Mean-square stability of numerical schemes for stochastic differential systems

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Abstract

Stochastic differential equations (SDEs) represent physical phenomena dominated by stochastic processes. As for deterministic ordinary differential equations (ODEs), various numerical schemes are proposed for SDEs. We have proposed the mean-square stability of numerical schemes for a scalar SDE, that is, the numerical stability with respect to the mean-square norm. However we studied it for only scalar SDEs because of difficulty and complexity in SDE systems. In the present note we will consider a 2-dimensional linear system with one multiplicative noise and try to analyze them.

1 Introduction

We have proposed the numerical mean-square stability (MS-stability) for a scalar stochastic differential equation (SDE) with one multiplicative noise [7]. However we studied it for only scalar SDEs. Komori and Mitsui [4, 5] analyzed numerical MS-stability for a 2-dimensional SDE with special case, that is, simultaneously diagonalizable case. In this note we will try to analyze numerical MS-stability of the Euler-Maruyama scheme for general 2-dimensional SDE systems.

Consider the SDE of Itô-type given by

$$dX(t) = f(t, X)dt + g(t, X)dW(t)$$ (1)

with $f(0, t) = g(0, t) = 0$ so that the steady state $X(t) = 0$ is the equilibrium solution. The Euler-Maruyama scheme for the discrete approximate solution $\{\bar{X}_n\}$ is

$$\bar{X}_{n+1} = \bar{X}_n + f(t_n, \bar{X}_n)h + g(t_n, \bar{X}_n)\Delta W_n$$

where $h$ and $\Delta W_n$ stand for the step-size and the increment of the Wiener process, respectively. Then we can give the definition of the MS-stability.

**Definition 1** Steady solution $X(t) \equiv 0$ is asymptotically stable in mean-square if

$$\forall \varepsilon > 0, \exists \delta > 0; \quad E(\|X(t)\|^2) < \varepsilon \quad for \ all \ t \geq 0 \ and \ \|X_0\| < \delta$$

and

$$\exists \delta_0; \quad \lim_{t \to \infty} E(\|X(t)\|^2) = 0 \quad for \ all \ \|X_0\| < \delta_0$$
Here the norm $\|x\|$ is the Euclidean norm of a vector $x \in \mathbb{R}^2$.

We will consider three types of linear SDE systems, and try to analyze them. In the next section we describe the results of MS-stability for three types of the SDE system. Section 3 shows the results of numerical MS-stability of the Euler-Maruyama scheme corresponding to results in Section 2. In Section 4 we will show the numerical experiments confirming our stability analysis in Section 3. Finally we will describe our conclusion and future aspects.

2 MS-stability

We will restrict the SDE (1) to an Ito-type 2-dimensional linear SDE system with one multiplicative noise, which has the form

\[
\begin{cases}
    dX(t) = DX(t)dt + BX(t)dW(t), \\
    X(0) = 1.
\end{cases}
\]

Here the real constant matrices $D$ and $B$ are given by

\[
D = \begin{bmatrix}
    \lambda_1 & 0 \\
    0 & \lambda_2
\end{bmatrix}, \quad
B = \begin{bmatrix}
    \alpha_1 & \beta_1 \\
    \beta_2 & \alpha_2
\end{bmatrix}.
\]

Komori and Mitsui [4, 5] analyzed MS-stability for SDE system (2) with $\beta_1 = 0$ and $\beta_2 = 0$ (simultaneously diagonalizable case). We will consider more general SDE system, namely $\beta_1 \neq 0$ and $\beta_2 \neq 0$. First we will introduce the conventional and the logarithmic norms of matrices for stability analysis of the SDE system (2).

**Definition 2** Corresponding to the vector norms $l^1$, $l^2$ and $l^\infty$ in $\mathbb{R}^n$, we define the subordinate matrix norms of square $n \times n$ matrix $A = (a_{ij})$ by

\[
\|A\|_1 = \max_j \{\sum_{i=1}^n |a_{ij}|\}, \quad \|A\|_\infty = \max_i \left\{\sum_{j=1}^n |a_{ij}|\right\},
\]

\[
\|A\|_2 = \left\{\text{maximum eigenvalue of } A^T A\right\}^{1/2}.
\]

**Definition 3** Logarithmic matrix norm $\mu[A]$ (see [1, 6]) is defined by

\[
\mu[A] = \lim_{h \to 0^+} (\|I + hA\| - 1)/h
\]

where $I$ is the unit matrix and $h \in \mathbb{R}$.

For the matrix norms $\| \cdot \|_1$, $\| \cdot \|_\infty$ and $\| \cdot \|_2$, the following identities are well known to evaluate the logarithmic norms.

\[
\mu_1[A] = \max_j \left\{a_{jj} + \sum_{i \neq j} |a_{ij}|\right\}, \quad \mu_\infty[A] = \max_i \left\{a_{ii} + \sum_{j \neq i} |a_{ij}|\right\},
\]

\[
\mu_2[A] = \text{maximum eigenvalue of } (A + A^T)/2.
\]
Let \( P(t) = \mathbb{E}(X(t)X(t)^T) \) be the \( 2 \times 2 \) matrix-valued second moment of the solution of (2). Then \( P(t) \) obeys the initial value problem of the following matrix ordinary differential equation (ODE)

\[
\frac{dP}{dt} = DP + PD^T + BPB^T \quad (t > 0),
\]

with \( P(0) = X_0X_0^T \). Due to the symmetry of the matrix \( P \) we have its governing ODEs of 3-dimension

\[
\frac{dY}{dt} = \mathcal{M}Y
\]

where

\[
Y(t) = (Y^1(t), Y^2(t), Y^3(t)), \quad Y^1(t) = \mathbb{E}(X^1(t))^2,
\]

\[
Y^2(t) = \mathbb{E}(X^2(t))^2, \quad Y^3(t) = \mathbb{E}(X^1(t)X^2(t)).
\]

We can readily obtain the following lemma owing to the logarithmic matrix norm \( \mu \).

**Lemma 1** The linear test system with the unit initial value is asymptotically MS-stable w.r.t. logarithmic norm \( \mu \) iff

\[
\mu(\mathcal{M}) < 0
\]

We will study MS-stability for the following three types of the test system. Drift matrix \( D \) in (2) is fixed with real numbers \( \lambda_1 < \lambda_2 < 0 \) and diffusion matrices \( B \) are either

Type I: \( \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \), Type II: \( \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} \), or Type III: \( \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \).

Here real numbers \( \alpha \) and \( \beta \) are non-negative.

**Theorem 1** In Type I the matrix in (4) is given by

\[
\mathcal{M} = \begin{bmatrix} 2\lambda_1 + \alpha^2 & 0 & 0 \\ 0 & 2\lambda_2 + \alpha^2 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 + \alpha^2 \end{bmatrix}.
\]

Henceforth the stability criterion w.r.t. \( \mu_2, \mu_\infty \) and \( \mu_1 \) yields

\[
\max\{2\lambda_1 + \alpha^2, 2\lambda_2 + \alpha^2\} < 0.
\]

(5)

We employed the following identity to derive (5).

\[
\lambda_1 + \lambda_2 + \alpha^2 = \frac{2\lambda_1 + \alpha^2 + 2\lambda_2 + \alpha^2}{2}
\]

(6)

Type II has the following
**Theorem 2** The coefficient matrix in Type II is given by

\[
\mathcal{M} = \begin{bmatrix}
2\lambda_1 & \beta^2 & 0 \\
\beta^2 & 2\lambda_2 & 0 \\
0 & 0 & \lambda_1 + \lambda_2 + \beta^2
\end{bmatrix},
\]

which implies the stability criterion w.r.t. \( \mu_\infty \) and \( \mu_1 \) as

\[
\max\{2\lambda_1 + \beta^2, 2\lambda_2 + \beta^2\} < 0.
\]

Again we employed (6).

Note that the condition represented by \( \mu_\infty \) is a sufficient condition for the convergence to the zero solution. We will show this through the following example.

**Example 1** The combination with

\[
D = \begin{bmatrix}
-100 & 0 \\
0 & -1
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
0 & 2 \\
2 & 0
\end{bmatrix}
\]

yields

\[
\mathcal{M} = \begin{bmatrix}
-200 & 4 & 0 \\
4 & -2 & 0 \\
0 & 0 & -97
\end{bmatrix},
\]

whose logarithmic norms are

\[
\mu_\infty(\mathcal{M}) = 2 > 0 \quad \text{but} \quad \mu_2(\mathcal{M}) = -101 + \sqrt{9817} < 0.
\]

Finally we will study Type III as the composition of Types I and II. We conclude with the theorem.

**Theorem 3** Type III has the coefficient matrix given by

\[
\mathcal{M} = \begin{bmatrix}
2\lambda_1 + \alpha^2 & \beta^2 & 2\alpha\beta \\
\beta^2 & 2\lambda_2 + \alpha^2 & 2\alpha\beta \\
\alpha\beta & \alpha\beta & \lambda_1 + \lambda_2 + \alpha^2 + \beta^2
\end{bmatrix},
\]

which brings the stability condition w.r.t. \( \mu_\infty \) as

\[
\max\{2\lambda_1 + (|\alpha| + |\beta|)^2, 2\lambda_2 + (|\alpha| + |\beta|)^2\} < 0
\]

Note that the stability criterion for Type III is given only in \( \mu_\infty \).
3 MS-stability of Euler-Maruyama scheme

We now ask what conditions must be imposed in order that the numerical solution \( \overline{X}_n \) of (2) generated by a numerical scheme satisfies

\[
\overline{Y}_n = \mathrm{E}|\overline{X}_n|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{7}
\]

When we apply a numerical scheme to (2) and calculate the components of the second moment of \( \overline{X}_n \), we obtain a one-step difference equation of the form

\[
\overline{Y}_{n+1} = \overline{\mathcal{M}} \overline{Y}_n \tag{8}
\]

where

\[
\overline{Y}_n = (\overline{Y}_n^1, \overline{Y}_n^2, \overline{Y}_n^3), \quad \overline{Y}_n^1 = \mathrm{E}(\overline{X}_n^1)^2, \quad \overline{Y}_n^2 = \mathrm{E}(\overline{X}_n^2)^2, \quad \overline{Y}_n^3 = \mathrm{E}(\overline{X}_n^1 \overline{X}_n^2).
\]

We shall call \( \overline{\mathcal{M}} \) the stability matrix of the scheme. Note that \( \overline{Y}_n \rightarrow 0 \) as \( n \rightarrow \infty \) if

\[
||\overline{\mathcal{M}}|| < 1. \tag{9}
\]

**Definition 4** The numerical scheme is said to be MS-stable w.r.t. \( ||\cdot|| \) if it has \( \overline{\mathcal{M}} \) satisfying \( ||\overline{\mathcal{M}}|| < 1 \).

We will calculate the stability matrices \( \overline{\mathcal{M}} \) and MS-stability conditions w.r.t. \( ||\cdot|| \) of the Euler-Maruyama scheme for Type I, II and III. Let \( r(x) \) be \( 1 + x \) in the following theorems.

**Theorem 4** For Type I we obtain

\[
\overline{\mathcal{M}} = \begin{bmatrix}
    r^2(\lambda_1 h) + \alpha^2 h & 0 & 0 \\
    0 & r^2(\lambda_2 h) + \alpha^2 h & 0 \\
    0 & 0 & r(\lambda_1 h)r(\lambda_2 h) + \alpha^2 h
\end{bmatrix},
\]

which yields the stability condition w.r.t. \( ||\cdot||_2, ||\cdot||_\infty \) and \( ||\cdot||_1 \) as

\[
\max\{(1 + \lambda_1 h)^2 + \alpha^2 h, (1 + \lambda_2 h)^2 + \alpha^2 h\} < 1. \tag{10}
\]

The inequality

\[
r(\lambda_1 h)r(\lambda_2 h) + \alpha^2 h \leq \frac{r^2(\lambda_1 h) + r^2(\lambda_2 h) + 2\alpha^2 h}{2} \tag{11}
\]

is utilized to derive the above result. When we observe the left-hand side in the MS-stability condition (10), we conclude to check the numerical MS-stability whether the pair \((\overline{h}, k) = (\lambda h, \alpha^2/\lambda)\) satisfying \( |R(\overline{h}, k)| < 1 \) for every \( \lambda_1 \) and \( \lambda_2 \). Namely we should check \((\overline{h}_1, k_1) = (\lambda_1 h, \alpha^2/\lambda_1), (\overline{h}_2, k_2) = (\lambda_2 h, \alpha^2/\lambda_2) \in \mathcal{R}_{\mathrm{EM}} \). Here \( \mathcal{R}_{\mathrm{EM}} \) is the MS-stability region of the Euler-Maruyama scheme in scalar case. We will show the region in Fig. 1.

Next we will focus on Type II. We will calculate the \( \overline{\mathcal{M}} \) and stability condition as same as Type I.
Theorem 5 Type II has the stability matrix given by
\[
\overline{\mathcal{M}} = \begin{bmatrix}
r^2(\lambda_1 h) & \beta^2 h & 0 \\
\beta^2 h & r^2(\lambda_2 h) & 0 \\
0 & 0 & r(\lambda_1 h)r(\lambda_2 h) + \beta^2 h
\end{bmatrix},
\] (12)
which brings the stability condition w.r.t. \( \| \cdot \|_{\infty} \) and \( \| \cdot \|_1 \) as
\[
\max\{(1 + \lambda_1 h)^2 + |\beta^2 h|, (1 + \lambda_2 h)^2 + |\beta^2 h|\} < 1.
\]
We result in stability function of the Euler-Maruyama scheme (scalar case), namely \( R(\bar{h}, k) \) again applicable by \( \bar{h} = \lambda h, \ k = \beta^2/\lambda \) like as Type I.

Finally we try to analyze Type III.

Theorem 6 For Type III we have
\[
\overline{\mathcal{M}} = \begin{bmatrix}
r^2(\lambda_1 h) + \alpha^2 h & \beta^2 h & 2\alpha\beta h \\
\beta^2 h & r^2(\lambda_2 h) + \alpha^2 h & 2\alpha\beta h \\
\alpha\beta h & \alpha\beta h & r(\lambda_1 h)r(\lambda_2 h) + (\alpha^2 + \beta^2)h
\end{bmatrix},
\]
which implies the stability condition w.r.t. \( \| \cdot \|_{\infty} \) as
\[
\max\{(1 + \lambda_1 h)^2 + (|\alpha| + |\beta|)^2 h, (1 + \lambda_2 h)^2 + (|\alpha| + |\beta|)^2 h\} < 1.
\]
Like as Type I and II, we conclude that stability function of the Euler-Maruyama scheme (scalar case) \( R(\bar{h}, k) \) again applicable with \( \bar{h} = \lambda h, \ k = (|\alpha| + |\beta|)^2/\lambda \).

4 Numerical experiments

In this section we will show the confirmation for our MS-stability of the Euler-Maruyama scheme through numerical experiments. We will describe four examples corresponding to Type I, II, and III (2 examples) as follows.

Example 2 (Type I)
\[
dX = \begin{bmatrix}
-200 & 0 \\
0 & -100
\end{bmatrix} X dt + \begin{bmatrix}
10 & 0 \\
0 & 10
\end{bmatrix} X dW(t)
\] (13)

\( h = 0.005, \ (\bar{h}, k) = (-1, -0.5), (-0.5, -1) : \text{stable} \)
\( h = 0.01, \ (\bar{h}, k) = (-2, -0.5), (-1, -1) : \text{unstable} \)
\( h = 0.02, \ (\bar{h}, k) = (-4, -0.5), (-2, -1) : \text{unstable} \)
\( h = 0.05, \ (\bar{h}, k) = (-10, -0.5), (-5, -1) : \text{unstable} \)

Example 3 (Type II)
\[
dX = \begin{bmatrix}
-200 & 0 \\
0 & -100
\end{bmatrix} X dt + \begin{bmatrix}
0 & 10 \\
10 & 0
\end{bmatrix} X dW(t)
\]

\( h = 0.005, \ (\bar{h}, k) = (-1, -0.5), (-0.5, -1) : \text{stable} \)
\( h = 0.01, \ (\bar{h}, k) = (-2, -0.5), (-1, -1) : \text{unstable} \)
\( h = 0.02, \ (\bar{h}, k) = (-4, -0.5), (-2, -1) : \text{unstable} \)
\( h = 0.05, \ (\bar{h}, k) = (-10, -0.5), (-5, -1) : \text{unstable} \)
Example 4 (Type III)
\[
\frac{\mathrm{d}X}{\mathrm{d}t} = \begin{bmatrix} -200 & 0 \\ 0 & -100 \end{bmatrix} X + \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix} \frac{\mathrm{d}X}{\mathrm{d}W(t)}
\]

\( h = 0.005 \), \((\bar{h}, k) = (-1, -0.625), (-0.5, -1.25) : \text{stable} \)
\( h = 0.01 \), \((\bar{h}, k) = (-2, -0.625), (-1, -1.25) : \text{unstable} \)

Example 5 (Type III)
\[
\frac{\mathrm{d}X}{\mathrm{d}t} = \begin{bmatrix} -200 & 0 \\ 0 & -100 \end{bmatrix} X + \begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix} \frac{\mathrm{d}X}{\mathrm{d}W(t)}
\]

\( h = 0.005 \), \((\bar{h}, k) = (-1, -0.625), (-0.5, -1.25) : \text{stable} \)
\( h = 0.01 \), \((\bar{h}, k) = (-2, -0.625), (-1, -1.25) : \text{unstable} \)

We took the initial value \( X(0) = (1, 1) \) and 10,000 samples. We will show the results of Example 2 to Fig. 2, Example 3 to Fig. 3, Example 4 to Fig. 4 and Example 5 to Fig. 5.

5 Conclusions and Future aspects

We extended numerical MS-stability for a scalar SDE with one multiplicative noise to it for a 2-dimensional SDE system with one multiplicative noise. We will analyze MS-stability for general pair of the matrices \( D \) and \( B \), and more dimensional case. And we will investigate the relation of the MS-stability conditions in matrix norms, for example, between \( \| \cdot \|_{\infty} \) and \( \| \cdot \|_2 \).

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References


Figure 1: MS-stability region of Euler-Maruyama scheme
$S \succ \succ Y, -Y, -Y^2 \cdots \cdots$.

Figure 2: Example 1 (upper left: $h = 0.005$, upper right: $h = 0.01$, lower left: $h = 0.02$, lower right: $h = 0.05$)
Figure 3: Example 2 (upper left: $h = 0.005$, upper right: $h = 0.01$, lower left: $h = 0.02$, lower right: $h = 0.05$)

Figure 4: Example 3 (left: $h = 0.005$, right: $h = 0.01$)
Figure 5: Example 4 (left: $h = 0.005$, right: $h = 0.01$)