A Market Game with Infinitely Many Players

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1 Introduction

A market game that derive from an exchange economy in which the finite number of traders have continuous concave monetary utility functions was studied fully in [4] and a market game with infinitely many traders described with a non-atomic measure space was extensively investigated in [1]. The non-atomic measure space played a crucial role to remove the concavity of utility functions from the assumption in [4]. In this paper, we shall study a market game with infinite traders described with a general measure space preserving the concavity assumption for utilities. It will be shown that such a market game has properties parallel to those of an exact game studied in [3] and each member of the core of a market game has an outcome density with respect to the measure.

Let $(\Omega, \mathcal{F})$ be a measurable space. A game $v$ is a nonnegative real valued function, defined on the $\sigma$-field $\mathcal{F}$, which maps the empty set to zero. An outcome of a game $v$ is a finitely additive real valued function $\alpha$ on $\mathcal{F}$ such that $\alpha(\Omega) = v(\Omega)$. For an outcome $\alpha$ of $v$, an integrable function $f$ satisfying $\int_S f \, d\mu = \alpha(S)$ for all $S \in \mathcal{F}$ is said to be an outcome density of $\alpha$ with respect to $\mu$. An outcome indicates outcomes to each coalitions while an outcome density designates outcomes to every players. The core of $v$ is the set of outcomes $\alpha$ satisfying $\alpha(S) \geq v(S)$ for all $S \in \mathcal{F}$.

To every game $v$ we associate an extended real number $|v|$ defined by

$$ |v| = \sup \left\{ \sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_\Omega \right\},$$

where $n = 1, 2, \ldots, S_i \in \mathcal{F}, \lambda_i$ is a real number. The notation $\chi_A$ denotes the characteristic function of a subset $A$ of $\Omega$. For a game $v$ with $|v| < \infty$, \ldots
we define two games $\overline{v}$ and $\hat{v}$ by

$$
\overline{v}(S) = \sup \left\{ \sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_S \right\}, \quad S \in \mathcal{F},
$$

$$
\hat{v}(S) = \min \{ \alpha(S) : \alpha \text{ is additive}, \alpha \geq v, \alpha(\Omega) = |v| \}, \quad S \in \mathcal{F},
$$

following [3]. A game $v$ is said to be balanced if $v(\Omega) = |v|$, totally balanced if $v = \overline{v}$ and exact if $v = \hat{v}$, respectively. It is proved in [3] that the core of a game is nonempty if and only if it is balanced, every exact game is totally balanced, and every totally balanced game is balanced.

A game $v$ is said to be monotone if $S \subset T$ implies $v(S) \leq v(T)$. A game $v$ is said to be inner continuous at $S \in \mathcal{F}$ if it follows that

$$
\lim_{n \to \infty} v(S_n) = v(S)
$$

for any nondecreasing sequence $\{S_n\}$ of measurable sets such that $\bigcup_{n=1}^{\infty} S_n = S$. Similarly, a game $v$ is said to be outer continuous at $S \in \mathcal{F}$ if it follows that $\lim_{n \to \infty} v(S_n) = v(S)$ for any nonincreasing sequence $\{S_n\}$ of measurable sets such that $\bigcap_{n=1}^{\infty} S_n = S$. A game $v$ is continuous at $S \in \mathcal{F}$ if it is both inner and outer continuous at $S$.

2 Market Games

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space throughout this paper. We denote utilities of players by a Carathéodory type function $u$ defined on $\Omega \times R_+^l$ to $R_+$, where $R_+^l$ denotes the nonnegative orthant of the $l$-dimensional Euclidean space $R_+^l$, and $R_+$ is the set of nonnegative real numbers. The nonnegative number $u(\omega, x)$ designates the density of the utility of a player $\omega$ getting goods $x$. We always use the ordinary coordinatewise order when having concern with an order in $R_+^l$. We suppose that the function $u : \Omega \times R_+^l \to R_+$ satisfies the conditions:

1. The function $\omega \mapsto u(\omega, x)$ is measurable for all $x \in R_+^l$;

2. The function $x \mapsto u(\omega, x)$ is continuous, concave, nondecreasing, and $u(\omega, 0) = 0$, for almost all $\omega$ in $\Omega$;

3. $\sigma \equiv \sup\{u(\omega, x) : (\omega, x) \in \Omega \times B_+\} < \infty$, where $B_+ = \{x \in R_+^l : ||x|| \leq 1\}$, and $||x||$ denotes the Euclidean norm of $x \in R_+^l$.

For any measurable set $S \in \mathcal{F}$, the set of integrable functions on $S$ to $R_+^l$ is denoted by $L_1(S, R_+^l)$. We take an element $e$ of $L_1(S, R_+^l)$ as the
density of initial endowments for the players. For any \( S \in \mathcal{F} \), define
\[
v(S) \equiv \sup \left\{ \int_S u(\omega, x(\omega)) \, d\mu : x \in L_1(S, R^l_+), \int_S x \, d\mu = \int_S e \, d\mu \right\}.
\]  
(4)

The set function \( v \) defined above is called a market game derived from the market \((\Omega, \mathcal{F}, \mu, u, e)\).

We shall confirm that the market game \( v \) is actually a game in the rest of this section. It is well known that the function \( \omega \mapsto u(\omega, x(\omega)) \) is measurable for any \( x \in L_1(S, R^l_+) \). Moreover we need to show that the mapping \( \omega \mapsto u(\omega, x(\omega)) \) is integrable in order to define \( v(S) \) as a real number.

**Lemma 1** If \( x \in L_1(S, R^l_+) \), then \( u(\cdot, x(\cdot)) \in L_1(S, R^l) \) for any \( S \in \mathcal{F} \) and the map \( x \mapsto u(\cdot, x(\cdot)) \) is continuous with respect to the norm topologies of \( L_1(S, R^l_+) \) and \( L_1(S, R^l) \).

**Proof** Let \( x \in L_1(S, R^l_+) \). Since \( u(\omega, \cdot) \) is concave, for any \( x \in R^l_+ \) with \( \|x\| > 1 \), we have the inequality
\[
\frac{u(\omega, x) - u(\omega, x/\|x\|)}{\|x - x/\|x\||} \leq \frac{u(\omega, x/\|x\|) - u(\omega, 0)}{\|x/\|x\||},
\]  
(5)
and hence we have \( u(\omega, x) \leq \|x\|\sigma \). It is obvious from the definition of \( \sigma \) that \( u(\omega, x) \leq \sigma \) for all \( x \) with \( \|x\| \leq 1 \). Thus we have \( u(\omega, x) \leq \sigma(1 + \|x\|) \) for any \( x \in R^l_+ \) and this leads to the inequalities
\[
\int_S u(\omega, x(\omega)) \, d\mu \leq \int_S \sigma(1 + \|x(\omega)\|) \, d\mu = \sigma \left( \mu(S) + \int_S \|x(\omega)\| \, d\mu \right) < \infty.
\]  
(6)

Thus it follows that \( u(\cdot, x(\cdot)) \in L_1(S, R^l) \). The second part of the assertion is verified in Theorem 2.1 of [2]. Although Theorem 2.1 of [2] is proved under the hypotheses that \( \Omega \) is a measurable set in \( R^l \) and the second argument \( x \) of the function \( u \) runs over \( R \), the proof of Theorem 2.1 of [2] is valid even in our setting. Thus the map \( x \mapsto u(\cdot, x(\cdot)) \) is norm continuous. Q.E.D.

**Remark 1** The assumption of the finiteness of \( \sigma \) is necessary to prove Lemma 1. The following example violates the assumption and shows that \( u \) does not necessarily convey an integrable function to an integrable function.
Example 1 Let \( l = 1 \) and \( \Omega = (0, 1) \). Define \( u : (0, 1) \times R_+ \rightarrow R_+ \) by \( u(\omega, x) = \sqrt{x}/\omega \). Then, for the function \( x(\omega) = 1 \) for all \( \omega \in (0, 1) \), it follows \( u(\omega, x(\omega)) = 1/\omega \), and obviously it is not integrable.

Lemma 2 A market game \( v \) is actually a game and is monotone.

Proof It is obvious \( v(\emptyset) = 0 \). The finiteness of \( v(S) \) follows since the inequalities

\[
\int_S u(\omega, x(\omega)) \, d\mu(\omega) \leq \sigma \int_S (1 + \|x\|) \, d\mu \\
\leq \sigma \left( \mu(S) + \sum_{i=1}^{l} \int_S x^i \, d\mu \right) = \sigma \left( \mu(S) + \sum_{i=1}^{l} \int_S e^i \, d\mu \right)
\]

(7)

hold if

\[
\int_S x \, d\mu = \int_S e \, d\mu,
\]

(8)

where \( x^i \) and \( e^i \) are the \( i \)-th coordinate functions of \( x \) and \( e \), respectively. Moreover \( v \) is monotone because the function \( x \mapsto u(\omega, x) \) is nondecreasing for almost all \( \omega \in \Omega \). Q.E.D.

Remark 2 The supremum in the definition of a market game cannot be replaced by maximum in general as the following example shows.

Example 2 \([1, \text{pp. 204}]\) Let \( l = 1 \), \( \Omega = [0, 1] \) and \( \mu \) be the Lebesgue measure. Define \( u : [0, 1] \times R_+ \rightarrow R_+ \) by \( u(\omega, x) = \omega x \) and let \( e(\omega) = 1 \) for all \( \omega \in \Omega \). Then \( v([0,1]) = 1 \) but, for any \( x \in L_1([0,1], R_+) \) with \( \int_0^1 x \, d\mu = 1, \int_0^1 \omega x(\omega) \, d\mu(\omega) \) never reaches 1.

3 Cores of Market Games

We shall investigate properties of the cores of the market games in this section. We start with a lemma on concave functions.

Lemma 3 If \( f : R_+^l \rightarrow R \) is concave and \( f(0) = 0 \), then for any \( x_1, \ldots, x_n \in R_+^l \) and \( \lambda_1, \ldots, \lambda_n \geq 0 \) with \( \sum_{i=1}^n \lambda_i \leq 1 \), it follows that

\[
\sum_{i=1}^n \lambda_i f(x_i) \leq f(\sum_{i=1}^n \lambda_i x_i).
\]

(9)
Proof We can assume that $\lambda = \sum_{i=1}^{n} \lambda_i > 0$ without loss of generality. It follows that
\[
\sum_{i=1}^{n} \lambda_i f(x_i) = \lambda \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} f(x_i)
\]
\[
\leq \lambda f\left(\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} x_i\right)
\]
\[
= (1 - \lambda)f(0) + \lambda f\left(\frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i x_i\right)
\]
\[
\leq f\left(\sum_{i=1}^{n} \lambda_i x_i\right).
\]
Q.E.D.

Let $S'$ and $S$ be measurable sets with $S' \subset S$. For any $x \in L_1(S', R_+^l)$, define an extension $\overline{x} \in L_1(S, R_+^l)$ of $x$ to $S$ by
\[
\overline{x}(\omega) = \begin{cases} x(\omega), & \text{if } \omega \in S'; \\ 0, & \text{if } \omega \in S \setminus S'. \end{cases}
\]
(14)

Proposition 1 A market game $v$ is totally balanced.

Proof Take any $S \in \mathcal{F}$ and $S_i \in \mathcal{F}$ and $\lambda_i > 0$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} \lambda_i x_{S_i} \leq x_S$. We can assume that $\mu(S) > 0$ without loss of generality.

Let $\epsilon$ be an arbitrary positive number. Take $x_i \in L_1(S_i, R_+^l)$ such that
\[
\int_{S_i} x_i \, d\mu = \int_{S_i} e \, d\mu \quad \text{and} \quad v(S_i) - \frac{\epsilon}{n} < \int_{S_i} u(\omega, x_i(\omega)) \, d\mu(\omega),
\]
(15)
and define $y \in L_1(S, R_+^l)$ by
\[
y = \sum_{i=1}^{n} \lambda_i \overline{x}_i.
\]
(16)
Then we have the following:

\[
\int_S y \, d\mu = \sum_{i=1}^{n} \lambda_i \int_S \overline{x}_i \, d\mu \tag{17}
\]

\[
= \sum_{i=1}^{n} \lambda_i \int_{S_i} e \, d\mu \tag{18}
\]

\[
= \int_{S} e \sum_{i=1}^{n} \lambda_i \chi_{S_i} \, d\mu \tag{19}
\]

\[
\leq \int_{S} e \, d\mu. \tag{20}
\]

Define \( y' \in L_1(S, R_+^l) \) by

\[
y' = y + \frac{1}{\mu(S)} \left( \int_S e \, d\mu - \int_S y \, d\mu \right). \tag{21}\]

Then it is easily seen that \( \int_S y' \, d\mu = \int_S e \, d\mu. \)

On the other hand, let \( A \) be the family of all nonempty subsets \( A \) of \( \{1, \ldots, n\} \) such that \( T_A \equiv \bigcap_{i \in A} S_i \cap \bigcap_{j \in A^c} (S \setminus S_j) \neq \emptyset \). Then it is easily seen that \( S_i = \bigcup_{A \in A} T_A \) for \( i = 1, \ldots, n \) and \( \{T_A : A \in A\} \) is a partition of \( \bigcup_{i=1}^{n} S_i \), and \( \sum_{i \in A} \lambda_i \leq 1 \) for all \( A \in A \). For any \( i \) and \( A \) with \( i \in A \in A \), define \( x_i^A = x_i|_{T_A} \), the restriction of \( x_i \) to \( T_A \). Then we have

\[
\overline{x}_i = \sum_{A \in i} x_i^A \quad \text{and} \quad y = \sum_{A \in A} \sum_{i \in A} \lambda_i \overline{x}_i^A. \tag{22}\]
Thus we have

\[ \sum_{i=1}^{n} \lambda_i v(S_i) - \epsilon < \sum_{i=1}^{n} \lambda_i \int_{S_i} u(\omega, x_i(\omega)) \, d\mu(\omega) \] (23)

\[ = \sum_{i=1}^{n} \sum_{A \ni i} \lambda_i \int_{T_A} u(\omega, x_i^A(\omega)) \, d\mu(\omega) \] (24)

\[ = \sum_{A \in A} \sum_{i \in A} \lambda_i \int_{T_A} u(\omega, x_i^A(\omega)) \, d\mu(\omega) \] (25)

\[ = \sum_{A \in A} \int_{T_A} \sum_{i \in A} \lambda_i u(\omega, x_i^A(\omega)) \, d\mu(\omega) \] (26)

\[ \leq \sum_{A \in A} \int_{T_A} u(\omega, \sum_{i \in A} \lambda_i x_i^A(\omega)) \, d\mu(\omega) \text{ by Lemma 3} \] (27)

\[ = \int_S u(\omega, \sum_{A \in A} \sum_{i \in A} \lambda_i x_i^A(\omega)) \, d\mu(\omega) \text{ by } u(\omega, 0) = 0 \] (28)

\[ = \int_S u(\omega, y(\omega)) \, d\mu(\omega) \] (29)

\[ \leq \int_S u(\omega, y'(\omega)) \, d\mu(\omega) \text{ by monotonicity of } u(\omega, \cdot) \] (30)

\[ \leq v(S). \] (31)

Therefore, we have

\[ \sum_{i=1}^{n} \lambda_i v(S_i) \leq v(S). \] (32)

Thus \( \overline{v}(S) \leq v(S) \) and the reverse inequality is obvious. Hence we have \( \overline{v} = v \). Q.E.D.

A market game has a continuity property by nature.

**Proposition 2** A market game \( v \) is inner continuous at any \( S \) in \( \mathcal{F} \).

**Proof** Let \( \{S_n\} \) be a sequence of measurable sets with \( \bigcup_{n=1}^{\infty} S_n = S \) and \( \epsilon \) an arbitrary positive number. Then, there is \( x \in L_1(S, \mathbb{R}^l) \) such that

\[ v(S) - \epsilon < \int_S u(\omega, x(\omega)) \, d\mu(\omega) \quad \text{and} \quad \int_S x \, d\mu = \int_S e \, d\mu. \] (33)
Let $x_n$ be the restriction $x|_{S_n}$ and define a sequence $\{y_n\}$ of functions in $L_1(S_n, R^l_+)$ by

$$y_n^i = \begin{cases} \frac{\int_{S_n} e^i d\mu}{\int_{S_n} x_n^i d\mu} x_n^i, & \text{if } \int_{S_n} x_n^i d\mu > \int_{S_n} e^i d\mu; \\ x_n^i + \frac{1}{\mu(S_n)} \left( \int_{S_n} e^i d\mu - \int_{S_n} x_n^i d\mu \right), & \text{if } \int_{S_n} x_n^i d\mu \leq \int_{S_n} e^i d\mu, \end{cases}$$

for $i = 1, \ldots, l$. It is obvious that

$$\int_{S_n} y_n d\mu = \int_{S_n} e d\mu. \quad (35)$$

On the other hand, since

$$\lim_{n \to \infty} \int_{S_n} |y_n^i - x_n^i| d\mu = \lim_{n \to \infty} \left| \int_{S_n} e^i d\mu - \int_{S_n} x_n^i d\mu \right| = 0, \quad (36)$$

for $i = 1, \ldots, l$, we have

$$\lim_{n \to \infty} \int_S \|y_n - x\| d\mu = \lim_{n \to \infty} \int_{S_n} \|y_n - x\| d\mu + \lim_{n \to \infty} \int_{S\setminus S_n} \|x\| d\mu = 0, \quad (37)$$

and hence $y_n$ converges to $x$ with respect to the norm topology of $L_1(S, R^l_+)$. Therefore, by Lemma 1, it follows that

$$\lim_{n \to \infty} \int_{S_n} u(\omega, y_n(\omega)) d\mu(\omega) = \lim_{n \to \infty} \int_S u(\omega, y_n(\omega)) d\mu(\omega) = \int_{S} u(\omega, x(\omega)) d\mu(\omega)$$

and hence, for sufficiently large $n$,

$$v(S) - \epsilon < \int_{S_n} u(\omega, y_n(\omega)) d\mu(\omega) \leq v(S_n). \quad (39)$$

Thus we have $\lim_{n \to \infty} v(S_n) = v(S)$. Q.E.D.

**Remark 3** Every exact game which is continuous at $\Omega$, equivalently inner continuous at $\Omega$, is continuous at every $S \in \mathcal{F}$ according to [3]. A market game, however, is not necessarily continuous at each $S \in \mathcal{F}$. Consider again the market game in Example 2. The game is not outer continuous at each $S \in \mathcal{F}$ with $0 < \mu(S) < \mu(\Omega)$ according to [1].

Now we have reached our main theorem combining Proposition 1 and Proposition 2.
Theorem 1 A market game $v$ has a nonempty core, and every element $\alpha$ of the core is countably additive and has a unique outcome density $f \in L_1(\Omega, R_+)$, and hence it follows that

$$\alpha(S) = \int_S f \ d\mu, \ S \in \mathcal{F}. \quad (40)$$

Proof The core is nonempty by Proposition 1. Each element $\alpha$ of the core is continuous at $\Omega$ by Proposition 2, and hence $\alpha$ is countably additive. To prove existence of an outcome density for $\alpha$, it is sufficient to show that $\alpha$ is absolutely continuous with respect to $\mu$ by virtue of the Radon-Nikodym theorem. If $\mu(S) = 0$, then $v(S^c) = v(\Omega)$ by the definition of the game $v$, and hence we have $\alpha(S^c) \geq v(S^c) = v(\Omega) = \alpha(\Omega)$, that is, $\alpha(S) = 0$. Q.E.D.

Remark 4 Similar to the assertion of Theorem 1, an exact game which is continuous at $\Omega$ has a nonempty core and every member of the core is countably additive. Moreover, there is a measure $\lambda$ on $\mathcal{F}$ such that every member of the core is absolutely continuous with respect to $\lambda$ according to [3]. The following example shows that there is a market game which is not exact, and hence Theorem 1 is independent of the results of [3].

Example 3 [[1], pp. 192] Let $l = 1$, $\Omega = [0,1]$ and $\mu$ be the Lebesgue measure. Define $u : [0,1] \times R_+ \to R_+$ by

$$u(\omega, x) = \sqrt{x + \omega} - \sqrt{\omega} \quad \text{and} \quad e(\omega) = \frac{1}{32} \text{ for all } \omega \in [0,1]. \quad (41)$$

According to [1], the core of the market game has only one member $\alpha$ and the outcome density $f$ of $\alpha$ is given by

$$f(\omega) = \begin{cases} \left(\frac{1}{2} - \sqrt{\omega}\right)^2 + \frac{1}{32}, & \text{if } \omega \in [0, \frac{1}{4}] \\ \frac{1}{32}, & \text{if } \omega \in \left[\frac{1}{4}, 1\right]. \end{cases} \quad (42)$$

Thus it follows $\alpha([\frac{1}{2}, 1]) = \frac{1}{64}$, and hence $\hat{v}(\frac{1}{2}, 1]) = \frac{1}{64}$. On the other hand, we have

$$\sqrt{x + \omega} - \sqrt{\omega} \leq \sqrt{x + \frac{1}{2}} - \sqrt{\frac{1}{2}} \leq \sqrt{\frac{1}{2}} x \quad (43)$$

for $1/2 \leq \omega \leq 1$ and $x \geq 0$. Thus, if $x \in L_1([0,1], R_+)$ satisfies

$$\int_{\frac{1}{2}}^{1} x \ d\mu = \int_{\frac{1}{2}}^{1} e \ d\mu = \frac{1}{64}, \quad (44)$$

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Therefore we have $v([\frac{1}{2}, 1]) < \hat{v}([\frac{1}{2}, 1])$ and $v$ is not exact.

4 Concluding Remark

We have shown that every member of the core of a market game is countably additive and hence has an outcome density, and an exact game which is continuous at $\Omega$ has these properties as written in Remark 4. If we proved that every totally balanced game that is continuous at $\Omega$ is a game derived from a market in our sense, then we could deduce from Theorem 1 that every totally balanced game that is continuous at $\Omega$ has a nonempty core whose members are all countably additive and have outcome densities. This problem is the infinite version of the problem solved in [4], but it is still open.

References


