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LEXICOGRAPHIC GRÖBNER BASES OF TORIC IDEALS
ARISING FROM ROOT SYSTEMS

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Abstract. The present paper is a brief draft based on a joint work with Takayuki Hibi. Gröbner bases of toric ideals arising from root systems are studied.

Introduction
Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set and let $K[t, t^{-1}, s] = K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s]$ denote the Laurent polynomial ring over a field $K$. We associate each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with the monomial $t^\alpha s = t_1^{\alpha_1} \cdots t_n^{\alpha_n} s \in K[t, t^{-1}, s]$ and write $\mathcal{R}_K[\mathcal{A}]$ for the subalgebra of $K[t, t^{-1}, s]$ generated by all monomials $t^\alpha s$ with $\alpha \in \mathcal{A}$. Let $K[x] = K[\{ x_\alpha ; \alpha \in \mathcal{A} \}]$ denote the polynomial ring in $\#(\mathcal{A})$ variables over $K$ and $I_\mathcal{A} \subset K[x]$ the kernel of the surjective homomorphism $\pi : K[x] \to \mathcal{R}_K[\mathcal{A}]$ defined by setting $\pi(x_\alpha) = t^\alpha s$ for all $\alpha \in \mathcal{A}$. The ideal $I_\mathcal{A}$ is called the toric ideal of the configuration $\mathcal{A}$. It is known [9] that if $I_\mathcal{A}$ possesses a squarefree initial ideal, then the convex hull of $\mathcal{A}$ possesses a unimodular triangulation.

Fix $n \geq 2$. Let $e_i$ denote the $i$-th unit coordinate vector of $\mathbb{R}^n$. We write $\mathbb{A}_{n-1}^+$, $\mathbb{B}_n^+$, $\mathbb{C}_n^+$, $\mathbb{D}_n^+$ and $\mathbb{BC}_n^+$ for the set of positive roots of root systems $\mathbb{A}_{n-1}$, $\mathbb{B}_n$, $\mathbb{C}_n$, $\mathbb{D}_n$ and $\mathbb{BC}_n$, respectively ([3, pp. 64 - 65]):

\[
\begin{align*}
\mathbb{A}_{n-1}^+ &= \{ e_i - e_j ; 1 \leq i < j \leq n \}; \\
\mathbb{B}_n^+ &= \{ e_i ; 1 \leq i \leq n \} \cup \{ e_i + e_j ; 1 \leq i < j \leq n \} \cup \{ e_i - e_j ; 1 \leq i < j \leq n \}; \\
\mathbb{C}_n^+ &= \{ 2e_i ; 1 \leq i \leq n \} \cup \{ e_i + e_j ; 1 \leq i < j \leq n \} \cup \{ e_i - e_j ; 1 \leq i < j \leq n \}; \\
\mathbb{D}_n^+ &= \{ e_i + e_j ; 1 \leq i < j \leq n \} \cup \{ e_i - e_j ; 1 \leq i < j \leq n \}; \\
\mathbb{BC}_n^+ &= \mathbb{B}_n^+ \cup \mathbb{C}_n^+.
\end{align*}
\]

Let, in addition, $\bar{\Phi}^+ = \Phi^+ \cup \{(0, 0, \ldots, 0)\}$, where $\Phi = \mathbb{A}_{n-1}, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n$ or $\mathbb{BC}_n$ and where $(0, 0, \ldots, 0)$ is the origin of $\mathbb{R}^n$.

In their combinatorial study of hypergeometric functions associated with root systems, Gelfand, Graev and Postnikov [2, Theorem 6.3] discovered a squarefree quadratic initial ideal of the toric ideal $I_{\mathbb{A}_{n-1}^+}$ of $\mathbb{A}_{n-1}^+$. Moreover, for any subconfiguration $\mathcal{A}$ of $\mathbb{A}_{n-1}^+$, the configuration $\tilde{\mathcal{A}} = \mathcal{A} \cup (0, 0, \ldots, 0)$ possesses a regular unimodular triangulation ([7, Example 2.4 (a)]). Stanley [8, Exercise 6.31 (b), p. 234] computed the Ehrhart polynomial of the convex polytope $\mathrm{conv}(\mathbb{A}_{n-1}^+)$. Fong [1] constructed certain triangulations of the configurations $\tilde{\mathcal{B}}_n^+$ ($= \mathrm{conv}(\mathbb{D}_n^+) \cap \mathbb{Z}^n$).
and \( \text{conv}(\tilde{C}_n^+) \cap \mathbb{Z}^n = \hat{B}_n^+ \), and computes the Ehrhart polynomials of \( \text{conv}(\hat{B}_n^+) \) and \( \text{conv}(\tilde{C}_n^+) \). The triangulations studied in [1] are, however, non-unimodular. Motivated by their results, Ohsugi–Hibi [6] showed that

**Proposition 0.1.** Let \( \Phi \subset \mathbb{Z}^n \) be one of the root systems \( A_{n-1}, B_n, C_n, D_n \) and \( B \subset A_{n-1} \). Then, there exists a reverse lexicographic order such that the initial ideal of \( I_{\Phi^+} \) is generated by squarefree monomials.

Moreover, Ohsugi–Hibi [5] discussed subconfigurations \( \hat{A} = A \cup \{(0, 0, \ldots, 0)\} \) of \( \hat{B}_n^+ \cup \tilde{C}_n^+ \) which possesses a (regular) unimodular triangulation (i.e., \( I_{\hat{A}} \) which possesses a squarefree initial ideal).

Hence, it is natural to study the same problem as above for \( I_{\Phi^+} \) where \( \Phi \subset \mathbb{Z}^n \) is one of the root systems \( A_{n-1}, B_n, C_n, D_n \) and \( B \subset A_{n-1} \). (Then, \( I_{\Phi^+} \) is not generated by quadratic binomials if \( n \geq 6 \).)

1. **Squarefree lexicographic initial ideals**

Let \( \Phi^+ \subset \mathbb{Z}^n \) denote one of the configurations \( A_{n-1}^+, B_n^+, C_n^+, D_n^+ \) and \( B \subset A_{n-1}^+ \). Let \( K[A_{n-1}^+] \), \( K[B_n^+] \), \( K[C_n^+] \), \( K[D_n^+] \) and \( K[B \subset A_{n-1}^+] \) denote the polynomial rings

\[
K[A^+_{n-1}] = K[[a_{ij}^{1 \leq i < j \leq n}], \\
K[B_n^+] = K[[y_i^{1 \leq i \leq n} \cup \{e_{ij}^{1 \leq i < j \leq n} \cup \{f_{ij}^{1 \leq i < j \leq n}], \\
K[C_n^+] = K[[a_i^{1 \leq i \leq n} \cup \{e_{ij}^{1 \leq i < j \leq n} \cup \{f_{ij}^{1 \leq i < j \leq n}], \\
K[D_n^+] = K[[e_{ij}^{1 \leq i < j \leq n} \cup \{f_{ij}^{1 \leq i < j \leq n}], \\
K[B \subset A_{n-1}^+] = K[[a_i^{1 \leq i \leq n} \cup \{y_i^{1 \leq i \leq n} \cup \{e_{ij}^{1 \leq i < j \leq n} \cup \{f_{ij}^{1 \leq i < j \leq n}]
\]

over \( K \). Write \( \pi : K[\Phi^+] \rightarrow K[t, t^{-1}, s] \) for the homomorphism defined by setting

\[
\pi(a_i) = t_i^2s, \quad \pi(y_i) = t_i s, \quad \pi(e_{ij}) = t_i t_j s, \quad \pi(f_{ij}) = t_i t_j^{-1}s.
\]

Thus the kernel of \( \pi \) is the toric ideal \( I_{\Phi^+} \).

First, an explicit initial ideals of \( I_{A_{n-1}^+} \) generated by squarefree monomials of degree \( \leq 3 \) will be constructed. Let \( \leq \) be the lexicographic order induced by the ordering of variables

\[
f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n},
\]

and let \( <_{\text{rev}} \) be the reverse lexicographic order induced by the ordering of variables

\[
f_{n-1,n} > f_{n-2,n} > f_{n-2,n-1} > \cdots > f_{2,3} > f_{1,n} > \cdots > f_{1,3} > f_{1,2}.
\]

Then, the reduced Gröbner basis with respect to \( \leq \) (and \( <_{\text{rev}} \)) is as follows.

**Theorem 1.1** ([4]). The set of the binomials

\[
f_{i,j}f_{j,k} - f_{i,k}f_{j,t}, \quad i < j < k < \ell, \\
f_{i,j}f_{j,k} - f_{i,j+1}f_{i+1,k}, \quad i + 1 < j < k, \\
f_{i,j}f_{k+1,t} - f_{i,j+1}f_{k+1,t}, \quad i + 1 < j < k < \ell - 1,
\]

is the reduced Gröbner basis of the toric ideal \( I_{A_{n-1}^+} \) with respect to both \( \leq \) and \( <_{\text{rev}} \), where the initial monomial of each binomial is the first monomial.
Then, we can associate the initial ideal of $I_{\mathrm{A}_{n-1}^{+}}$ with respect to $\prec_{\text{lex}}$ with the regular unimodular triangulation $\Delta_{\prec_{\text{lex}}}$. A graph-theoretical characterization of the maximal faces of the triangulation $\Delta_{\prec_{\text{lex}}}$ is given in [4].

Second, we discuss the existence of squarefree initial ideals of the toric ideal $I_{\Phi^{+}}$ where $\Phi \subset \mathbb{Z}^{n}$ is one of the root systems $\mathbf{B}_{n}$, $\mathbf{C}_{n}$, $\mathbf{D}_{n}$ and $\mathbf{BC}_{n}$. The similar argument as in [5] plays an important role in the proof of Theorems 1.2 and 1.4.

Let $\prec_{\text{lex}}$ be the lexicographic order induced by the ordering of variables

$$a_{1} > a_{2} > \cdots > a_{n}$$
$$> f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n}$$
$$> e_{n-1,n} > e_{n-2,n-1} > e_{n-2,n} > \cdots > e_{1,2} > e_{1,3} > \cdots > e_{1,n}.$$

**Theorem 1.2.** The initial ideal of the toric ideal $I_{\mathbf{C}_{n}^{+}}$ with respect to $\prec_{\text{lex}}$ is generated by squarefree monomials.

Let $\prec_{\text{lex}}^{c}$ denote the lexicographic order obtained by restricting $\prec_{\text{lex}}$ to $K[\mathbf{D}_{n}^{+}]$. By the elimination property of the lexicographic order $\prec_{\text{lex}}$, we have the following corollary from Theorem 1.2.

**Corollary 1.3.** The initial ideal of the toric ideal $I_{\mathbf{D}_{n}^{+}}$ with respect to $\prec_{\text{lex}}^{c}$ is generated by squarefree monomials.

We now consider the root systems $\mathbf{B}_{n}$ and $\mathbf{BC}_{n}$. Let $\prec_{\text{lex}}^{bc}$ be the lexicographic order induced by the ordering of variables

$$a_{1} > a_{2} > \cdots > a_{n}$$
$$> e_{n-1,n} > e_{n-2,n-1} > e_{n-2,n} > \cdots > e_{1,2} > e_{1,3} > \cdots > e_{1,n}$$
$$> y_{1} > y_{2} > \cdots > y_{n}$$
$$> f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n}.$$

**Theorem 1.4.** The initial ideal of the toric ideal $I_{\mathbf{BC}_{n}^{+}}$ with respect to $\prec_{\text{lex}}^{bc}$ is generated by squarefree monomials.

Let $\prec_{\text{lex}}^{b}$ denote the lexicographic order obtained by restricting $\prec_{\text{lex}}^{bc}$ to $K[\mathbf{B}_{n}^{+}]$. By the elimination property of the lexicographic order $\prec_{\text{lex}}^{bc}$, we have the following corollary from Theorem 1.4.

**Corollary 1.5.** The initial ideal of the toric ideal $I_{\mathbf{B}_{n}^{+}}$ with respect to $\prec_{\text{lex}}^{b}$ is generated by squarefree monomials.

**Remark 1.6.** Let $n \geq 6$ and let $\Phi^{+}$ denote one of the configurations $\mathbf{A}_{n-1}^{+}$, $\mathbf{B}_{n}^{+}$, $\mathbf{C}_{n}^{+}$, $\mathbf{D}_{n}^{+}$ and $\mathbf{BC}_{n}^{+}$. Then $I_{\Phi^{+}}$ is not generated by quadratic binomials. Hence, in particular, $I_{\Phi^{+}}$ does not possess a quadratic Gröbner basis.
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