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Numerical semigroups of toric type of higher dimension

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Let $k$ be an algebraically closed field of characteristic 0 and $n$ an integer at least 2. We set $T = \mathbb{G}_m^n$ where $\mathbb{G}_m = \text{Spec } k[X, X^{-1}]$ is the multiplicative group. Moreover, we denote by $M$ (resp. $N$) the group $\text{Hom}_{\text{Alg.Groups}}(T, \mathbb{G}_m)$ of characters of $T$ (resp. the group $\text{Hom}_{\text{Alg.Groups}}(\mathbb{G}_m, T)$ of 1-parameter subgroups of $T$). Then we have a non-singular canonical pairing $\langle \ , \rangle : M \times N \to \mathbb{Z}$ where $\mathbb{Z}$ is the ring of integers. We set $N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes \mathbb{Z} \mathbb{R}$ where $\mathbb{R}$ is the set of real numbers. Let $\sigma$ be a strongly convex rational polyhedral cone in $N_\mathbb{R}$, i.e., there exist a finite number of vectors $x_i \in N_\mathbb{R}$ defined over the ring $\mathbb{Q}$ of rational numbers such that

$$\sigma = \{ \sum_{i=1}^{N'} \lambda_i x_i | \lambda_i \geq 0, \text{ all } i \} = \sum_{i=1}^{N'} \mathbb{R}_+ x_i$$

and it contains no line through the origin where $\mathbb{R}_+$ is the set of non-negative real numbers. We set

$$\check{\sigma} = \{ r \in M_\mathbb{R} | < r, a > \geq 0, \text{ all } a \in \sigma \}.$$ 

Then $\check{\sigma} \cap M$ becomes a subsemigroup of $M$. An $n$-dimensional affine toric variety is expressed as $\text{Spec } k[\check{\sigma} \cap M]$. Let $M(\check{\sigma} \cap M)$ be the minimal set of generators for the semigroup $\check{\sigma} \cap M$. Then we can embed the affine toric variety $X_\sigma = \text{Spec } k[\check{\sigma} \cap M]$ into the affine $m$-space $\mathbb{A}_m^m = \text{Spec } k[Y_1, \ldots, Y_m]$ using the $k$-algebra homomorphism $k[Y_1, \ldots, Y_m] \to k[\check{\sigma} \cap M]$ which sends $Y_i$ to $T^h_i$ where we set $M(\check{\sigma} \cap M) = \{ b_1, \ldots, b_m \}$.

Let $H$ be a numerical semigroup, i.e., a subsemigroup of the additive semigroup $\mathbb{N}$ of non-negative integers such that its complement in $\mathbb{N}$ is finite. We denote by $g(H)$ the cardinality of $\mathbb{N} \setminus H$, which is called the genus of $H$. We set

$$c(H) = \text{Min}\{ c \in \mathbb{N} | c + \mathbb{N} \subseteq H \},$$

which is called the conductor of $H$. Then we get $c(H) \leq 2g(H)$. Let $M(H)$ be the minimal set of generators for $H$. If $M(H) = \{ a_1, a_2, \ldots, a_l \}$, then we set

$$\alpha_i = \text{Min}\{ \alpha | \alpha a_i \in < a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_l > \}$$

for $i = 1, \ldots, l$, where for any positive integers $b_1, \ldots, b_l$ we denote by $< b_1, \ldots, b_l >$ the subsemigroup of $\mathbb{N}$ generated by $b_1, \ldots, b_l$. Moreover, $H$ is called a Weierstrass semigroup if there exist a complete non-singular irreducible algebraic curve $C$ over $k$ and its point $P$ such that

$$H = \{ \nu \in \mathbb{N} | \text{there is a rational function } f \text{ on } C \text{ such that } (f)_\infty = \nu P \}.$$
Let \( \lambda \in \sigma \cap N \) such that \( <r, \lambda > > 0 \) for any non-zero \( r \in \hat{\sigma} \cap M \). Take a numerical semigroup \( H \) containing the semigroup \( < \hat{\sigma} \cap M, \lambda > \). We can define the morphism \( \mathbb{A}^{l} \rightarrow \mathbb{A}^{m} \) by the \( k \)-algebra homomorphism
\[
k[Y_{1}, \ldots, Y_{m}] \rightarrow k[X_{1}, \ldots, X_{l}]
\]
which sends \( Y_{i} \) to \( x^{<b_{i}, \lambda>} = X_{1}^{n_{1}} \cdots X_{l}^{n_{l}} \) where \( M(H) = \{a_{1}, \ldots, a_{l}\} \) and \( < b_{i}, \lambda > = \nu_{1}a_{1} + \cdots + \nu_{l}a_{l} \) for some non-negative integers \( \nu_{i} \)'s. The above morphism \( \mathbb{A}^{l} \rightarrow \mathbb{A}^{m} \) is said to be induced by \( \lambda \). A numerical semigroup \( H \) is constructed from \( X_{\sigma} \) and \( \lambda \) if \( \# M(H) = \# M(\hat{\sigma} \cap M) - n + 1 \) and \( \text{Spec} k[H] \) is isomorphic to the fiber product
\[
\mathbb{A}^{l} \times_{\mathbb{A}^{l+n-1}} \text{Spec} k[\hat{\sigma} \cap M]
\]
where \( l = \# M(H) \), \( \text{Spec} k[\hat{\sigma}_{a,b} \cap M] \rightarrow \mathbb{A}^{l+n-1} \) is the embedding using \( M(\hat{\sigma} \cap M) \), and \( \mathbb{A}^{l} \rightarrow \mathbb{A}^{l+n-1} \) is the morphism induced by \( \lambda \). In this case we also call \( H \) a numerical semigroup of \((n\text{-dimensional}) \) toric type. Then we can show that \( H \) is Weierstrass (see Komeda [2]). Here we pose the following problem:

**Problem 1.** Let \( X_{\sigma} \) be an affine toric variety. Give a numerical semigroup \( H \) which is constructed from \( X_{\sigma} \) and some \( \lambda \in \sigma \cap N \).

In the case where \( X_{\sigma} \) is 2-dimensional we get the following:

**Fact 2.** Let \( X_{\sigma} \) be a 2-dimensional affine toric variety. Then \( \sigma \) is expressed as \( \sigma = \mathbb{R}_{+}(1, 0) + \mathbb{R}_{+}(a, b) \) where \( a \) and \( b \) are integers with \( b > 0 \) and \( (a, b) = 1 \). If \( b = 1 \), then we may assume that \( a = 0 \). If \( b > 1 \), then we may assume that \( 0 < a < b \). The above cone \( \sigma \) is denoted by \( \sigma_{a,b} \). If \( a \leq 9 \), we can give a numerical semigroup \( H_{a,b} \) which is constructed from \( X_{\sigma_{a,b}} \) and \( \lambda = (a^{2}, (a-1)b) \) (see Komeda [3]).

We would like to consider Problem 1 in a higher dimensional case. This paper is aimed at the following:

**Aim 3.** For any \( n \geq 3 \) we give a numerical semigroup \( H \) of \( n \)-dimensional toric type. Namely, we find an \( n \)-dimensional affine toric variety \( X_{\sigma} \) such that there exists a numerical semigroup \( H \) which is constructed from \( X_{\sigma} \) and some \( \lambda \in \sigma \cap N \).

**Example 4.** Consider the 4-dimensional cone
\[
\sigma = \mathbb{R}_{+}(1, 0, 0, 0) + \mathbb{R}_{+}(0, 0, 1, 1) + \mathbb{R}_{+}(1, 0, 0, 0) + \mathbb{R}_{+}(0, 1, 1, 0) + \mathbb{R}_{+}(0, 1, 0, 1).
\]
Let \( X_{\sigma} = \text{Spec} k[\hat{\sigma} \cap M] \) be the 4-dimensional affine toric variety associated to \( \sigma \). We note that
\[
\hat{\sigma} \cap M = (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, -1, 0), (0, -1, 1, 1).
\]
Let \( H \) be a numerical semigroup with \( M(H) = \{a_{1}, a_{2}, a_{3}\} \). Assume that \( c(H) < 2g(H) \), i.e., \( H \) is non-symmetric. Then we have
\[
\alpha_{1}a_{1} = \alpha_{12}a_{2} + \alpha_{13}a_{3}, \quad \alpha_{2}a_{2} = \alpha_{21}a_{1} + \alpha_{23}a_{3}, \quad \alpha_{3}a_{3} = \alpha_{31}a_{1} + \alpha_{32}a_{2},
\]
where $0 < \alpha_{ij} < \alpha_{ij}$, $\alpha_{1} = \alpha_{12} + \alpha_{31}$, $\alpha_{2} = \alpha_{12} + \alpha_{32}$ and $\alpha_{3} = \alpha_{13} + \alpha_{23}$ (see Herzog [1]). Take $\lambda = (\alpha_{12}a_{1}, \alpha_{12}a_{1}, \alpha_{23}a_{2}, \alpha_{32}a_{2}) \in \sigma \cap N$. Then we can show that the numerical semigroup $H$ is of 4-dimensional toric type which is constructed from $X_{\sigma}$ and $\lambda$.

We can generalize the above cone to an $n$-dimensional cone $\sigma$ such that there exists a numerical semigroup which is constructed from $X_{\sigma}$ and some $\lambda \in \sigma \cap N$.

**Proposition 5.** Let $n \geq 4$. For any $i$ with $1 \leq i \leq n$ let $e_{i}$ be the vector in $\mathbb{R}^{n}$ whose $j$-th component is $\delta_{ij}$ where $\delta_{ij}$ is Kronecker symbol. We set

$$\sigma = \mathbb{R}_{+}e_{1} + \sum_{i=4}^{n}\mathbb{R}_{+}e_{i} + \mathbb{R}_{+}(0, 1, 0, 1, 0, \ldots, 0) + \mathbb{R}_{+}(0, 1, 0, 1, 0, \ldots, 1).$$

Consider the $n$-dimensional affine toric variety $X_{\sigma} = \text{Spec } k[\check{\sigma} \cap M]$. We note that $\check{\sigma} \cap M = \langle e_{1} \mid 1 \leq i \leq n \rangle$, where $e_{-2,3,i}$ is the vector in $\mathbb{R}^{n}$ whose second component is $-1$, third and $j$-th components are 1, and the other components are 0. Let $H_{n}$ be a numerical semigroup with

$$M(H_{n}) = \{a_{1} = n, a_{2} = n + 1, a_{3} = 2n + 3, a_{4} = 2n + 4, \ldots, a_{n-1} = 2n + n - 1\}.$$ 

Then we have relations

$$\alpha_{1}a_{1} = 4a_{1} = a_{2} + a_{n-1}, \quad \alpha_{2}a_{2} = 3a_{2} = a_{1} + a_{3}, \quad \alpha_{3}a_{3} = 2a_{3} = 2a_{2} + a_{4},$$

$$\alpha_{i}a_{i} = 2a_{i} = a_{i-1} + a_{i+1} \quad (4 \leq i \leq n - 2), \quad \alpha_{n-1}a_{n-1} = 2a_{n-1} = 3a_{1} + a_{n-2}.$$ 

Take $\lambda = (3a_{1}, a_{2}, a_{2}, a_{3}, a_{4}, \ldots, a_{n-3}, a_{n-2})$. Then $\lambda \in \sigma \cap N$. We can show that the numerical semigroup $H_{n}$ is of $n$-dimensional toric type which is constructed from $X_{\sigma}$ and $\lambda$.

A desired $3$-dimensional affine toric variety is given by the following:

**Example 6.** Let $\sigma_{1,1,2} = \mathbb{R}_{+}(1,0,0) + \mathbb{R}_{+}(0,1,0) + \mathbb{R}_{+}(1,1,2)$. Consider the $3$-dimensional affine toric variety $X_{\sigma} = \text{Spec } k[\check{\sigma} \cap M]$. We note that $\check{\sigma} \cap M = \langle (1, 0, 0), (0, 1, 0), (0, 0, 1), (2,0, -1), (1,1,-1), (0, 2, -1) \rangle$.

For any $m \in \mathbb{N}$ with $m \geq 1$, let $H$ be a numerical semigroup with

$$M(H) = \{a_{1} = 4, a_{2} = 4m + 1, a_{3} = 4m + 3, a_{4} = 4m + 2\}.$$ 

Then we have relations

$$\alpha_{1}a_{1} = (2m + 1)a_{1} = a_{2} + a_{3}, \quad \alpha_{2}a_{2} = 2a_{2} = ma_{1} + a_{4},$$

$$\alpha_{3}a_{3} = 2a_{3} = (m+1)a_{1} + a_{4}, \quad \alpha_{4}a_{4} = 2a_{4} = a_{2} + a_{3}.$$ 

Take $\lambda = (4m + 1, 4m + 3, 4m + 2)$. Then $\lambda \in \sigma \cap N$. We can show that the numerical semigroup $H$ is of $3$-dimensional toric type which is constructed from $X_{\sigma_{1,1,2}}$ and $\lambda$. 

References

