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Schematization of homotopy types and realizations

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1 Introduction

Using ground breaking results of Simpson in a joint work with T. Pantev and B.Toen [KTP] we have found new homotopy invariants of a topological space X, related to the action of $\pi_1(X,x)$ on the higher homotopy groups $\pi_i(X)\otimes\mathbb{C}$, $i>1$. We have explained the construction of a Hodge decomposition on the schematized homotopy type $(X\otimes\mathbb{C})^{\mathrm{sch}}$ of a smooth projective variety X. We construct some new examples of homotopy types which are not realizable as complex projective manifolds. This is an account of the work in [KTP] given at the UCI-RIMS conference in RIMS Kyoto. We also describe future directions of research.

2 The construction of the Hodge filtration

The construction of the Hodge filtration in rational homotopy theory, utilizes far going ideas of Beilinson and Deligne and is based on fact that the cochain algebra $C^{\bullet}_{dR}(X,\mathbb{C})$ of a simply connected projective X can be used as an algebraic model of the complex homotopy type of X. In particular, the Hodge decomposition on the level of differential forms yields a

Hodge decomposition on the commutative dga $C^\bullet(X, \mathbb{C})$. The main technical tool is the Schematization idea of Toen [Toen].

Before we can apply the same reasoning to the general setting we need to resolve the following

Problem Find an algebraic model for $(X \otimes \mathbb{C})^{\mathrm{sch}}$.

The construction is intimately related to the way one does algebraic geometry in the category of stacks $\mathrm{Ho}(\mathrm{SPr}(\mathbb{C}))$.

We need the notion of an *affine stack* which should be thought of as a derived version of the notion of an affine scheme.

Given a cdga A one defines a simplicial presheaf $\mathrm{Spec}(A) \in \mathrm{SPr}(\mathbb{C})$ by setting

$$\begin{aligned} \mathrm{Spec}(A) \; : \qquad & (\mathrm{Aff}\,/\mathbb{C}) \longrightarrow (\mathrm{SSet}) \\ & \mathrm{Spec}(B) \longrightarrow \mathrm{Hom}_{\mathrm{dga}}(A, B) \end{aligned}$$

Furthermore, for any cdga A one defines a stack $\mathbb{R}\,\mathrm{Spec}(A) \in \mathrm{Ho}(\mathrm{SPr}(\mathbb{C}))$ by setting

$$\mathbb{R}\,\mathrm{Spec}(A) := \mathrm{Spec}(\acute{A}),$$

where $\acute{A}$ is a cofibrant replacement of A in the cmc of cdga.

Definition A stack $F \in \mathrm{Ho}(\mathrm{SPr}(\mathbb{C}))$ is called *affine* if there exists a commutative dga A concentrated in non-negative degrees so that

$$F \cong \mathbb{R}\,\mathrm{Spec}(A)$$

General fact: Any simply connected schematic homotopy type is an affine stack.

Comments: • A cdga A is *cofibrant* if it is built by a transfinite sequence of pushouts of free dga.

• A stack F is affine iff the dga $\mathbb{L}\mathcal{O}(F)$ of cochains of F with coefficients in $\mathbb{G}_a$ is 'small' and the natural map $F \to \mathbb{R}\,\mathrm{Spec}(\mathbb{L}\mathcal{O}(F))$ is an equivalence of simplicial presheaves.

Using the general fact above we get the desired algebraic model of $(X \otimes \mathbb{C})^{\mathrm{sch}}$:

Consider the natural map

$$\begin{array}{ccc} (X \otimes \mathbb{C})^{\mathrm{sch}} & \xrightarrow{p} & K(\pi_1((X \otimes \mathbb{C})^{\mathrm{sch}}), 1) \\ & & \| \\ & & K(\pi_1(X)^{\mathrm{alg}}, 1), \end{array}$$

and let $\widetilde{X \otimes \mathbb{C}}$ be the homotopy fiber of p. Then $\widetilde{X \otimes \mathbb{C}}$ is a simply connected schematic homotopy type and so one can find a cdga A, so that $\widetilde{X \otimes \mathbb{C}} \cong \mathbb{R}\,\mathrm{Spec}(A)$.

Conclusion: $\pi_1(X)^{\mathrm{alg}}$ acts on $\mathbb{R}\,\mathrm{Spec}(A)$ and $(X \otimes \mathbb{C})^{\mathrm{sch}} \cong [\mathbb{R}\,\mathrm{Spec}(A)/\pi_1(X)^{\mathrm{alg}}]$.

To make this algebraic model explicit we need to calculate A.

The answer is given by

Theorem (Katzarkov, Pantev, Toen)) *Let X be a pointed connected homotopy type and let G be the pro-reductive completion of $\pi_1(X)$ over $\mathbb{C}$. View the algebra of functions*

$\mathcal{O}(G)$ together with its $\pi_1(X)$ action as a local system of algebras on X and let $C^\bullet(X, \mathcal{O}(G))$ be the algebra of cochains on X with coefficients in $\mathcal{O}(G)$. Then

$$(X \otimes \mathbb{C})^{\mathrm{sch}} \cong [\mathbb{R}\,\mathrm{Spec}(C^\bullet(X, \mathcal{O}(G)))/G].$$

Remark: The theorem remains true if we take G to be the pro-algebraic completion of $\pi_1(X)$, rather than the pro-reductive one.

A great advantage of the model

$$[\mathbb{R}\,\mathrm{Spec}(C^\bullet(X, \mathcal{O}(G)))/G]$$

is that it is related to the geometry of X.

If X is a smooth projective variety $C^\bullet(X, \mathcal{O}(G))$ can be computed from the de Rham complex of the local system $\mathcal{O}(G)$.

Using Simpson's non-abelian Hodge correspondence one can then relate this dga to the Čech cochain algebra computing the cohomology of certain Higgs bundles on X.

Combined with the rescaling action of $\mathbb{C}^\times$ on Higgs bundles this yields the Hodge decomposition on the schematic homotopy type, i.e. provides an action of the group $\mathbb{C}^{\times\delta}$ on the stack $(X \otimes \mathbb{C})^{\mathrm{sch}}$.

This result is summarized in the following:

Theorem (Katzarkov, Pantev ,Toen)

Let X be a pointed projective manifold. There exists an action of $\mathbb{C}^{\times\delta}$ on $(X \otimes \mathbb{C})^{\mathrm{sch}}$ so that:

- *The induced action of $\mathbb{C}^{\times\delta}$ on the cohomology groups $H^\bullet((X \otimes \mathbb{C})^{\mathrm{sch}}, \mathbb{G}_a) = H^\bullet(X, \mathbb{C})$ is compatible with the Hodge decomposition.*

- *The induced action of $\mathbb{C}^{\times\delta}$ on $\pi_1(X)^{\mathrm{red}}$ coincides with the one defined by Simpson.*

- *If X is simply connected, then the induced action of $\mathbb{C}^{\times\delta}$ on*

$$\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}})(\mathbb{C}) \cong \pi_i(X) \otimes \mathbb{C}$$

  coincides with the Hodge decomposition defined by Deligne-Griffiths-Morgan-Sullivan.

- *If $R_n := \mathrm{Hom}(\pi_1(X), GL_n(\mathbb{C}))/GL_n(\mathbb{C})$, then the induced action of $\mathbb{C}^\times$ on R_n is continuous in the analytic topology.*

For any pointed connected homotopy type X we can use the schematization $(X \otimes \mathbb{C})^{\mathrm{sch}}$ to construct a new homotopy invariant of X which is related in a subtle way to the action of $\pi_1(X)$ on the higher homotopy groups of X.

Let F be a pointed schematic homotopy type. By definition $\pi_1(F, *)$ is an affine group scheme and one can show that $\pi_i(F, *)$ are all abelian unipotent group schemes for $i > 0$.

Let $\pi_1(F,*)^{\mathrm{red}}$ be the maximal reductive quotient of $\pi_1(F,*)$ considered as a subgroup of $\pi_1(F,*)$ via the Levi decomposition.

Since $\pi_i(F,*)$ is a linearly compact vector space and $\pi_1(F,*)^{\mathrm{red}}$ is an affine reductive group scheme acting on it we get a decomposition as a (possibly infinite) product

$$\pi_i(F,*) = \prod_{\rho \in R} \pi_i(F,*)^\rho.$$

Comments:

• R denotes the set of isomorphism classes of finite dimensional simple linear representations of $\pi_1(F,*)$ and $\pi_i(F,*)^\rho$ is a (possibly infinite) product of representations of class ρ.

• Using the fact that the Levi decomposition is unique up to an inner automorphism one can check that the set

$$\{\rho \in R | \pi_i(F,*)^\rho \neq 0\}$$

is well defined and independent of the choice of the embedding $\pi_1(F)^{\mathrm{red}} \subset \pi_1(F)$.

Definition Let F be a pointed schematic homotopy type. The subset

$$\mathrm{Supp}(\pi_i(F,*)) := \{\rho \in R | \pi_i(F,*)^\rho \neq 0\} \subset R$$

is called the support of $\pi_i(F,*)$ for every $i > 1$.

The naturality of the construction of the Hodge decomposition on $(X \otimes \mathbb{C})^{\mathrm{sch}}$ now gives the following

Lemma *For any pointed projective manifold (X,x) and any $i > 1$ the subset*

$$\mathrm{Supp}((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$$

is invariant under the $\mathbb{C}^\times$ action on R.

Note: In this geometric case the identification of

$$R(\pi_1((X \otimes \mathbb{C})^{\mathrm{sch}}, x)) \cong R(\pi_1(X,x)),$$

allows us to view $R(\pi_1((X \otimes \mathbb{C})^{\mathrm{sch}}, x))$ as a geometric object as well. Explicitly if $n > 0$ the component R_n of $R(\pi_1(X,x)$ consisting of simple representations of rank n is in a natural way an algebraic variety. Indeed, we can identify R_n with the geometric quotient $\mathrm{Hom}(\pi_1(X), GL_n(\mathbb{C}))^s / GL_n(\mathbb{C})$.

As a consequence we get the following

Theorem (Katzarkov, Pantev, Toen)

Let (X,x) be a pointed projective manifold, then:

- *If $\rho \in \mathrm{Supp}((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$ is an isolated point (for the natural topology on $R(\pi_1(X,x))$), then the local system on X corresponding to ρ underlies a polarizable complex variation of Hodge structures.*

- *If $\pi_i((X \otimes \mathbb{C})^{\text{sch}}, x)$ is an affine group scheme of finite type, then each simple factor of the semi-simplification of the $\pi_1(X, x)$-module $\pi_i((X \otimes \mathbb{C})^{\text{sch}}, x)$ underlies a polarizable* $\mathbb{C}$ VHS.

- *Suppose that $\pi_1(X, x)$ is abelian. Then each isolated character*

$$\chi \in \text{Supp}((X \otimes \mathbb{C})^{\text{sch}}, x)$$

  must be unitary.

Warning: The support invariants

$$\text{Supp}((X \otimes \mathbb{C})^{\text{sch}}, x)$$

are related to the action of $\pi_1(X, x)$ on $\pi_i(X, x) \otimes \mathbb{C}$ in a highly non-trivial way, which at the moment can be understood only in very special cases.

Nevertheless, the previous theorem can be used to produce explicit new examples of homotopy types which are not realizable by smooth projective varieties.

In order to construct these examples we will need to compute the support invariants explicitly, at least in some cases.

Definition A finitely generated group Γ is called *algebraically good* (relative to $\mathbb{C}$) if the natural morphism of pointed stacks

$$(K(\Gamma, 1) \otimes \mathbb{C})^{\text{sch}} \to K(\Gamma^{\text{alg}}, 1)$$

is an isomorphism.

Theorem (Katzarkov, Pantev ,Toen)

Let $n > 1$ and let (Y, y) be a pointed connected homotopy type so that $\pi_1(Y, y) =: \Gamma$ is algebraically good, $\pi_i(Y, y)$ are finitely generated for $1 < i \leq n$ and $\pi_i(Y, y) = 0$ for $i > n$. Let $\rho : \Gamma \to GL_m(\mathbb{Z})$, and let $\rho_1, \ldots, \rho_r$ be the simple factors of the semi-simplification of $\rho_{\mathbb{C}}$.

Let $Z = K(\Gamma, \mathbb{Z}^m, n) \times_{K(\Gamma,1)} Y$. If there is a $X \in \mathcal{P}$ so that $\tau_{\leq n} X \cong \tau_{\leq n} Z$, then the real Zariski closure of the image of each ρ_j is a group of Hodge type.

Examples:

(a) Let $\Gamma = \mathbb{Z}^{2g}$ and let ρ be any integral reductive representation, such that $\rho_{\mathbb{C}}$ is not unitary. Then at least one of the characters ρ_j is not unitary and so the real Zariski closure of the image of ρ_j is not of Hodge type.

(b) Let Γ be the fundamental group of a compact Riemann surface of genus $g > 2$ and let $m > 2$. Let $\rho : \Gamma \to SL_m(\mathbb{C})$ be a surjective homomorphism. Then the real Zariski closure of the image of ρ is $SL_m(\mathbb{R})$, which is not a group of Hodge type.

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Remark: Γ is algebraically good iff for any finite dimensional complex representation V of Γ, the natural map $\Gamma \to \Gamma^{\mathrm{alg}}$ induces an isomorphism

$$HH^\bullet(\Gamma^{\mathrm{alg}}, V) \cong H^\bullet(\Gamma, V).$$

Here $HH^\bullet(\Gamma^{\mathrm{alg}}, V)$ denotes the Hochschild cohomology of the affine group scheme Γ^{alg}.

Examples: Finite groups, free groups of finite type, finitely generated abelian groups and the fundamental groups of compact Riemann surfaces are algebraically good groups.

3 The weight filtration

The Hodge decomposition we constructed on $(X \otimes \mathbb{C})^{\mathrm{sch}}$ is only a part of a schematic mixed Hodge structure which includes a *weight filtration*:

Theorem (Katzarkov, Pantev, Toen)

For any pointed projective manifold (X, x) there exists a natural $\mathbb{C}^{\times\delta}$-equivariant tower of pointed schematic homotopy types

$$(X \otimes \mathbb{C})^{\mathrm{sch}} \to \ldots \to \mathbb{L}W^{(1)}(X \otimes \mathbb{C}) \to \mathbb{L}W^{(0)}(X \otimes \mathbb{C}),$$

satisfying:

- *There is a natural isomorphism of $\mathbb{C}^{\times\delta}$-equivariant stacks*

$$\mathbb{L}W^{(0)}(X \otimes \mathbb{C}) = B\pi_1(X, x)^{\mathrm{red}}.$$

- *The homotopy fiber $\mathrm{Gr}_W^{(m)}(X \otimes \mathbb{C})$ of the morphism*

$$\mathbb{L}W^{(m)}(X \otimes \mathbb{C}) \to \mathbb{L}W^{(m-1)}(X \otimes \mathbb{C})$$

is representable by a cosimplicial abelian unipotent group scheme.

- *The action of $\mathbb{C}^\times$ on the stacks $\mathrm{Gr}_W^{(m)}(X \otimes \mathbb{C})$ satisfies a purity condition.*

The relationship between X and $(X \otimes \mathbb{C})^{\mathrm{sch})}$ is quite difficult to understand. This is related to the existence of groups which are not algebraically good.

If X is a complex algebraic manifold, then every point has a Zariski neighborhood whose underlying homotopy type is a $K(\pi, 1)$, where π is a group built by successive extensions of free groups of finite type (M.Artin).

It seems likely that such π's are algebraically good (Serre had shown that these groups are pro-finitely good). Also, Beilinson has proven that the limit of all Zariski neighborhoods of a point $x \in X$ is of the form $K(\pi, 1)$ with π being an algebraically group.

Thus the schematization is relatively easy to understand locally in the Zariski topology.

This justifies the search for a schematic Van Kampen theorem.

In fact, observe that the schematization functor is left adjoint and thus commutes with homotopy colimits. Therefore one expects that for any open hypecover U_* of X we will have an equivalence

$$(X \otimes \mathbb{C})^{\mathrm{sch}} \cong \mathrm{hocolim}_{[n]\in\Delta}(U_n \otimes \mathbb{C})^{\mathrm{sch}}.$$

Modulo (hard) technical details this result makes $(X \otimes \mathbb{C})^{\mathrm{sch}}$ computable.

In [KTPW] we introduce a weight filtration on $X \otimes \mathbb{C})^{\mathrm{sch}}$. We hope that the "Whitehead products" will constitute new restrictions on the non-Kähler homotopy types. Building on ideas of Deligne and Ihara we can look at different realizations - in particular one can use an l -adic version of $X \otimes \mathbb{C})^{\mathrm{sch}}$ in order to study existence of rational points on X.

4 Realizations

For X a scheme over $\mathbb{Q}$, one can consider $(X')^l$ the l-adic homotopy type of $X' := X \otimes \overline{Q}$ (extension of X over the algebraic closure of $\mathbb{Q}$). This is a schematic homotopy type over $SpecZ_l$, which comes with an action of the Galois group of $\mathbb{Q}$ (the Galois group acts on the scheme X'). Now, if X has a rational point, it induces a non-trivial fixed point of the l-adic homtopy type $(X')^l$ by the Galois group. Therefore, if one knows that the Galois group does not have any fixed points on $(X')^l$, one also knows that X does not have any rational point.

It would be very interesting to know a statement in the other direction. When does a fixed point of $(X')^l$ come from a rational point of X ? This question seems related to Grothendieck's anabelian geometry, and was answered when X is a smooth curve of genus $g > 1$. Indeed, in this case $(X')^l$ is nothing else but the geometric fundamental group of X' together with the action of $Gal(Q)$. The following problem seems reasonable:

Problem 1. Find an extension of Grothendieck's anabelian geometry to the case of the whole etale homotopy type (or any reasonable homotopy type, like DR, Dol ...) or their schematizations.

For example using whole etale homotopy type one can try to prove some anabelian theorem for varieties, where the fundamental group does not suffice. This approach will be continuation of the ideas of Ihara and Nakamura [IN]. The examples from [IN] show that the classical Grothendieck's anabelian approach failes on them but they indeed have different etale homotopy types and schematizations.

The proof of Tamagawa's theorem is based on previous works of Neukirch and Uchida, which use in an essential way class field theory. This suggests the following:

Problem 2. Find a connection between extended anabelian geometry (to the case of the l-adic homotopy type) and higher class field theory.

This connection can already be seen in the works of Bogomolov and Pop on "birational anabelian geometry" (i.e. function fields are characterized by their Galois groups).

Inspired by that we can go back to $\mathbb{C}$ and make the following conjecture suggested by above considerations - we see this conjecture as "DR realization" of the above discussion:

Conjecture The derived category of coherent algebraic D- modules on a smooth projective variety X over $\mathbb{C}$ recovers X.

The above conjecture suggests the most general anabelian approach we can think of at the moment.

Problem 3. Find a generalized "hyperbolicity condition" under which the schematization of homotopy type of a smooth projective variety X and its realizations recover X. Find generalized nonabelian obstructions to the Hasse principle given by schematization of homotopy type of a smooth projective variety X and its realizations.

5 Geometric approach to Nonabelian Mixed Hodge structures

In a different more geometric approach initiated by Simpson [Sim], [Sim97] together with Pantev and Simpson [KPSM] we study models the homotopy type of $\mathbb{CP}^1$. During our original research on this question, we actually started out by looking at the homotopy type of $\mathbb{CP}^1$ and then worked backwards from there to deduce what conditions needed to be required of a nonabelian mixed Hodge structure. In particular, one notices from the $\mathbb{CP}^1$ example:

1. The fact that one need to include the "homotopy weight" in the picture (although this was known from another context of Hodge III;

2. The condition that the Whitehead products induce zero on the associated-graded of the weight filtration, which suggested the general definition that the associated-graded of the weight filtration should have a structure of a "spectrum" or equivalently a perfect complex.

This approach has been completely worked out in [KPSM]. We will further develop it in a book [KTPB].

I state a conjecture about representability in the simplyconnected case which we are working on at the moment. This conjecture (if it turns out to be true) states how to define the "mixed Hodge structure on the homotopy type of X".

Conjecture 5.1 *Suppose X is a simply connected smooth projective variety. Then there is a* universal morphism *to a simply connected nonabelian mixed Hodge structure*

$$X_M \to \mathcal{Y} = \mathcal{MHS}(X)$$

with the property that for any nonabelian mixed Hodge structure $\mathcal{V}$ the resulting morphism

$$\underline{Hom}(\mathcal{Y}, \mathcal{V}) \to \underline{Hom}(X_M, \mathcal{V})$$

is an equivalence. Furthermore this representing object specializes to the n-stack representing the de Rham (resp. Betti) shape of X, i.e.
Y_{DR} is the n-stack representing the very presentable shape of X_{DR};
$Tot^F(Y_{DR})$ is the Hodge filtration on the de Rham representing object and
$Y_{B,\mathbf{C}}$ is the n-stack representing the very presentable shape of X_B.
Finally, the mixed Hodge structures on the homotopy vector spaces $\pi_i(\mathcal{Y}) = \pi_i(X_B) \otimes \mathbf{C}$ coincide with those defined by Morgan and Hain.

As a consequence of the functoriality and the universality in this last conjecture 3.1 $\mathcal{MHS}(X)$ would be functorial in X. In other words, if $X \to Z$ is a morphism of smooth projective varieties then we would get a morphism of nonabelian mixed Hodge structures

$$\mathcal{MHS}(X) \to \mathcal{MHS}(Z).$$

It is natural to ask which morphisms of nonabelian mixed Hodge structures, come from morphisms of varieties. This seems to be a subtle question, which we are trying to address.

Another natural question is how does one compare the above approaches. A way to do that is to develop a relative version of $(X \otimes \mathbb{C})^{sch}$ - the shape of X. This will allow us to define a nonabelian verion of Abel Jacobi map. As a result we have the following nonabelian Torelli type of statement.

Conjecture If X is a smooth projective variety of general type then the shape of X recovers X.

References

[G] A. Grothendieck Pursuing stacks, unpublished manuscript.

[Ha1] R. Hain Mixed Hodge structure on homotopy Bull. AMS **14** (1986), 111-114.

[Ha2] R. Hain The de Rham homotopy theory of complex algebraic varieties, I, K-Theory **1** (1987), 271-324.

[IN] Y. Ihara, H . Nakamura Some illustrative examples for anabelian geometry in high dimensions,Geometric Galois actions, 1, 127–138, London Math. Soc. Lecture Note Ser., 242, Cambridge Univ. Press, Cambridge, 1997.,

[Kat97] L. Katzarkov. On the Shafarevich maps. In *Algebraic geometry—Santa Cruz 1995*, pages 173–216. Amer. Math. Soc., Providence, RI, 1997.

[KPS98] L. Katzarkov, T. Pantev, C. Simpson. Density of monodromy actions on non-abelian cohomology to appear in Advances in Mathematics.

[KPSM] L. Katzarkov, T. Pantev, C. Simpson. Nonabelian mixed Hodge structures math.AG/0006213.

[KTPB] L. Katzarkov, T. Pantev, C. Simpson, B. Toen. Nonabelian Hodge theory book in preparation

[KTPW] L. Katzarkov, T. Pantev, B. Toen. Schematic homotopy types and non-abelian Hodge theory II: The weight filtration. in preparation.

[KTP] L. Katzarkov, T. Pantev, B. Toen. Schematic homotopy types and non-abelian Hodge theory I: The Hodge decomposition. math.AG/0107129

[PP] L. Katzarkov, T. Pantev Non abelian p, p cycles to appear in Motives , polylogs and nonabelian Hodge theory 1998 IP lectures.

[Sim] C. Simpson. Mixed twistor structures, math.AG/9705006. (alg-geom/9705006).

[Sim97] C. Simpson. The Hodge filtration on nonabelian cohomology. In *Algebraic geometry—Santa Cruz 1995*, pages 217–281. Amer. Math. Soc., Providence, RI, 1997.

[Toen] B. Toen. Schematization of homotopy types AG/0012219

[Ta] Z. Tamsamani Sur des notions de n-categories et n-groupoide non strictes via des ensembles multi-simpliciaux. K-Theory **16** (1999), 51-99.

[To2] B.Toen Dualité de Tannaka supérieure I: Structures monnoidales. MPI preprint, 2000.

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