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Kyoto University
Stability analysis of numerical methods for delay integro-differential equations

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Abstract

Stability of \( \theta \)-methods for delay integro-differential equations (DIDEs) is studied on the basis of the linear equation

\[
\frac{du}{dt} = \lambda u(t) + \mu u(t - \tau) + \kappa \int_{t-\tau}^{t} u(\sigma)d\sigma,
\]

where \( \lambda, \mu, \kappa \) are complex numbers and \( \tau \) is a constant delay. It is shown that every \( A \)-stable \( \theta \)-method possesses a similar stability property to \( P \)-stability, i.e., the method preserves the delay-independent stability of the exact solution under the condition that \( \tau/h \) is an integer, where \( h \) is a step-size. It is also shown that the method does not possess the same property if \( \tau/h \) is not an integer. As a result, any \( \theta \)-method cannot possess a similar stability property to \( GP \)-stability with respect to DIDEs.

1. Introduction

We study stability of (2-stage) \( \theta \)-methods for delay integro-differential equations (DIDEs) on the basis of the linear equation

\[
\frac{du}{dt} = \lambda u(t) + \mu u(t - \tau) + \kappa \int_{t-\tau}^{t} u(\sigma)d\sigma,
\]

where \( \lambda, \mu, \kappa \) are complex numbers and \( \tau \) is a constant delay. When \( \kappa = 0 \), the equation (1.1) coincides with the test equation

\[
\frac{du}{dt} = \lambda u(t) + \mu u(t - \tau),
\]

which was proposed by Barwell [1] to examine stability of numerical methods for delay differential equations (DDEs). As described in [1], if \( \lambda, \mu \) satisfy

\[
| \mu | < -\text{Re} \lambda,
\]

the zero solution of (1.2) is asymptotically stable for any \( \tau \geq 0 \). This asymptotic property is called delay-independent stability, and analogous stability properties of numerical methods are considered on the basis of the condition (1.3). For example,
A numerical method for DDEs is said to be $P$-stable if every numerical solution to (1.2) tends to zero whenever $\lambda$, $\mu$ satisfy (1.3) and $\tau/h$ is an integer, where $h$ is the step-size. A numerical method is said to be $GP$-stable if the same holds for any constant step-size.

In the last two decades, various studies were carried out concerning stability properties of numerical methods for DDEs (see, e.g., [12]). In particular, an earliest study by Watanabe and Roth [10] has revealed that every $A$-stable $\theta$-method is $GP$-stable. To the contrary, little is known about stability properties of numerical methods for DIDEs. It is quite recent that we studied delay-independent stability of linear DIDEs [7], and even stability of $\theta$-methods for (1.1) remains to be investigated.

By Theorem 2 of [7], the zero solution of (1.1) is asymptotically stable for any $\tau \geq 0$ if and only if $\lambda$, $\mu$, $\kappa$ satisfy

\begin{align}
\lambda + \mu + \kappa \tau &\neq 0 \quad \text{for any } \tau \geq 0, \\
z^2 - z\lambda - \kappa &= 0, \ z \in C', \ z \neq 0 \quad \Rightarrow \quad \text{Re} \ z < 0, \\
\left| \frac{\mu z - \kappa}{z^2 - z\lambda - \kappa} \right| &< 1 \quad \text{for any } \text{Re} \ z = 0 \text{ with } z \neq 0.
\end{align}

Moreover, the conditions (1.5), (1.6) are rewritten as

\begin{align}
\text{Re} \lambda < 0 \quad \text{and} \quad \left( \text{Re} \lambda \text{Re}(\lambda\kappa) + (\text{Im} \kappa)^2 < 0 \text{ or } \kappa = 0 \right), \\
\text{Im}[(\lambda + \mu)\kappa] = 0 \quad \text{and} \quad \left[ |\mu|^2 < (\text{Re} \lambda)^2 + 2 \text{Re} \kappa \\
\quad \text{or} \quad \text{Im} \lambda = 0, \ |\mu|^2 = (\text{Re} \lambda)^2 + 2 \text{Re} \kappa \right],
\end{align}

respectively (Sect. 3 in [7]). When $\lambda$, $\mu$, $\kappa$ are all real and $\kappa \neq 0$, these conditions are reduced to the simple condition

\begin{align}
\lambda < 0, \quad \kappa < 0, \quad \mu^2 \leq \lambda^2 + 2\kappa.
\end{align}

We study stability properties of $\theta$-methods by comparing the region determined by these conditions with a kind of stability regions of the methods.

**Fig. 1** Delay-independent v.s. delay-dependent stability regions
It should be noted that a considerable number of papers [2, 3, 4, 6, 9] are devoted to stability analysis of \( \theta \)-methods for DDEs, which does not seem strange from a practical viewpoint. Some important instances of stiff DDEs are obtained from the space-descritization of partial functional differential equations (see, e.g., [13]). The \( \theta \)-methods have practicality in such a situation.

2. Stability regions of \( \theta \)-methods

Consider delay integro-differential equations (DIDEs) with a constant delay,

\[
\frac{du}{dt} = f\left(t, u(t), u(t - \tau), \int_{t-\tau}^{t} g(t, \sigma, u(\sigma))d\sigma\right).
\]

(2.1)

For a given step-size \( h > 0 \), let \( m \) be the smallest integer greater than or equal to \( \tau/h \). Then, the delay \( \tau \) is represented in the form

\[
\tau = (m - \delta)h, \quad 0 \leq \delta < 1,
\]

(2.2)

and the relation

\[
t_{n} - \tau = t_{n-m} + \delta h
\]

(2.3)

holds for the step points \( t_{n} = t_{0} + nh, \ n \in \mathbb{Z} \).

By approximating the delayed argument and the integrand in (2.1) with linear interpolation, we can adapt a \( \theta \)-method to (2.1) as follows:

\[
u_{n+1} = u_{n} + h(1 - \theta)f(t_{n}, u_{n}, v_{n}, G_{n}) + h\theta f(t_{n+1}, u_{n+1}, v_{n+1}, G_{n+1}),
\]

(2.4)

where, \( 0 \leq \theta \leq 1 \), \( u_{n} \) is an approximate value of \( u(t_{n}) \), and

\[
v_{n} = (1-\delta)u_{n-m} + \delta u_{n-m+1},
\]

(2.5)

\[
G_{n} = \frac{h(1-\delta)^{2}}{2}g(t_{n}, t_{n-m}, u_{n-m}) + \frac{h(2-\delta^{2})}{2}g(t_{n}, t_{n-m+1}, u_{n-m+1}) +
\sum_{k=2}^{m-1}g(t_{n}, t_{n-m+k}, u_{n-m+k}) + \frac{1+\theta}{2}u_{n} + \frac{\theta}{2}u_{n+1},
\]

(2.6)

As a result, the integral term of (2.1) is approximated with the trapezoidal rule. When \( \theta = 1/2 \) and \( \delta = 0 \), the formula (2.4)-(2.6) determines a method that belongs to a class of Runge-Kutta methods discussed in [7]. But, when \( \theta \neq 1/2 \), it gives another type of numerical method.

In the case of the test equation (1.1), the formula (2.4)-(2.6) is reduced to

\[
u_{n+1} = u_{n} + (1 - \theta)\alpha u_{n} + \theta\alpha u_{n+1}
\]

\[
+\beta\left[(1-\delta)(1-\theta)u_{n-m} + (\delta + \theta - 2\delta\theta)u_{n-m+1} + \delta\theta u_{n-m+2}\right]
\]

\[
+\gamma\left[\frac{(1-\delta)^{2}(1-\theta)}{2}u_{n-m} + \frac{(2-\delta^{2})(1-\theta) + (1-\delta)^{2}\theta}{2}u_{n-m+1}
\right.
\]

\[
\left. + \frac{2-\delta^{2}\theta}{2}u_{n-m+2} + \sum_{k=3}^{m-1}u_{n-m+k} + \frac{1+\theta}{2}u_{n} + \frac{\theta}{2}u_{n+1}\right].
\]

(2.7)
\[ \alpha = h \lambda, \quad \beta = h \mu, \quad \gamma = h^2 \kappa. \quad (2.8) \]

The characteristic equation of (2.7) is written as
\[
z^{m+1} - z^m - (1 - \theta)\alpha z^m - \theta \alpha z^{m+1} - \beta \left[ (1 - \delta)(1 - \theta) + (\delta + \theta - 2\delta \theta)z + \delta \theta z^2 \right] - \gamma \left[ \frac{(1 - \delta)^2(1 - \theta)}{2} + \frac{(2 - \delta^2)(1 - \theta) + (1 - \delta)^2 \theta}{2} z + \frac{2 - \delta^2 \theta}{2} z^2 + \sum_{k=3}^{m-1} z^k + \frac{1 + \theta}{2} z^m + \frac{\theta}{2} z^{m+1} \right] = 0. \quad (2.9)\]

Using (2.9) we define the sets \( S_{\theta,m}^{(\delta)} \) and \( S_{\theta}^{(\delta)} \) for \( 0 \leq \delta < 1 \) by
\[
S_{\theta,m}^{(\delta)} = \{ (\alpha, \beta, \gamma) \in C^3 : \text{all the roots of (2.9) satisfy } |z| < 1 \}, \quad (2.10)
\]
\[
S_{\theta}^{(\delta)} = \cap_{m \geq 1} S_{\theta,m}^{(\delta)}. \quad (2.11)
\]

The set \( S_{\theta}^{(\delta)} \) is an analogue of the \( \delta \)-stability region of the \( \theta \)-method [4].

When \( z = 1 \), the left hand side of (2.9) is equal to \(-[\alpha + \beta + (m - \delta)\gamma]\). Hence, for any \( m \geq 1 \), \( z = 1 \) is not a root of (2.9) if and only if
\[
(C_0) \quad \alpha + \beta + (m - \delta)\gamma \neq 0 \text{ for any } m \geq 1.
\]

Substituting \( \sum_{k=3}^{m-1} z^k = (z^3 - z^m)/(1 - z) \) into (2.9) and multiplying \( (1 - z) \), we get
\[
z^m q(z) - p(z) = 0, \quad (2.12)
\]
\[
q(z) = q_0 z^2 + q_1 z + q_2, \quad (2.13)
\]
\[
p(z) = p_0 z^3 + p_1 z^2 + p_2 z + p_3, \quad (2.14)
\]

where
\[
q_0 = \theta \alpha + \frac{\theta}{2} \gamma - 1, \quad q_1 = (1 - 2\theta)\alpha + \frac{\gamma}{2} + 2,
\]
\[
q_2 = -(1 - \theta)\alpha + \frac{1 - \theta}{2} \gamma - 1,
\]
\[
p_0 = -\delta \theta \beta + \frac{\delta^2 \theta}{2} \gamma, \quad p_1 = (3\delta \theta - \delta - \theta)\beta + \frac{-3\delta^2 \theta + \delta^2 + 2\delta \theta + \theta}{2} \gamma,
\]
\[
p_2 = (-3\delta \theta + 2\delta + 2\theta - 1)\beta + \frac{3\delta^2 \theta - 2\delta^2 - 4\delta \theta - 2\delta^2 + 2\delta + 1}{2} \gamma,
\]
\[
p_3 = (\delta \theta - \delta - \theta + 1)\beta + \frac{-\delta^2 \theta + \delta^2 + 2\delta \theta - 2\delta - \theta + 1}{2} \gamma.
\]

Moreover, we set
\[
r(z) = p(z)/q(z), \quad (2.15)
\]
and consider the following conditions.
(a) \( q(z) \neq 0 \) for any \( |z| \geq 1 \).

(a) \( q(z) \neq 0 \) for any \( |z| > 1 \).

(b) \(|r(z)| < 1\) for any \(|z| = 1\) with \(z \neq 1\).

(b) \(|r(z)| \leq 1\) for any \(|z| = 1\).

These are regarded as conditions for \(\alpha, \beta, \gamma\). We also write

(c) \((\alpha, \beta, \gamma) \in S_{\theta}^{(\delta)}\).

Under this notation, we can characterize \(S_{\theta}^{(\delta)}\) as follows.

**Theorem 2.1** The following implications hold:

\[(\mathrm{C}_0)\) and (a) and (b) \(\Rightarrow\) (c) \(\Rightarrow\) (a) and (b).

If, in addition,

\[(\mathrm{C}_1)\) \(p(z), q(z)\) have no common zero on \(|z| = 1\),

then (c) implies (a).

**Proof.** Assume \((\mathrm{C}_0)\), (a) and (b). We first show that \(\hat{r}(z) = r(z)/z\) satisfies
\[|\hat{r}(z)| < 1\) for any \(|z| \geq 1\) with \(z \neq 1\).

The linear fractional transformation
\[z = \frac{w + 1}{w - 1}\] (2.16)

maps \(\text{Re} w > 0\) conformally onto \(|z| > 1\), with \(w = \infty\) corresponding to \(z = 1\). The function \(\hat{R}(w) = \hat{r}[(w + 1)/(w - 1)]\) is represented in the form

\[
P(w) = \hat{P}(w)/\hat{Q}(w),
\]

\[\hat{P}(w) = \left[\gamma w^2 + (-2\beta + 2\delta\gamma - \gamma)w + 2(1 - 2\delta)\beta - 2\delta(1 - \delta)\gamma\right] \times \left[w - (1 - 2\theta)\right],\] (2.18)

\[\hat{Q}(w) = (w + 1)\left\{\gamma w^2 + \left[2\alpha - (1 - 2\theta)\gamma\right]w - 2(1 - 2\theta)\alpha - 4\right\}.\] (2.19)

Then, it follows from (a) that \(\hat{R}(w)\) is a bounded, holomorphic function in \(\text{Re} w > 0\). Hence, by the Phragmén-Lindelöf theorem (see, e.g., [8], p. 168), it follows from (b) that \(|\hat{R}(w)| < 1\) for any \(\text{Re} w > 0\), which implies that \(|\hat{r}(z)| < 1\) for any \(|z| \geq 1\) with \(z \neq 1\).

If \(|z| \geq 1\) and \(z \neq 1\), then

\[z^m q(z) - p(z) = q(z)z \left[z^{m-1} - \hat{r}(z)\right] \neq 0,\]
which, together with \((\text{C}_0)\), implies \((c)\).

Assume \((c)\). If \(q(z_0) = 0\) for some \(|z_0| > 1\), then there exists \(\varepsilon > 0\) such that \(C(z_0, \varepsilon) \subset \{|z| > 1\}\) and \(q(z) \neq 0\) on \(C(z_0, \varepsilon)\), where

\[
C(z_0, \varepsilon) = \{z \in \mathcal{C} : |z - z_0| = \varepsilon\}.
\]

By Rouché's theorem, the polynomial \(z^m q(z) - p(z)\) has a root in the interior of \(C(z_0, \varepsilon)\) for sufficiently large \(m\), which contradicts \((c)\). Therefore, \((\hat{a})\) holds.

Moreover, if \(|r(z_0)| > 1\) for some \(|z_0| = 1\), then there exists \(\varepsilon > 0\) such that \(C(z_0, \varepsilon) \subset \{|z| > 1\}\) and \(q(z) \neq 0\) on \(C(z_0, \varepsilon)\), where \(\varepsilon > 0\) such that \(|r(z)| > 1\) for any \(z \in \overline{V}_\varepsilon\), where \(V_\varepsilon = \{z \in \mathcal{O} : |z - (1 + \varepsilon)z_0| < \varepsilon\}\). Hence,

\[
\rho = \max_{z \in \overline{V}_\varepsilon} |\psi(z)| < 1,
\]

and \(1 \in \mathcal{C} \setminus B(0, \rho)\), where \(B(0, \rho) = \{z \in \mathcal{C} : |z| \leq \rho\}\). On the other hand, we have

\[
\mathcal{C} \setminus B(0, \rho) \subset \bigcup_{m \geq 1} \left\{z^m \psi(z) : z \in V_\varepsilon\right\},
\]

by Proposition 7 of [11]. Since \(|z| > 1\) for any \(z \in V_\varepsilon\), it follows from (2.20) that \(z^m = r(z)\) holds for some \(m \geq 1\) and \(|z| > 1\), which contradicts \((c)\). Therefore, \((\hat{b})\) holds.

It is easy to see that \((\hat{a})\) and \((\hat{b})\) imply \((a)\) under the condition \((\text{C}_1)\). \(\square\)

3. Stability regions in the case \(\delta = 0\)

We consider the case \(\delta = 0\). Since \(q(1) = \gamma\), \(z = 1\) satisfies \(q(z) = 0\) if and only if \(\gamma = 0\). We assume that \(\gamma \neq 0\) for a while, and rewrite the conditions \((a)\), \((\hat{a})\), \((b)\), \((\hat{b})\) by making use of the linear fractional transformation (2.16).

The function \(R(w) = r[(w + 1)/(w - 1)]\) is represented in the form

\[
R(w) = P(w)/Q(w), \quad P(w) = (\gamma w - 2\beta \left[w - (1 - 2\theta)\right], \quad Q(w) = \gamma w^2 + [2\alpha - (1 - 2\theta)\gamma] w - 2(1 - 2\theta)\alpha - 4.
\]

Hence, \((a)\), \((\hat{a})\), \((b)\), \((\hat{b})\) are equivalent to

\[
\begin{align*}
(A) & \quad Q(w) \neq 0 \text{ for any } \text{Re} w \geq 0, \\
(A) & \quad Q(w) \neq 0 \text{ for any } \text{Re} w > 0, \\
(B) & \quad |R(w)| < 1 \text{ for any } \text{Re} w = 0, \\
(\hat{B}) & \quad |R(w)| \leq 1 \text{ for any } \text{Re} w = 0,
\end{align*}
\]
When $\alpha, \gamma$ are real, (A), (A) are reduced to

\begin{align}
\gamma\left[2\alpha - (1-2\theta)\gamma\right] > 0, & \quad \gamma\left[-4 - 2(1-2\theta)\alpha\right] > 0, \quad (3.4) \\
\gamma\left[2\alpha - (1-2\theta)\gamma\right] \geq 0, & \quad \gamma\left[-4 - 2(1-2\theta)\alpha\right] \geq 0, \quad (3.5)
\end{align}

respectively. In addition, putting $w = i y, \ y \in \mathbb{R}$, we have

\begin{align}
|Q(w)|^2 - |P(w)|^2 &= 4 \text{Im}\left[(\alpha + \beta)\overline{\gamma}\right] y^3 + 4\left(|\alpha|^2 - |\beta|^2 + 2\text{Re} \gamma\right)y^2 \\
&+ \left\{16 \text{Im} \alpha + 4(1-2\theta)^2 \text{Im}\left[(\alpha + \beta)\overline{\gamma}\right]\right\}y \\
&+ 4 + 2(1-2\theta)\alpha \geq 0, \quad (3.6)
\end{align}

When $\alpha, \beta, \gamma$ are real, it is reduced to

\begin{align}
|Q(w)|^2 - |P(w)|^2 &= 4(\alpha^2 - \beta^2 + 2\gamma)y^2 + 4\eta, \quad (3.7) \\
\eta &= \left[(1 - 2\theta)(\alpha + \beta) + 2\right] \left[(1 - 2\theta)(\alpha - \beta) + 2\right]. \quad (3.8)
\end{align}

Hence, in this case, (B), (B) are equivalent to

\begin{align}
\beta^2 &\leq \alpha^2 + 2\gamma, \quad \eta > 0, \quad (3.9) \\
\beta^2 &\leq \alpha^2 + 2\gamma, \quad \eta \geq 0, \quad (3.10)
\end{align}

respectively.

\textbf{Fig. 2} $\gamma$-section of $S^{(0)}_{\theta} \cap \mathbb{R}^3$ ($0 \leq \theta < 1/2$)
Let $\alpha < 0$ and $\gamma < 0$. The conditions (3.4), (3.9) are reduced to
\[ \alpha > -\frac{2}{1-2\theta}, \quad \gamma > \frac{2\alpha}{1-2\theta}, \quad \beta^2 \leq \alpha^2 + 2\gamma, \quad |\beta| < \alpha + \frac{2}{1-2\theta}, \] (3.11)
when $0 \leq \theta < 1/2$ (Fig. 2), and
\[ \beta^2 \leq \alpha^2 + 2\gamma, \] (3.12)
when $1/2 \leq \theta \leq 1$. If $\alpha(0), \beta$ satisfy $\beta^2 \leq \alpha^2 + 2\gamma$ for $\gamma < 0$, then $\alpha + \beta < 0$, and (C$_0$) holds. Hence, by Theorem 2.1, these determine the region
\[ S_\theta^{(0)} \cap \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha < 0, \gamma < 0\}, \] (3.13)
except for ambiguity of the boundary.

We now denote by $\Omega$ the set of all the triplicate $(\lambda, \mu, \kappa)$ for which the zero solution of (1.1) is asymptotically stable for any $\tau \geq 0$, i.e.,
\[ \Omega = \{(\lambda, \mu, \kappa) \in \mathfrak{D}^3 : \text{(1.4), (1.5), (1.6 are satisfied)}\}. \] (3.14)
It is easy to see that
\[ (\lambda, \mu, \kappa) \in \Omega \implies (h\lambda, h\mu, h^2\kappa) \in \Omega \text{ for any } h > 0. \] (3.15)
The following theorem shows that $A$-stable $\theta$-methods possess a similar stability property to $P$-stability with respect to DIDEs.

**Theorem 3.2** If $1/2 \leq \theta \leq 1$, then $\Omega \subset S_\theta^{(0)}$.

**Proof.** The inclusion $\Omega \cap \{\gamma = 0\} \subset S_\theta^{(0)}$ follows from the known result as in the case of DDEs (see, e.g., Theorem 2.6 in [6]). We consider the case $\gamma \neq 0$.

Let $(\alpha, \beta, \gamma) \in \Omega$. The condition (C$_0$) follows from (1.4). Moreover, it follows from (3.6) and $\text{Im}[(\alpha + \beta)\bar{\gamma}] = 0$ that for $w = iy, y \in \mathbb{R},$
\[ |Q(w)|^2 - |P(w)|^2 = \eta_0 y^2 + 2\eta_1 y + \eta_2, \] (3.16)
\[ \eta_0 = 4\left( |\alpha|^2 - |\beta|^2 + 2 \text{Re}\gamma \right), \quad \eta_1 = 8 \text{Im} \alpha, \]
\[ \eta_2 = |2(1 - 2\theta)\alpha + 4|^2 - |2(1 - 2\theta)\beta|^2. \]
Since
\[ \eta_2 = 16 + 16(1 - 2\theta) \text{Re} \alpha + 4(1 - 2\theta)^2 \left( |\alpha|^2 - |\beta|^2 \right) \geq 16, \] (3.17)
\[ \eta_0^2 - \eta_0 \eta_2 \leq 64(\text{Im} \alpha)^2 - 64 \left( |\alpha|^2 - |\beta|^2 + 2 \text{Re}\gamma \right) \]
\[ = -64 \left( \text{Re} \alpha)^2 + 2 \text{Re}\gamma - |\beta|^2 \right), \] (3.18)
we have
\[ |Q(w)| > |P(w)| \text{ for any } \text{Re} w = 0, \] (3.19)
which implies (B).

When $\theta = 1/2$, it holds that

$$Q(w) = \gamma w^2 + 2\alpha w - 4 = -w^2 \left[ (2/w)^2 - \alpha(2/w) - \gamma \right].$$

(3.20)

Hence, (A) for $\theta = 1/2$ follows from (1.5).

The condition (A) for $\theta = 1/2$, together with (3.19), implies (A) for $1/2 < \theta \leq 1$. In fact, if $Q(w) = 0$ has a solution with $\text{Re} \ w \geq 0$ for some $1/2 < \theta \leq 1$, then it follows from (A) for $\theta = 1/2$ that there exists $1/2 < \theta_0 \leq \theta$ such that $Q(w) = 0$ for $\theta = \theta_0$ has a solution with $\text{Re} \ w = 0$. But this is impossible by (3.19). \qed

4. Stability regions in the case $\delta \neq 0$

The same result as in Theorem 3.2 does not hold in the case $\delta \neq 0$. As a result, any $\theta$-method cannot possess a similar stability property with respect to DIDEs.

Theorem 4.3 If $0 < \delta < 1$, there exists $(\alpha, \beta, \gamma) \in \Omega$ which does not belong to $S_\theta^{(\delta)}$.

Proof. The function $R(w) = r[(w + 1)/(w - 1)]$ can be written as

$$R(w) = \frac{\tilde{P}(w)}{\tilde{Q}(w)},$$

(4.1)

$$\tilde{P}(w) = \left[ \gamma w^2 + (-2\beta + 2\delta\gamma - \gamma)w + 2(1 - 2\delta)\beta - 2\delta(1 - \delta)\gamma \right] \times \left[ w - (1 - 2\theta) \right],$$

(4.2)

$$\tilde{Q}(w) = (w - 1) \left\{ \gamma w^2 + [2\alpha - (1 - 2\theta)\gamma]w - 2(1 - 2\theta)\alpha - 4 \right\}.$$  

(4.3)

When $\alpha, \beta, \gamma$ are real, we have for $w = iy, \ y \in \mathbb{R},$

$$|\tilde{Q}(w)|^2 - |\tilde{P}(w)|^2 = 4(y^2 + 1) \left[ (\alpha^2 - \beta^2 + 2\gamma)y^2 + \eta \right] + 4\delta(1 - \delta)(2\beta - \delta\gamma)[2\beta + (1 - \delta)\gamma] y^2 + (1 - 2\theta)^2 \right],$$

(4.4)

$$\eta = \left[ (1 - 2\theta)(\alpha + \beta) + 2 \right] \left[ (1 - 2\theta)(\alpha - \beta) + 2 \right].$$

(4.5)

When $\alpha = -\sqrt{-2\gamma}$ and $\beta = 0$, (4.4) is a quadratic function of $y$ and the coefficient of $y^2$ is given by

$$4 \left[ -(1 - 2\theta)\sqrt{-2\gamma} + 2 \right]^2 - 4\delta^2(1 - \delta)^2 \gamma ^2.$$  

(4.6)

If $0 < \delta < 1$ and $-\gamma$ is sufficiently large, the value (4.6) is negative. This implies that (B) does not hold near $(\alpha, \beta) = (-\sqrt{-2\gamma}, 0)$, a point on the hyperbola $\beta^2 = \alpha^2 + 2\gamma$, if $-\gamma$ is sufficiently large. Therefore, by Theorem 2.1, there are points in $\Omega$ which do not belong to $S_\theta^{(\delta)}$. \qed
In some cases, the region $S_{\theta}^{(\delta)} \cap \mathbb{R}^3$ is determined on the basis of Theorem 2.1. Let $1/2 \leq \theta \leq 1$, and assume that $\alpha < 0$ and $\gamma < 0$. Then, (a), which does not depend on $\delta$, is satisfied, and $(C_0)$ holds if $\beta^2 \leq \alpha^2 + 2\gamma$. Moreover, (b) is rewritten as

\[
|\tilde{Q}(w)| > |\tilde{P}(w)| \quad \text{for any } \text{Re} w = 0.
\]

Fig. 3 Examples of $\gamma$-sections of $S_{\theta}^{(\delta)} \cap \mathbb{R}^3$ ($\delta = 1/2$)

In the case $\theta = 1/2$ (the trapezoidal rule), we have for $w = iy$, $y \in \mathbb{R}$,

\[
|\tilde{Q}(w)|^2 - |\tilde{P}(w)|^2 = 4\left[(y^2 + 1)(ay^2 + 4) + by^2\right], \tag{4.7}
\]
\[
a = \alpha^2 - \beta^2 + 2\gamma, \tag{4.8}
\]
\[
b = \delta(1 - \delta)(2\beta - \delta\gamma)\left[2\beta + (1 - \delta)\gamma\right]. \tag{4.9}
\]

From (4.7) it is easy to verify that (B) holds if and only if $a \geq 0$ and

$\alpha + b + 4 \geq 0$, \; or \; $\left[a + b + 4 < 0 \text{ and } 16a > (a + b + 4)^2\right]. \tag{4.10}$

When $\delta = 1/2$, this condition is represented as

\[
\begin{cases}
\beta^2 \leq \alpha^2 + 2\gamma \quad \left(\alpha^2 \geq \frac{\gamma^2}{16} - 2\gamma - 4\right), \\
\beta^2 < \frac{1}{16} \left[15(\alpha^2 + 2\gamma) - \frac{\gamma^2}{16} - 4\right] \quad \left(\alpha^2 < \frac{\gamma^2}{16} - 2\gamma - 4\right).
\end{cases} \tag{4.11}
\]

In the case $\theta = 1$ (the backward Euler method), we have for $w = iy$, $y \in \mathbb{R}$,

\[
|\tilde{Q}(w)|^2 - |\tilde{P}(w)|^2 = 4(y^2 + 1)(ay^2 + c), \tag{4.12}
\]
\[
c = (2 - \alpha)^2 - \beta^2 + \delta(1 - \delta)(2\beta - \delta\gamma)\left[2\beta + (1 - \delta)\gamma\right]. \tag{4.13}
\]
The condition \((\tilde{B})\) holds if and only if \(a \geq 0, \ c > 0\), which is equivalent to
\[
\beta^2 \leq \alpha^2 + 2\gamma, \quad \alpha < \frac{\gamma}{4} + 2,
\]
when \(\delta = 1/2\).

References


