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Author(s)
Koto, Toshiyuki

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Stability analysis of numerical methods for delay integro-differential equations

TOSHIYUKI KOTO 小藤 俊幸

Department of Computer Science
The University of Electro-Communications
電気通信大学
e-mail: koto@im.uec.ac.jp

Abstract

Stability of $\theta$-methods for delay integro-differential equations (DIDEs) is studied on the basis of the linear equation

$$\frac{du}{dt} = \lambda u(t) + \mu u(t - \tau) + \kappa \int_{t-\tau}^{t} u(\sigma)d\sigma,$$

where $\lambda$, $\mu$, $\kappa$ are complex numbers and $\tau$ is a constant delay. It is shown that every $A$-stable $\theta$-method possesses a similar stability property to $P$-stability, i.e., the method preserves the delay-independent stability of the exact solution under the condition that $\tau/h$ is an integer, where $h$ is a step-size. It is also shown that the method does not possess the same property if $\tau/h$ is not an integer. As a result, any $\theta$-method cannot possess a similar stability property to $GP$-stability with respect to DIDEs.

1. Introduction

We study stability of (2-stage) $\theta$-methods for delay integro-differential equations (DIDEs) on the basis of the linear equation

$$\frac{du}{dt} = \lambda u(t) + \mu u(t - \tau) + \kappa \int_{t-\tau}^{t} u(\sigma)d\sigma,$$  \hspace{1cm} (1.1)

where $\lambda$, $\mu$, $\kappa$ are complex numbers and $\tau$ is a constant delay. When $\kappa = 0$, the equation (1.1) coincides with the test equation

$$\frac{du}{dt} = \lambda u(t) + \mu u(t - \tau),$$  \hspace{1cm} (1.2)

which was proposed by Barwell [1] to examine stability of numerical methods for delay differential equations (DDEs). As described in [1], if $\lambda$, $\mu$ satisfy

$$|\mu| < -\text{Re} \lambda,$$  \hspace{1cm} (1.3)

the zero solution of (1.2) is asymptotically stable for any $\tau \geq 0$. This asymptotic property is called delay-independent stability, and analogous stability properties of numerical methods are considered on the basis of the condition (1.3). For example,
a numerical method for DDEs is said to be $P$-stable if every numerical solution to (1.2) tends to zero whenever $\lambda$, $\mu$ satisfy (1.3) and $\tau/h$ is an integer, where $h$ is the step-size. A numerical method is said to be $GP$-stable if the same holds for any constant step-size.

In the last two decades, various studies were carried out concerning stability properties of numerical methods for DDEs (see, e.g., [12]). In particular, an earliest study by Watanabe and Roth [10] has revealed that every $A$-stable $\theta$-method is $GP$-stable. To the contrary, little is known about stability properties of numerical methods for DIDEs. It is quite recent that we studied delay-independent stability of linear DIDEs [7], and even stability of $\theta$-methods for (1.1) remains to be investigated.

By Theorem 2 of [7], the zero solution of (1.1) is asymptotically stable for any $\tau \geq 0$ if and only if $\lambda$, $\mu$, $\kappa$ satisfy

$$\lambda + \mu + \kappa \tau \neq 0 \quad \text{for any } \tau \geq 0,$$

$$z^2 - z\lambda - \kappa = 0, \quad z \in \mathbb{C}, \quad z \neq 0 \implies \operatorname{Re} z < 0,$$

$$\left| \frac{\mu z - \kappa}{z^2 - z\lambda - \kappa} \right| < 1 \quad \text{for any } \operatorname{Re} z = 0 \text{ with } z \neq 0.$$  \hspace{1cm} (1.5)

Moreover, the conditions (1.5), (1.6) are rewritten as

$$\operatorname{Re} \lambda < 0 \quad \text{and} \quad \left( \operatorname{Re} \lambda \operatorname{Re}(\lambda \overline{\kappa}) + (\operatorname{Im} \kappa)^2 < 0 \text{ or } \kappa = 0 \right),$$

$$\operatorname{Im}[(\lambda + \mu)\overline{\kappa}] = 0 \quad \text{and} \quad \left[ |\mu|^2 < (\operatorname{Re} \lambda)^2 + 2 \operatorname{Re} \kappa \quad \text{or} \quad (\operatorname{Im} \lambda = 0, \quad |\mu|^2 = (\operatorname{Re} \lambda)^2 + 2 \operatorname{Re} \kappa \right],$$

respectively (Sect. 3 in [7]). When $\lambda$, $\mu$, $\kappa$ are all real and $\kappa \neq 0$, these conditions are reduced to the simple condition

$$\lambda < 0, \quad \kappa < 0, \quad \mu^2 \leq \lambda^2 + 2 \kappa.$$  \hspace{1cm} (1.9)

We study stability properties of $\theta$-methods by comparing the region determined by these conditions with a kind of stability regions of the methods.

![Fig. 1 Delay-independent v.s. delay-dependent stability regions](image-url)
It should be noted that a considerable number of papers [2, 3, 4, 6, 9] are devoted to stability analysis of $\theta$-methods for DDEs, which does not seem strange from a practical viewpoint. Some important instances of stiff DDEs are obtained from the space-descriptization of partial functional differential equations (see, e.g., [13]). The $\theta$-methods have practicality in such a situation.

2. Stability regions of $\theta$-methods

Consider delay integro-differential equations (DIDEs) with a constant delay,

$$\frac{du}{dt} = f\left(t, u(t), u(t - \tau), \int_{t-\tau}^{t} g(t, \sigma, u(\sigma)) d\sigma\right). \quad (2.1)$$

For a given step-size $h > 0$, let $m$ be the smallest integer greater than or equal to $\tau/h$. Then, the delay $\tau$ is represented in the form

$$\tau = (m - \delta)h, \quad 0 \leq \delta < 1,$$

and the relation

$$t_n - \tau = t_{n-m} + \delta h \quad (2.2)$$

holds for the step points $t_n = t_0 + nh, \ n \in Z$.

By approximating the delayed argument and the integrand in (2.1) with linear interpolation, we can adapt a $\theta$-method to (2.1) as follows:

$$u_{n+1} = u_n + h(1-\theta)f(t_n, u_n, v_n, G_n) + h\theta f(t_{n+1}, u_{n+1}, v_{n+1}, G_{n+1}), \quad (2.4)$$

where, $0 \leq \theta \leq 1$, $u_n$ is an approximate value of $u(t_n)$, and

$$v_n = (1-\delta)u_{n-m} + \delta u_{n-m+1}, \quad (2.5)$$

$$G_n = \frac{h(1-\delta)^2}{2}g(t_n, t_{n-m}, u_{n-m}) + \frac{h(2-\delta^2)}{2}g(t_n, t_{n-m+1}, u_{n-m+1})$$

$$+ h \sum_{k=2}^{m-1} g(t_n, t_{n-m+k}, u_{n-m+k}) + \frac{h}{2}g(t_n, t_n, u_n) + \frac{1+\theta}{2}u_{n+1} + \frac{\theta}{2}u_{n+1} \quad (2.6)$$

As a result, the integral term of (2.1) is approximated with the trapezoidal rule. When $\theta = 1/2$ and $\delta = 0$, the formula (2.4)-(2.6) determines a method that belongs to a class of Runge-Kutta methods discussed in [7]. But, when $\theta \neq 1/2$, it gives another type of numerical method.

In the case of the test equation (1.1), the formula (2.4)-(2.6) is reduced to

$$u_{n+1} = u_n + (1-\theta)\alpha u_n + \theta\alpha u_{n+1}$$

$$+ \beta \left[ (1-\delta)(1-\theta)u_{n-m} + (\delta + \theta - 2\delta\theta)u_{n-m+1} + \delta\theta u_{n-m+2} \right]$$

$$+ \gamma \left[ \frac{(1-\delta)^2(1-\theta)}{2} u_{n-m} + \frac{(2-\delta^2)(1-\theta) + (1-\delta)^2\theta}{2} u_{n-m+1} \right.$$

$$\left. + \frac{2 - \delta^2\theta}{2} u_{n-m+2} + \sum_{k=3}^{m-1} u_{n-m+k} + \frac{1+\theta}{2} u_n + \frac{\theta}{2} u_{n+1} \right], \quad (2.7)$$
\( \alpha = h \lambda, \quad \beta = h \mu, \quad \gamma = h^2 \kappa. \) \tag{2.8} 

The characteristic equation of (2.7) is written as

\[
\begin{align*}
 z^{m+1} - z^m - (1-\theta)\alpha z^m - \theta\alpha z^{m+1} \\
&- \beta \left[ (1-\delta)(1-\theta) + (\delta + \theta - 2\delta \theta)z + \delta \theta z^2 \right] \\
&- \gamma \left[ \frac{(1-\delta)^2(1-\theta)}{2} + \frac{(2-\delta^2)(1-\theta) + (1-\delta)^2\theta}{2}z \\
&+ \frac{2-\delta^2\theta}{2} z^2 + \sum_{k=3}^{m-1} z^k + \frac{1+\theta}{2} z^m + \frac{\theta}{2} z^{m+1} \right] = 0. \tag{2.9}
\end{align*}
\]

Using (2.9) we define the sets \( S^{(\delta)}_{\theta,m} \) and \( S^{(\delta)}_{\theta} \) for \( 0 \leq \delta < 1 \) by

\[
S^{(\delta)}_{\theta,m} = \{(\alpha, \beta, \gamma) \in \mathcal{C}^3 : \text{all the roots of (2.9) satisfy } |z| < 1\},
\tag{2.10}
\]

\[
S^{(\delta)}_{\theta} = \bigcap_{m \geq 1} S^{(\delta)}_{\theta,m}. \tag{2.11}
\]

The set \( S^{(\delta)}_{\theta} \) is an analogue of the \( \delta \)-stability region of the \( \theta \)-method [4].

When \( z = 1 \), the left hand side of (2.9) is equal to \(-[\alpha + \beta + (m-\delta)\gamma]\). Hence, for any \( m \geq 1 \), \( z = 1 \) is not a root of (2.9) if and only if

\((C_0)\) \( \alpha + \beta + (m-\delta)\gamma \neq 0 \) for any \( m \geq 1 \).

Substituting \( \sum_{k=3}^{m-1} z^k = (z^3 - z^m)/(1-z) \) into (2.9) and multiplying \((1-z)\), we get

\[
\begin{align*}
 z^m q(z) - p(z) &= 0, \tag{2.12} \\
 q(z) &= q_0 z^2 + q_1 z + q_2, \tag{2.13} \\
 p(z) &= p_0 z^3 + p_1 z^2 + p_2 z + p_3, \tag{2.14}
\end{align*}
\]

where

\[
\begin{align*}
 q_0 &= \theta \alpha + \frac{1-\theta}{2} \gamma - 1, \quad q_1 = (1-2\theta)\alpha + \frac{\gamma}{2} + 2, \\
 q_2 &= -(1-\theta)\alpha + \frac{1-\theta}{2} \gamma - 1, \\
 p_0 &= -\delta \theta \beta + \frac{\delta^2 \theta}{2} \gamma, \quad p_1 = (3\delta \theta - \delta - \theta)\beta + \frac{-3\delta^2 \theta + \delta^2 + 2\delta \theta + \theta}{2} \gamma, \\
 p_2 &= -(3\delta \theta + 2\delta + 2\theta - 1)\beta + \frac{3\delta^2 \theta - 2\delta^2 - 4\delta \theta - 2\delta^2 + 2\delta + 1}{2} \gamma, \\
 p_3 &= (\delta \theta - \delta - \theta + 1)\beta + \frac{-\delta^2 \theta + \delta^2 + 2\delta \theta - 2\delta - \theta + 1}{2} \gamma.
\end{align*}
\]

Moreover, we set

\[
r(z) = p(z)/q(z), \tag{2.15}
\]

and consider the following conditions.
(a) \( q(z) \neq 0 \) for any \( |z| \geq 1 \).

(a) \( q(z) \neq 0 \) for any \( |z| > 1 \).

(b) \( |r(z)| < 1 \) for any \( |z|=1 \) with \( z \neq 1 \).

(b) \( |r(z)| \leq 1 \) for any \( |z|=1 \).

These are regarded as conditions for \( \alpha, \beta, \gamma \). We also write

(c) \( (\alpha, \beta, \gamma) \in S_{\delta}^{(\delta)} \).

Under this notation, we can characterize \( S_{\delta}^{(\delta)} \) as follows.

**Theorem 2.1** The following implications hold:

- (C\(_0\)) and (a) and (b) \( \Rightarrow \) (c) \( \Rightarrow \) (a) and (b).

If, in addition,

- (C\(_1\)) \( p(z), q(z) \) have no common zero on \( |z|=1 \),

then (c) implies (a).

**Proof.** Assume (C\(_0\)), (a) and (b). We first show that \( \hat{r}(z) = r(z)/z \) satisfies

\[ |\hat{r}(z)| < 1 \] for any \( |z| \geq 1 \) with \( z \neq 1 \).

The linear fractional transformation

\[ z = \frac{w+1}{w-1} \] maps \( \text{Re} \ w > 0 \) conformally onto \( |z| > 1 \), with \( w = \infty \) corresponding to \( z = 1 \). The function \( \hat{R}(w) = \hat{r}[(w+1)/(w-1)] \) is represented in the form

\[ \hat{R}(w) = \frac{\hat{P}(w)/\hat{Q}(w)}{w-(1-2\theta)}, \] \[ \hat{P}(w) = \left[ \gamma w^2 + (-2\beta + 2\delta \gamma - \gamma)w + 2(1 - 2\delta)\beta - 2\delta(1 - \delta)\gamma \right] \times \left[ w - (1 - 2\theta) \right], \] \[ \hat{Q}(w) = (w+1) \left\{ \gamma w^2 + \left[ 2\alpha - (1 - 2\theta)\gamma \right]w - 2(1 - 2\theta)\alpha - 4 \right\}. \]

Then, it follows from (a) that \( \hat{R}(w) \) is a bounded, holomorphic function in \( \text{Re} \ w > 0 \). Hence, by the Phragmén-Lindelöf theorem (see, e.g., [8], p. 168), it follows from (b) that \( |\hat{R}(w)| < 1 \) for any \( \text{Re} \ w > 0 \), which implies that \( |\hat{r}(z)| < 1 \) for any \( |z| \geq 1 \) with \( z \neq 1 \).

If \( |z| \geq 1 \) and \( z \neq 1 \), then

\[ z^m q(z) - p(z) = q(z)z \left[ z^{m-1} - \hat{r}(z) \right] \neq 0, \]
which, together with \((C_0)\), implies \((c)\).

Assume \((c)\). If \(q(z_0) = 0\) for some \( |z_0| > 1 \), then there exists \( \varepsilon > 0 \) such that 
\[
C(z_0, \varepsilon) \subset \{ |z| > 1 \} \quad \text{and} \quad q(z) \neq 0 \quad \text{on} \quad C(z_0, \varepsilon),
\]
where 
\[
C(z_0, \varepsilon) = \{ z \in \mathbb{C} : |z - z_0| = \varepsilon \}.
\]
By Rouché's theorem, the polynomial \( z^m q(z) - p(z) \) has a root in the interior of \( C(z_0, \varepsilon) \) for sufficiently large \( m \), which contradicts \((c)\). Therefore, \((\hat{a})\) holds.

Moreover, if \( |r(z_0)| > 1 \) for some \( |z_0| = 1 \), then the equation \( z^m = r(z) \) has a solution with \( |z| > 1 \) for sufficiently large \( m \). This is verified by applying Proposition 7 of [11] to \( \psi(z) = 1/r(z) \). In fact, there exists \( \varepsilon > 0 \) such that \( |r(z)| > 1 \) for any \( z \in \overline{V_\varepsilon} \), where 
\[
V_\varepsilon = \{ z \in \mathcal{O} : |z - (1 + \varepsilon)z_0| < \varepsilon \}.
\]
Hence,
\[
\rho = \max_{z \in \overline{V_\varepsilon}} |\psi(z)| < 1,
\]
and \( 1 \in \mathcal{O} \setminus B(0, \rho) \), where \( B(0, \rho) = \{ z \in \mathcal{O} : |z| \leq \rho \} \). On the other hand, we have
\[
\mathcal{O} \setminus B(0, \rho) \subset \bigcup_{m \geq 1} \{ z^m \psi(z) : z \in V_\varepsilon \}, \tag{2.20}
\]
by Proposition 7 of [11]. Since \( |z| > 1 \) for any \( z \in V_\varepsilon \), it follows from (2.20) that \( z^m = r(z) \) holds for some \( m \geq 1 \) and \( |z| > 1 \), which contradicts \((c)\). Therefore, \((\hat{b})\) holds.

It is easy to see that \((\hat{a})\) and \((\hat{b})\) imply \((a)\) under the condition \((C_1)\). \(\square\)

3. Stability regions in the case \( \delta = 0 \)

We consider the case \( \delta = 0 \). Since \( q(1) = \gamma \), \( z = 1 \) satisfies \( q(z) = 0 \) if and only if \( \gamma = 0 \). We assume that \( \gamma \neq 0 \) for a while, and rewrite the conditions \((a)\), \((\hat{a})\), \((b)\), \((\hat{b})\) by making use of the linear fractional transformation (2.16).

The function \( R(w) = r[(w+1)/(w-1)] \) is represented in the form
\[
R(w) = P(w)/Q(w), \tag{3.1}
\]
\[
P(w) = (\gamma w - 2\beta)[w - (1 - 2\theta)], \tag{3.2}
\]
\[
Q(w) = \gamma w^2 + \left[2\alpha - (1 - 2\theta)\gamma\right]w - 2(1 - 2\theta)\alpha - 4. \tag{3.3}
\]
Hence, \((a)\), \((\hat{a})\), \((b)\), \((\hat{b})\) are equivalent to
\[
\text{(A)} \quad Q(w) \neq 0 \quad \text{for any} \quad \text{Re} \ w \geq 0,
\]
\[
\text{(A)} \quad Q(w) \neq 0 \quad \text{for any} \quad \text{Re} \ w > 0,
\]
\[
\text{(B)} \quad |R(w)| < 1 \quad \text{for any} \quad \text{Re} \ w = 0,
\]
\[
\text{(B)} \quad |R(w)| \leq 1 \quad \text{for any} \quad \text{Re} \ w = 0,
\]
respectively. When \( \alpha, \gamma \) are real, (A), (A) are reduced to
\[
\gamma [2\alpha - (1 - 2\theta)\gamma] > 0, \quad \gamma [-4 - 2(1 - 2\theta)\alpha] > 0, \quad (3.4)
\]
\[
\gamma [2\alpha - (1 - 2\theta)\gamma] \geq 0, \quad \gamma [-4 - 2(1 - 2\theta)\alpha] \geq 0, \quad (3.5)
\]
respectively. In addition, putting \( w = iy, \ y \in \mathbb{R} \), we have
\[
|Q(w)|^2 - |P(w)|^2 = 4 \text{Im}[(\alpha + \beta)\overline{\gamma}]y^3 + 4(\alpha^2 - \beta^2 + 2\text{Re}\gamma)y^2 + \left\{ 16\text{Im}\alpha + 4(1 - 2\theta)^2 \text{Im}[(\alpha + \beta)\overline{\gamma}] \right\} y + 4 + 2(1 - 2\theta)\alpha^2 - |2(1 - 2\theta)\beta|^2. \quad (3.6)
\]
When \( \alpha, \beta, \gamma \) are real, it is reduced to
\[
|Q(w)|^2 - |P(w)|^2 = 4(\alpha^2 - \beta^2 + 2\gamma)y^2 + 4\eta, \quad (3.7)
\]
\[
\eta = \left[ (1 - 2\theta)(\alpha + \beta) + 2 \right]\left[ (1 - 2\theta)(\alpha - \beta) + 2 \right]. \quad (3.8)
\]
Hence, in this case, (B), (B) are equivalent to
\[
\beta^2 \leq \alpha^2 + 2\gamma, \quad \eta > 0, \quad (3.9)
\]
\[
\beta^2 \leq \alpha^2 + 2\gamma, \quad \eta \geq 0, \quad (3.10)
\]
respectively.

Fig. 2 \( \gamma \)-section of \( S_{\theta}^{(0)} \cap \mathbb{R}^3 \) \( 0 \leq \theta < 1/2 \)
Let \( \alpha < 0 \) and \( \gamma < 0 \). The conditions (3.4), (3.9) are reduced to
\[
\alpha > -\frac{2}{1 - 2\theta}, \quad \gamma > \frac{2\alpha}{1 - 2\theta}, \quad \beta^2 \leq \alpha^2 + 2\gamma, \quad |\beta| < \alpha + \frac{2}{1 - 2\theta},
\]
(3.11)
when \( 0 \leq \theta < 1/2 \) (Fig. 2), and
\[
\beta^2 \leq \alpha^2 + 2\gamma,
\]
(3.12)
when \( 1/2 \leq \theta \leq 1 \). If \( \alpha < 0, \beta \) satisfy \( \beta^2 \leq \alpha^2 + 2\gamma \) for \( \gamma < 0 \), then \( \alpha + \beta < 0 \), and (C_0) holds. Hence, by Theorem 2.1, these determine the region
\[
S_{\theta}^{(0)} \cap \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha < 0, \gamma < 0\},
\]
(3.13)
extcept for ambiguity of the boundary.

We now denote by \( \Omega \) the set of all the triplicate \((\lambda, \mu, \kappa)\) for which the zero solution of (1.1) is asymptotically stable for any \( \tau \geq 0 \), i.e.,
\[
\Omega = \{(\lambda, \mu, \kappa) \in \mathbb{R}^3 : (1.4), (1.5), (1.6) \text{ are satisfied}\}.
\]
(3.14)
It is easy to see that
\[
(\lambda, \mu, \kappa) \in \Omega \implies (h\lambda, h\mu, h^2\kappa) \in \Omega \text{ for any } h > 0.
\]
(3.15)
The following theorem shows that A-stable \( \theta \)-methods possess a similar stability property to \( P \)-stability with respect to DIDEs.

**Theorem 3.2** If \( 1/2 \leq \theta \leq 1 \), then \( \Omega \subset S_{\theta}^{(0)} \).

**Proof.** The inclusion \( \Omega \cap \{\gamma = 0\} \subset S_{\theta}^{(0)} \) follows from the known result as in the case of DDEs (see, e.g., Theorem 2.6 in [6]). We consider the case \( \gamma \neq 0 \).

Let \((\alpha, \beta, \gamma) \in \Omega\). The condition (C_0) follows from (1.4). Moreover, it follows from (3.6) and \( \text{Im}[(\alpha + \beta)\bar{\gamma}] = 0 \) that for \( w = iy, y \in \mathbb{R} \),
\[
|Q(w)|^2 - |P(w)|^2 = \eta_0 y^2 + 2\eta_1 y + \eta_2,
\]
(3.16)
\[
\eta_0 = 4\left(|\alpha|^2 - |\beta|^2 + 2\text{Re}\gamma\right), \quad \eta_1 = 8\text{Im}\alpha,
\]
\[
\eta_2 = |2(1 - 2\theta)\alpha + 4|^2 - |2(1 - 2\theta)\beta|^2.
\]
Since
\[
\eta_2 = 16 + 16(1 - 2\theta)\text{Re}\alpha + 4(1 - 2\theta)^2\left(|\alpha|^2 - |\beta|^2\right) \geq 16,
\]
(3.17)
\[
\eta_1^2 - \eta_0 \eta_2 \leq 64(\text{Im}\alpha)^2 - 64\left(|\alpha|^2 - |\beta|^2 + 2\text{Re}\gamma\right)
\]
\[
= -64\left[(\text{Re}\alpha)^2 + 2\text{Re}\gamma - |\beta|^2\right],
\]
(3.18)
we have
\[
|Q(w)| > |P(w)| \text{ for any } \text{Re} w = 0,
\]
(3.19)
which implies (B).

When \( \theta = 1/2 \), it holds that
\[
Q(w) = \gamma w^2 + 2\alpha w - 4 = -w^2 \left[ (2/w)^2 - \alpha(2/w) - \gamma \right].
\] (3.20)

Hence, (A) for \( \theta = 1/2 \) follows from (1.5).

The condition (A) for \( \theta = 1/2 \), together with (3.19), implies (A) for \( 1/2 < \theta \leq 1 \). In fact, if \( Q(w) = 0 \) has a solution with \( \text{Re} w \geq 0 \) for some \( 1/2 < \theta \leq 1 \), then it follows from (A) for \( \theta = 1/2 \) that there exists \( 1/2 < \theta_0 \leq \theta \) such that \( Q(w) = 0 \) for \( \theta = \theta_0 \) has a solution with \( \text{Re} w = 0 \). But this is impossible by (3.19).

4. Stability regions in the case \( \delta \neq 0 \)

The same result as in Theorem 3.2 does not hold in the case \( \delta \neq 0 \). As a result, any \( \theta \)-method cannot possess a similar stability property to \( GP \)-stability with respect to DIDEs.

**Theorem 4.3** If \( 0 < \delta < 1 \), there exists \( (\alpha, \beta, \gamma) \in \Omega \) which does not belong to \( S_\theta^{(\delta)} \).

**Proof.** The function \( R(w) = r[(w+1)/(w-1)] \) can be written as
\[
R(w) = \frac{\tilde{P}(w)}{\tilde{Q}(w)},
\]
\[
\tilde{P}(w) = \left( \gamma w^2 + (-2\beta + 2\delta \gamma - \gamma)w + 2(1-2\delta)\beta - 2\delta(1-\delta)\gamma \right) \times \left( w - (1-2\theta) \right),
\]
\[
\tilde{Q}(w) = (w-1) \left\{ \gamma w^2 + \left[ 2\alpha - (1-2\theta)\gamma \right] w - 2(1-2\theta)\alpha - 4 \right\}.
\] (4.1) \hspace{1cm} (4.2) \hspace{1cm} (4.3)

When \( \alpha, \beta, \gamma \) are real, we have for \( w = iy, \ y \in \mathbb{R} \),
\[
| \tilde{Q}(w) |^2 - | \tilde{P}(w) |^2 = 4(y^2 + 1) \left[ (\alpha^2 - \beta^2 + 2\gamma)y^2 + \eta \right] + 4\delta(1-\delta)(2\beta - \delta \gamma) \left[ (2\beta + (1-\delta)\gamma) y^2 + (1-2\theta)^2 \right],
\]
\[
\eta = \left[ (1-2\theta)(\alpha + \beta) + 2 \right] \left[ (1-2\theta)(\alpha - \beta) + 2 \right].
\] (4.4) \hspace{1cm} (4.5)

When \( \alpha = -\sqrt{-2\gamma} \) and \( \beta = 0 \), (4.4) is a quadratic function of \( y^2 \) and the coefficient of \( y^2 \) is given by
\[
4 \left[ -(1-2\theta)\sqrt{-2\gamma} + 2 \right]^2 - 4\delta^2(1-\delta)^2\gamma^2.
\] (4.6)

If \( 0 < \delta < 1 \) and \( -\gamma \) is sufficiently large, the value (4.6) is negative. This implies that \( \text{(b)} \) does not hold near \( (\alpha, \beta) = (-\sqrt{-2\gamma}, 0) \), a point on the hyperbola \( \beta^2 = \alpha^2 + 2\gamma \), if \( -\gamma \) is sufficiently large. Therefore, by Theorem 2.1, there are points in \( \Omega \) which do not belong to \( S_\theta^{(\delta)} \). \( \square \)
In some cases, the region $S^{(\delta)}_{\theta} \cap \mathbb{R}^{3}$ is determined on the basis of Theorem 2.1. Let $1/2 \leq \theta \leq 1$, and assume that $\alpha < 0$ and $\gamma < 0$. Then, (a), which does not depend on $\delta$, is satisfied, and (C0) holds if $\beta^{2} \leq \alpha^{2} + 2\gamma$. Moreover, (b) is rewritten as

$$| \tilde{Q}(w) | > | \tilde{P}(w) | \quad \text{for any } \text{Re } w = 0.$$  \hfill (B)

In the case $\theta = 1/2$ (the trapezoidal rule), we have for $w = iy$, $y \in \mathbb{R}$,

$$| \tilde{Q}(w) |^{2} - | \tilde{P}(w) |^{2} = 4 (y^{2} + 1)(ay^{2} + 4) + by^{2},$$  \hspace{1cm} (4.7)

$$a = \alpha^{2} - \beta^{2} + 2\gamma,$$  \hspace{1cm} (4.8)

$$b = \delta(1 - \delta)(2\beta - \delta\gamma)[2\beta + (1 - \delta)\gamma].$$  \hspace{1cm} (4.9)

From (4.7) it is easy to verify that (B) holds if and only if $a \geq 0$ and

$$a + b + 4 \geq 0, \quad \text{or} \quad \left[ a + b + 4 < 0 \text{ and } 16a > (a + b + 4)^{2} \right].$$  \hspace{1cm} (4.10)

When $\delta = 1/2$, this condition is represented as

$$\begin{cases} 
\beta^{2} \leq \alpha^{2} + 2\gamma & \left( \alpha^{2} \geq \frac{\gamma^{2}}{16} - 2\gamma - 4 \right), \\
\beta^{2} < \frac{1}{16} \left[ 15(\alpha^{2} + 2\gamma) - \frac{\gamma^{2}}{16} - 4 \right] & \left( \alpha^{2} < \frac{\gamma^{2}}{16} - 2\gamma - 4 \right). 
\end{cases}$$  \hspace{1cm} (4.11)

In the case $\theta = 1$ (the backward Euler method), we have for $w = iy$, $y \in \mathbb{R}$,

$$| \tilde{Q}(w) |^{2} - | \tilde{P}(w) |^{2} = 4(y^{2} + 1)(ay^{2} + c),$$  \hspace{1cm} (4.12)

$$c = (2 - \alpha)^{2} - \beta^{2} + \delta(1 - \delta)(2\beta - \delta\gamma)[2\beta + (1 - \delta)\gamma].$$  \hspace{1cm} (4.13)
The condition $(\tilde{B})$ holds if and only if $a \geq 0$, $c > 0$, which is equivalent to

$$\beta^2 \leq \alpha^2 + 2\gamma, \quad \alpha < \frac{\gamma}{4} + 2,$$

(4.14)

when $\delta = 1/2$.

References


