type function u defined on $\Omega \times R^l_+$ to R_+, where R^l_+ denotes the nonnegative orthant of the l-dimensional Euclidean space R^l, and R_+ is the set of nonnegative real numbers. The nonnegative number $u(\omega, x)$ designates the density of the utility of a player ω getting goods x. We always use the ordinary coordinatewise order when having concern with an order in R^l_+. We suppose that the function $u : \Omega \times R^l_+ \to R_+$ satisfies the conditions:

1. The function $\omega \mapsto u(\omega, x)$ is measurable for all $x \in R^l_+$;

2. The function $x \mapsto u(\omega, x)$ is continuous, concave, nondecreasing, and $u(\omega, 0) = 0$, for almost all ω in Ω;

3. $\sigma \equiv \sup\{u(\omega, x) : (\omega, x) \in \Omega \times B_+\} < \infty$, where $B_+ = \{x \in R^l_+ : \|x\| \leq 1\}$, and $\|x\|$ denotes the Euclidean norm of $x \in R^l_+$.

For any measurable set $S \in \mathscr{F}$, the set of integrable functions on S to R^l_+ is denoted by $L_1(S, R^l_+)$. We take an element e of $L_1(S, R^l_+)$ as the density of initial endowments for the players. For any $S \in \mathscr{F}$, define

$$v(S) \equiv \sup \left\{ \int_S u(\omega, x(\omega))\, d\mu(w) : x \in L_1(S, R^l_+),\ \int_S x\, d\mu = \int_S e\, d\mu \right\}. \tag{4}$$

The set function v defined above is called a *market game* derived from the market $(\Omega, \mathscr{F}, \mu, u, e)$.

We shall confirm that the market game v is actually a game in the rest of this section. It is well known that the function $\omega \mapsto u(\omega, x(\omega))$ is measurable for any $x \in L_1(S, R^l_+)$. Moreover we need to show that the mapping $\omega \mapsto u(\omega, x(\omega))$ is integrable in order to define $v(S)$ as a real number.

Lemma 1 If $x \in L_1(S, R^l_+)$, then $u(\cdot, x(\cdot)) \in L_1(S, R_+)$ for any $S \in \mathscr{F}$ and the map $x \mapsto u(\cdot, x(\cdot))$ is continuous with respect to the norm topologies of $L_1(S, R^l_+)$ and $L_1(S, R_+)$.

Proof Let $x \in L_1(S, R^l_+)$. Since $u(\omega, \cdot)$ is concave, for any $x \in R^l_+$ with $\|x\| > 1$, we have the inequality

$$\frac{u(\omega, x) - u(\omega, x/\|x\|)}{\|x - x/\|x\|\|} \leq \frac{u(\omega, x/\|x\|) - u(\omega, 0)}{\|x/\|x\|\|}, \tag{5}$$

and hence we have $u(\omega, x) \leq \|x\|\sigma$. It is obvious from the definition of σ that $u(\omega, x) \leq \sigma$ for all x with $\|x\| \leq 1$. Thus we have $u(\omega, x) \leq \sigma(1 + \|x\|)$ for any $x \in R^l_+$ and this leads to the inequalities

$$\int_S u(\omega, x(\omega))\, d\mu \leq \int_S \sigma(1 + \|x(\omega)\|)\, d\mu = \sigma \left(\mu(S) + \int_S \|x(\omega)\|\, d\mu \right) < \infty. \tag{6}$$

Thus it follows that $u(\cdot, x(\cdot)) \in L_1(S, R_+)$. The second part of the assertion is verified in Theorem 2.1 of [2]. Although Theorem 2.1 of [2] is proved under the hypotheses that Ω is a measurable set in R^l and the second argument x of the function u runs over R, the proof of Theorem 2.1 of [2] is valid even in our setting. Thus the map $x \mapsto u(\cdot, x(\cdot))$ is norm continuous. Q.E.D.

Remark 1 The assumption of the finiteness of σ is necessary to prove Lemma 1. The following example violates the assumption and shows that u does not necessarily convey an integrable function to an integrable function.

Example 1 Let $l = 1$ and $\Omega = (0,1)$. Define $u : (0,1) \times R_+ \to R_+$ by $u(\omega, x) = \sqrt{x}/\omega$. Then, for the function $x(\omega) = 1$ for all $\omega \in (0,1)$, it follows $u(\omega, x(\omega)) = 1/\omega$, and obviously it is not integrable.

Lemma 2 A market game v is actually a game and is monotone.

Proof It is obvious $v(\emptyset) = 0$. The finiteness of $v(S)$ follows since the inequalities

$$\int_S u(\omega, x(\omega))\, d\mu(\omega) \leq \sigma \int_S (1 + \|x\|)\, d\mu$$
$$\leq \sigma \left(\mu(S) + \sum_{i=1}^{l} \int_S x^i\, d\mu \right) = \sigma \left(\mu(S) + \sum_{i=1}^{l} \int_S e^i\, d\mu \right) \tag{7}$$

hold if

$$\int_S x\, d\mu = \int_S e\, d\mu, \tag{8}$$

where x^i and e^i are the i-th coordinate functions of x and e, respectively. Moreover v is monotone because the function $x \mapsto u(\omega, x)$ is nondecreasing for almost all $\omega \in \Omega$. Q.E.D.

Remark 2 The supremum in the definition of a market game cannot be replaced by maximum in general as the following example shows.

Example 2 [[1], pp. 204] Let $l = 1$, $\Omega = [0,1]$ and μ be the Lebesgue measure. Define $u : [0,1] \times R_+ \to R_+$ by $u(\omega, x) = \omega x$ and let $e(\omega) = 1$ for all $\omega \in \Omega$. Then $v([0,1]) = 1$ but, for any $x \in L_1([0,1], R_+)$ with $\int_0^1 x\, d\mu = 1$, $\int_0^1 \omega x(\omega)\, d\mu(\omega)$ never reaches 1.

3 Cores of Market Games

We shall investigate properties of the cores of the market games in this section. We start with a lemma on concave functions.

Lemma 3 If $f : R^l_+ \to R$ is concave and $f(0) = 0$, then for any $x_1, \ldots, x_n \in R^l_+$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i \leq 1$, it follows that

$$\sum_{i=1}^{n} \lambda_i f(x_i) \leq f(\sum_{i=1}^{n} \lambda_i x_i). \tag{9}$$

Proof We can assume that $\lambda = \sum_{i=1}^{n} \lambda_i > 0$ without loss of generality. It follows that

$$\sum_{i=1}^{n} \lambda_i f(x_i) = \lambda \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} f(x_i) \tag{10}$$

$$\leq \lambda f(\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} x_i) \tag{11}$$

$$= (1-\lambda) f(0) + \lambda f(\frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i x_i) \tag{12}$$

$$\leq f(\sum_{i=1}^{n} \lambda_i x_i). \tag{13}$$

Q.E.D.

Let S' and S be measurable sets with $S' \subset S$. For any $x \in L_1(S', R^l_+)$, define an extension $\overline{x} \in L_1(S, R^l_+)$ of x to S by

$$\overline{x}(\omega) = \begin{cases} x(\omega), & \text{if } \omega \in S'; \\ 0, & \text{if } \omega \in S \setminus S'. \end{cases} \tag{14}$$

Proposition 1 A market game v is totally balanced.

Proof Take any $S \in \mathscr{F}$ and $S_i \in \mathscr{F}$ and $\lambda_i > 0$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_S$. We can assume that $\mu(S) > 0$ without loss of generality.

Let ϵ be an arbitrary positive number. Take $x_i \in L_1(S_i, R^l_+)$ such that

$$\int_{S_i} x_i \, d\mu = \int_{S_i} e \, d\mu \quad \text{and} \quad v(S_i) - \frac{\epsilon}{n} < \int_{S_i} u(\omega, x_i(\omega)) \, d\mu(\omega), \tag{15}$$

and define $y \in L_1(S, R^l_+)$ by

$$y = \sum_{i=1}^{n} \lambda_i \overline{x}_i. \tag{16}$$

Then we have the following:

$$\int_S y\,d\mu = \sum_{i=1}^{n} \lambda_i \int_S \overline{x}_i\,d\mu \tag{17}$$

$$= \sum_{i=1}^{n} \lambda_i \int_{S_i} e\,d\mu \tag{18}$$

$$= \int_S e \sum_{i=1}^{n} \lambda_i \chi_{S_i}\,d\mu \tag{19}$$

$$\leq \int_S e\,d\mu. \tag{20}$$

Define $y' \in L_1(S, R^l_+)$ by

$$y' = y + \frac{1}{\mu(S)} \left(\int_S e\,d\mu - \int_S y\,d\mu \right). \tag{21}$$

Then it is easily seen that $\int_S y'\,d\mu = \int_S e\,d\mu$.

On the other hand, let $\mathcal{A}$ be the family of all nonempty subsets A of $\{1,\ldots,n\}$ such that $T_A \equiv \bigcap_{i\in A} S_i \cap \bigcap_{j\in A^c}(S \setminus S_j) \neq \emptyset$. Then it is easily seen that $S_i = \cup_{A\ni i} T_A$ for $i = 1,\ldots,n$ and $\{T_A : A \in \mathcal{A}\}$ is a partition of $\bigcup_{i=1}^{n} S_i$, and $\sum_{i\in A} \lambda_i \leq 1$ for all $A \in \mathcal{A}$. For any i and A with $i \in A \in \mathcal{A}$, define $x_i^A = x_i|_{T_A}$, the restriction of x_i to T_A. Then we have

$$\overline{x}_i = \sum_{A\in i} \overline{x}_i^A \quad \text{and} \quad y = \sum_{A\in\mathcal{A}} \sum_{i\in A} \lambda_i \overline{x}_i^A. \tag{22}$$

Thus we have

$$\sum_{i=1}^{n} \lambda_i v(S_i) - \epsilon < \sum_{i=1}^{n} \lambda_i \int_{S_i} u(\omega, x_i(\omega))\, d\mu(\omega) \tag{23}$$

$$= \sum_{i=1}^{n} \sum_{A \ni i} \lambda_i \int_{T_A} u(\omega, x_i^A(\omega))\, d\mu(\omega) \tag{24}$$

$$= \sum_{A \in \mathcal{A}} \sum_{i \in A} \lambda_i \int_{T_A} u(\omega, x_i^A(\omega))\, d\mu(\omega) \tag{25}$$

$$= \sum_{A \in \mathcal{A}} \int_{T_A} \sum_{i \in A} \lambda_i u(\omega, x_i^A(\omega))\, d\mu(\omega) \tag{26}$$

$$\leq \sum_{A \in \mathcal{A}} \int_{T_A} u(\omega, \sum_{i \in A} \lambda_i x_i^A(\omega))\, d\mu(\omega) \quad \text{by Lemma 3} \tag{27}$$

$$= \int_S u(\omega, \sum_{A \in \mathcal{A}} \sum_{i \in A} \lambda_i \overline{x}_i^A(\omega))\, d\mu(\omega) \quad \text{by } u(\omega, 0) = 0 \tag{28}$$

$$= \int_S u(\omega, y(\omega))\, d\mu(\omega) \tag{29}$$

$$\leq \int_S u(\omega, y'(\omega))\, d\mu(\omega) \quad \text{by monotonicity of } u(\omega, \cdot) \tag{30}$$

$$\leq v(S). \tag{31}$$

Therefore, we have

$$\sum_{i=1}^{n} \lambda_i v(S_i) \leq v(S). \tag{32}$$

Thus $\overline{v}(S) \leq v(S)$ and the reverse inequality is obvious. Hence we have $\overline{v} = v$. Q.E.D.

A market game has a continuity property by nature.

Proposition 2 A market game v is inner continuous at any S in $\mathscr{F}$.

Proof Let $\{S_n\}$ be a sequence of measurable sets with $\bigcup_{n=1}^{\infty} S_n = S$ and ϵ an arbitrary positive number. Then, there is $x \in L_1(S, R_+^l)$ such that

$$v(S) - \epsilon < \int_S u(\omega, x(\omega))\, d\mu(\omega) \quad \text{and} \quad \int_S x\, d\mu = \int_S e\, d\mu. \tag{33}$$

Let x_n be the restriction $x|_{S_n}$ and define a sequence $\{y_n\}$ of functions in $L_1(S_n, R^l_+)$ by

$$y^i_n = \begin{cases} \frac{\int_{S_n} e^i \, d\mu}{\int_{S_n} x^i_n \, d\mu} x^i_n, & \text{if } \int_{S_n} x^i_n \, d\mu > \int_{S_n} e^i \, d\mu; \\ x^i_n + \frac{1}{\mu(S_n)} \left(\int_{S_n} e^i \, d\mu - \int_{S_n} x^i_n \, d\mu \right), & \text{if } \int_{S_n} x^i_n \, d\mu \leq \int_{S_n} e^i \, d\mu, \end{cases} \tag{34}$$

for $i = 1, \ldots, l$. It is obvious that

$$\int_{S_n} y_n \, d\mu = \int_{S_n} e \, d\mu. \tag{35}$$

On the other hand, since

$$\lim_{n\to\infty} \int_{S_n} |y^i_n - x^i_n| \, d\mu = \lim_{n\to\infty} \left| \int_{S_n} e^i \, d\mu - \int_{S_n} x^i_n \, d\mu \right| = 0, \tag{36}$$

for $i = 1, \ldots, l$, we have

$$\lim_{n\to\infty} \int_S \|\overline{y}_n - x\| \, d\mu = \lim_{n\to\infty} \int_{S_n} \|y_n - x\| \, d\mu + \lim_{n\to\infty} \int_{S\setminus S_n} \|x\| \, d\mu = 0, \tag{37}$$

and hence $\overline{y}_n$ converges to x with respect to the norm topology of $L_1(S, R^l_+)$. Therefore, by Lemma 1, it follows that

$$\lim_{n\to\infty} \int_{S_n} u(\omega, y_n(\omega)) \, d\mu(\omega) = \lim_{n\to\infty} \int_S u(\omega, \overline{y}_n(\omega)) \, d\mu(\omega) = \int_S u(\omega, x(\omega)) \, d\mu(\omega) \tag{38}$$

and hence, for sufficiently large n,

$$v(S) - \epsilon < \int_{S_n} u(\omega, y_n(\omega)) \, d\mu(\omega) \leq v(S_n). \tag{39}$$

Thus we have $\lim_{n\to\infty} v(S_n) = v(S)$. Q.E.D.

Remark 3 Every exact game which is continuous at Ω, equivalently inner continuous at Ω, is continuous at every $S \in \mathscr{F}$ according to [3]. A market game, however, is not necessarily continuous at each $S \in \mathscr{F}$. Consider again the market game in Example 2. The game is not outer continuous at each $S \in \mathscr{F}$ with $0 < \mu(S) < \mu(\Omega)$ according to [1].

Now we have reached our main theorem combining Proposition 1 and Proposition 2.

Theorem 1 A market game v has a nonempty core, and every element α of the core is countably additive and has a unique outcome density $f \in L_1(\Omega, R_+)$, and hence it follows that

$$\alpha(S) = \int_S f\, d\mu, \quad S \in \mathscr{F}. \tag{40}$$

Proof The core is nonempty by Proposition 1. Each element α of the core is continuous at Ω by Proposition 2, and hence α is countably additive. To prove existence of an outcome density for α, it is sufficient to show that α is absolutely continuous with respect to μ by virtue of the Radon-Nikodym theorem. If $\mu(S) = 0$, then $v(S^c) = v(\Omega)$ by the definition of the game v, and hence we have $\alpha(S^c) \geq v(S^c) = v(\Omega) = \alpha(\Omega)$, that is, $\alpha(S) = 0$. Q.E.D.

Remark 4 Similar to the assertion of Theorem 1, an exact game which is continuous at Ω has a nonempty core and every member of the core is countably additive. Moreover, there is a measure λ on $\mathscr{F}$ such that every member of the core is absolutely continuous with respect to λ according to [3]. The following example shows that there is a market game which is not exact, and hence Theorem 1 is independent of the results of [3].

Example 3 [[1], pp. 192] Let $l = 1$, $\Omega = [0, 1]$ and μ be the Lebesgue measure. Define $u : [0, 1] \times R_+ \to R_+$ by

$$u(\omega, x) = \sqrt{x + \omega} - \sqrt{\omega} \quad \text{and} \quad e(\omega) = \frac{1}{32} \text{ for all } \omega \in [0, 1]. \tag{41}$$

According to [1], the core of the market game has only one member α and the outcome density f of α is given by

$$f(\omega) = \begin{cases} (\frac{1}{2} - \sqrt{\omega})^2 + \frac{1}{32}, & \text{if } \omega \in [0, \frac{1}{4}]; \\ \frac{1}{32}, & \text{if } \omega \in [\frac{1}{4}, 1]. \end{cases} \tag{42}$$

Thus it follows $\alpha([\frac{1}{2}, 1]) = \frac{1}{64}$, and hence $\hat{v}([\frac{1}{2}, 1]) = \frac{1}{64}$. On the other hand, we have

$$\sqrt{x + \omega} - \sqrt{\omega} \leq \sqrt{x + \frac{1}{2}} - \sqrt{\frac{1}{2}} \leq \sqrt{\frac{1}{2}} x \tag{43}$$

for $1/2 \leq \omega \leq 1$ and $x \geq 0$. Thus, if $x \in L_1([0, 1], R_+)$ satisfies

$$\int_{\frac{1}{2}}^{1} x\, d\mu = \int_{\frac{1}{2}}^{1} e\, d\mu = \frac{1}{64}, \tag{44}$$

$$\int_{\frac{1}{2}}^{1} u(\omega, x(\omega))\, d\mu(\omega) \leq \int_{\frac{1}{2}}^{1} \sqrt{\frac{1}{2}x}\, d\mu = \frac{1}{64\sqrt{2}} < \frac{1}{64}. \tag{45}$$

Therefore we have $v([\frac{1}{2}, 1]) < \hat{v}([\frac{1}{2}, 1])$ and v is not exact.

4 Concluding Remark

We have shown that every member of the core of a market game is countably additive and hence has an outcome density, and an exact game which is continuous at Ω has these properties as written in Remark 4. If we proved that every totally balanced game that is continuous at Ω is a game derived from a market in our sense, then we could deduce from Theorem 1 that every totally balanced game that is continuous at Ω has a nonempty core whose members are all countably additive and have outcome densities. This problem is the infinite version of the problem solved in [4], but it is still open.

References

[1] Aumann RJ and Shapley LS (1974) Values of Non-Atomic Games. Princeton University Press, Princeton

[2] Krasnosel'skii MA (1963) Topological Methods in the Theory of Non-linear Integral Equations, Pergmon Press, Oxford

[3] Schmeidler D (1972) Cores of Exact Games, I. J Math Anal Appl **40**: 214–225

[4] Shapley LS and Shubik M (1969) On Market Games. J Econ Theory **1**: 9–25.