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<th>LEXICOGRAPHIC GROBNER BASES OF TORIC IDEALS ARISING FROM ROOT SYSTEMS (Algorithms in Algebraic Systems and Computation Theory)</th>
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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2002), 1268: 73-76</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42137">http://hdl.handle.net/2433/42137</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
LEXICOGRAPHIC GRÖBNER BASES OF TORIC IDEALS
ARISING FROM ROOT SYSTEMS

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ABSTRACT. The present paper is a brief draft based on a joint work with Takayuki Hibi. Gröbner bases of toric ideals arising from root systems are studied.

INTRODUCTION

Let \( \mathcal{A} \subset \mathbb{Z}^n \) be a finite set and let \( K[t, t^{-1}, s] = K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s] \) denote the Laurent polynomial ring over a field \( K \). We associate each \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \) with the monomial \( t^{\alpha}s = t_1^{\alpha_1} \cdots t_n^{\alpha_n}s \in K[t, t^{-1}, s] \) and write \( R_K[\mathcal{A}] \) for the subalgebra of \( K[t, t^{-1}, s] \) generated by all monomials \( t^{\alpha}s \) with \( \alpha \in \mathcal{A} \). Let \( K[x] = K[[x_\alpha ; \alpha \in \mathcal{A}]] \) denote the polynomial ring in \#(\mathcal{A}) \) variables over \( K \) and \( I_\mathcal{A} \subset K[x] \) the kernel of the surjective homomorphism \( \pi : K[x] \to R_K[\mathcal{A}] \) defined by setting \( \pi(x_\alpha) = t^{\alpha}s \) for all \( \alpha \in \mathcal{A} \). The ideal \( I_\mathcal{A} \) is called the toric ideal of the configuration \( \mathcal{A} \). It is known [9] that if \( I_\mathcal{A} \) possesses a squarefree initial ideal, then the convex hull of \( \mathcal{A} \) possesses a unimodular triangulation.

Fix \( n \geq 2 \). Let \( e_i \) denote the \( i \)-th unit coordinate vector of \( \mathbb{R}^n \). We write \( \mathcal{A}_{n-1}^{+}, \mathcal{B}_{n}^{+}, \mathcal{C}_{n}^{+}, \mathcal{D}_{n}^{+} \) and \( \mathcal{BC}_{n}^{+} \) for the set of positive roots of root systems \( \mathcal{A}_{n-1}, \mathcal{B}_{n}, \mathcal{C}_{n}, \mathcal{D}_{n} \) and \( \mathcal{BC}_{n} \), respectively ([3, pp. 64 - 65]):

- \( \mathcal{A}_{n-1}^{+} = \{e_i - e_j ; 1 \leq i < j \leq n \} \);
- \( \mathcal{B}_{n}^{+} = \{e_i ; 1 \leq i \leq n \} \cup \{e_i + e_j ; 1 \leq i < j \leq n \} \cup \{e_i - e_j ; 1 \leq i < j \leq n \} \);
- \( \mathcal{C}_{n}^{+} = \{2e_i ; 1 \leq i \leq n \} \cup \{e_i + e_j ; 1 \leq i < j \leq n \} \cup \{e_i - e_j ; 1 \leq i < j \leq n \} \);
- \( \mathcal{D}_{n}^{+} = \{e_i + e_j ; 1 \leq i < j \leq n \} \cup \{e_i - e_j ; 1 \leq i < j \leq n \} \);
- \( \mathcal{BC}_{n}^{+} = \mathcal{B}_{n}^{+} \cup \mathcal{C}_{n}^{+} \).

Let, in addition, \( \tilde{\Phi}^{+} = \Phi^{+} \cup \{(0,0, \ldots, 0)\} \), where \( \Phi = \mathcal{A}_{n-1}, \mathcal{B}_{n}, \mathcal{C}_{n}, \mathcal{D}_{n} \) or \( \mathcal{BC}_{n} \) and where \( (0,0, \ldots, 0) \) is the origin of \( \mathbb{R}^n \).

In their combinatorial study of hypergeometric functions associated with root systems, Gelfand, Graev and Postnikov [2, Theorem 6.3] discovered a squarefree quadratic initial ideal of the toric ideal \( I_{\tilde{\mathcal{A}}_{n-1}^{+}} \) of \( \tilde{\mathcal{A}}_{n-1}^{+} \). Moreover, for any subconfiguration \( \mathcal{A} \) of \( \mathcal{A}_{n-1}^{+} \), the configuration \( \tilde{\mathcal{A}} = \mathcal{A} \cup (0,0, \ldots, 0) \) possesses a regular unimodular triangulation ([7, Example 2.4 (a)]). Stanley [8, Exercise 6.31 (b), p. 234] computed the Ehrhart polynomial of the convex polytope \( \text{conv}(\tilde{\mathcal{A}}_{n-1}^{+}) \). Fong [1] constructed certain triangulations of the configurations \( \tilde{\mathcal{B}}_{n}^{+} (= \text{conv}(\tilde{\mathcal{D}}_{n}^{+}) \cap \mathbb{Z}^n) \)
and $\text{conv} (\tilde{C}_n^+) \cap \mathbb{Z}^n = \tilde{B}C_n^+$, and computes the Ehrhart polynomials of $\text{conv} (\tilde{B}_n^+)$ and $\text{conv} (\tilde{C}_n^+)$. The triangulations studied in [1] are, however, non-unimodular. Motivated by their results, Ohsugi–Hibi [6] showed that

**Proposition 0.1.** Let $\Phi \subset \mathbb{Z}^n$ be one of the root systems $A_{n-1}$, $B_n$, $C_n$, $D_n$ and $BC_n$. Then, there exists a reverse lexicographic order such that the initial ideal of $I_{\Phi^+}$ is generated by squarefree quadratic monomials.

Moreover, Ohsugi–Hibi [5] discussed subconfigurations $\tilde{A} = A \cup \{(0,0,\ldots,0)\}$ of $\tilde{B}_n^+ \cup \tilde{C}_n^+$ which possesses a (regular) unimodular triangulation (i.e., $I_{\tilde{A}}$ which possesses a squarefree initial ideal).

Hence, it is natural to study the same problem as above for $I_{\Phi^+}$ where $\Phi \subset \mathbb{Z}^n$ is one of the root systems $A_{n-1}$, $B_n$, $C_n$, $D_n$ and $BC_n$. (Then, $I_{\Phi^+}$ is not generated by quadratic binomials if $n \geq 6$.)

1. **Squarefree lexicographic initial ideals**

Let $\Phi^+ \subset \mathbb{Z}^n$ denote one of the configurations $A_{n-1}^+, B_n^+, C_n^+, D_n^+$ and $BC_n^+$. Let $K[A_{n-1}^+]$, $K[B_n^+]$, $K[C_n^+]$, $K[D_n^+]$ and $K[BC_n^+]$ denote the polynomial rings

$$
K[A_{n-1}^+] = K\{f_{i,j}\}_{1 \leq i < j \leq n},
$$
$$
K[B_n^+] = K\{y_i\}_{1 \leq i \leq n} \cup \{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n},
$$
$$
K[C_n^+] = K\{a_i\}_{1 \leq i \leq n} \cup \{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n},
$$
$$
K[D_n^+] = K\{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n},
$$
$$
K[BC_n^+] = K\{a_i\}_{1 \leq i \leq n} \cup \{y_i\}_{1 \leq i \leq n} \cup \{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}
$$

over $K$. Write $\pi : K[\Phi^+] \rightarrow K[t,t^{-1},s]$ for the homomorphism defined by setting

$$
\pi(a_i) = t_i^2s, \quad \pi(y_i) = t_is, \quad \pi(e_{i,j}) = t_it_js, \quad \pi(f_{i,j}) = t_it_j^{-1}s.
$$

Thus the kernel of $\pi$ is the toric ideal $I_{\Phi^+}$.

First, an explicit initial ideals of $I_{A_{n-1}^+}$ generated by squarefree monomials of degree $\leq 3$ will be constructed. Let $<_{\text{lex}}$ be the lexicographic order induced by the ordering of variables

$$
f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n},
$$

and let $<_{\text{rev}}$ be the reverse lexicographic order induced by the ordering of variables

$$
f_{n-1,n} > f_{n-2,n} > f_{n-2,n-1} > \cdots > f_{2,3} > f_{1,n} > \cdots > f_{1,3} > f_{1,2}.
$$

Then, the reduced Gröbner basis with respect to $<_{\text{lex}}$ (and $<_{\text{rev}}$) is as follows.

**Theorem 1.1 ([4]).** The set of the binomials

$$
\begin{align*}
&f_{i,\ell}f_{\dot{j},k} - f_{i,k}f_{\dot{j},\ell}, & i < j < k < \ell, \\
&f_{i,j}f_{\dot{k},\ell} - f_{i,j+1}f_{i+1,\ell}, & i + 1 < j < k, \\
&f_{i,j}f_{k+1,\ell} - f_{i,j+1}f_{i+1,j}f_{k,\ell}, & i + 1 < j < k < \ell - 1,
\end{align*}
$$

is the reduced Gröbner basis of the toric ideal $I_{A_{n-1}^+}$ with respect to both $<_{\text{lex}}$ and $<_{\text{rev}}$, where the initial monomial of each binomial is the first monomial.
Then, we can associate the initial ideal of $I_{\mathrm{A}_{n-1}^{+}}$ with respect to $<_{\text{lex}}$ with the regular unimodular triangulation $\Delta <_{\text{lex}}$. A graph-theoretical characterization of the maximal faces of the triangulation $\Delta <_{\text{lex}}$ is given in [4].

Second, we discuss the existence of squarefree initial ideals of the toric ideal $I_\Phi^+$ where $\Phi \subset \mathbb{Z}^n$ is one of the root systems $\mathcal{B}_n$, $\mathcal{C}_n$, $\mathcal{D}_n$ and $\mathcal{BC}_n$. The similar argument as in [5] plays an important role in the proof of Theorems 1.2 and 1.4.

Let $<_{\text{lex}}^c$ be the lexicographic order induced by the ordering of variables

$$a_1 > a_2 > \cdots > a_n$$
$$> f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n}$$
$$> e_{n-1,n} > e_{n-2,n-1} > e_{n-2,n} > \cdots > e_{1,2} > e_{1,3} > \cdots > e_{1,n}.$$ 

**Theorem 1.2.** The initial ideal of the toric ideal $I_{\mathcal{C}_n^+}$ with respect to $<_{\text{lex}}^c$ is generated by squarefree monomials.

Let $<_{\text{lex}}^d$ denote the lexicographic order obtained by restricting $<_{\text{lex}}^c$ to $K[\mathcal{D}_n^+]$. By the elimination property of the lexicographic order $<_{\text{lex}}^c$, we have the following corollary from Theorem 1.2.

**Corollary 1.3.** The initial ideal of the toric ideal $I_{\mathcal{D}_n^+}$ with respect to $<_{\text{lex}}^d$ is generated by squarefree monomials.

We now consider the root systems $\mathcal{B}_n$ and $\mathcal{BC}_n$. Let $<_{\text{lex}}^b$ be the lexicographic order induced by the ordering of variables

$$a_1 > a_2 > \cdots > a_n$$
$$> e_{n-1,n} > e_{n-2,n-1} > e_{n-2,n} > \cdots > e_{1,2} > e_{1,3} > \cdots > e_{1,n}$$
$$> y_1 > y_2 > \cdots > y_n$$
$$> f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n}.$$ 

**Theorem 1.4.** The initial ideal of the toric ideal $I_{\mathcal{BC}_n^+}$ with respect to $<_{\text{lex}}^b$ is generated by squarefree monomials.

Let $<_{\text{lex}}^b$ denote the lexicographic order obtained by restricting $<_{\text{lex}}^b$ to $K[\mathcal{B}_n^+]$. By the elimination property of the lexicographic order $<_{\text{lex}}^b$, we have the following corollary from Theorem 1.4.

**Corollary 1.5.** The initial ideal of the toric ideal $I_{\mathcal{B}_n^+}$ with respect to $<_{\text{lex}}^b$ is generated by squarefree monomials.

**Remark 1.6.** Let $n \geq 6$ and let $\Phi^+$ denote one of the configurations $\mathcal{A}_{n-1}^+$, $\mathcal{B}_n^+$, $\mathcal{C}_n^+$, $\mathcal{D}_n^+$ and $\mathcal{BC}_n^+$. Then $I_{\Phi^+}$ is not generated by quadratic binomials. Hence, in particular, $I_{\Phi^+}$ does not possess a quadratic Gröbner basis.
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