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Average number of connected components and free resolutions of Stanley-Reisner rings

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Introduction

The lower bound theorem (see, Theorem 1.1) gives not only the lower bound for the number of faces among the simplicial polytopes, but also the numerical criterion of the stacked polytopes, if the dimension of the polytope is more than three. But in the case of dimension 3, all simplicial polytopes with n vertices have the same f-vectors, more precisely, $f_1 = 3n-6$, and $f_2 = 2n-4$, where f_i is the number of i-faces. Hence, we cannot characterize the stacked polytopes by their f-vectors in this case. For this purpose, we need a subtler quantity. We introduce the following graph-theoretical invariant.

DEFINITION. Let $G = (V, E)$ be a finite graph with $\sharp(V) = n$. For $W \subset V$ we denote by G_W the induced subgraph of G by W. Let $c(G_W)$ be the number of connected components of $c(G_W)$. We define for $1 \leq i \leq n$

$$c_i(G) = \frac{1}{\binom{n}{i}} \sum_{W \subset V,\ \sharp(W)=i} c(G_W),$$

which stands for the average number of connected components of the induced subgraphs by all i-element subsets W of V.

If G is j-connected, then $c_i(G) = 1$ for $n-j+1 \leq i \leq n$. Hence, the sequence $(c_1(G), c_2(G), \ldots, c_n(G))$ can be considered as a refined concept of connectedness.

For a simplicial complex Δ, we define $c_i(\Delta) = c_i(\Delta^{(1)})$, where $\Delta^{(1)}$ is the 1-skeleton of Δ. For a simplicial polytope P, we denote by $\Delta(P)$ the boundary complex of P. We define $c_i(P) = c_i(\Delta(P))$.

Using this, we give a nemerical criterion of the stacked polytopes.

THEOREM 0.1. *Let P be a simplicial polytope with dimension d (≥ 3) and with n ($\geq d+3$) vertices. Then:*
(1) *We have*

$$c_i(P) \leq \frac{(i-1)\binom{n-d}{i}}{\binom{n}{i}} + 1, \; i = 1, 2, \ldots, n.$$

(2) *The following conditions are equivalent:*
(a) *P is a stacked polytope.*
(b) $c_i(P) = \frac{(i-1)\binom{n-d}{i}}{\binom{n}{i}} + 1$ *for all i with $2 \leq i \leq n-d$.*
(c) $c_i(P) = \frac{(i-1)\binom{n-d}{i}}{\binom{n}{i}} + 1$ *for some i with $2 \leq i \leq n-d$* if $d \geq 4$, *and for some i with $3 \leq i \leq n-d$ if $d = 3$.*

To prove the theorem we consider a minimal free resolution of the Stanley-Reisner ring $k[\Delta]$ of a simplicial complex Δ. By Hochster's formula (see Theorem 1.2), we have

$$\binom{n}{i}(c_i(\Delta) - 1) = \beta_{i-1,i}(k[\Delta]), \; i \geq 1,$$

where $\beta_{i-1,i}(k[\Delta])$ is the $(i-1, i)$-Betti number of the minimal free resolution of $k[\Delta]$. Since $k[\Delta(P)]$ is a Gorenstein graded ring which has an Artinian reduction with the weak Lefschez property (cf.[St$_1$]), we can apply Migliore-Nagel theorem [Mi-Na] for (1) and (c)$\Rightarrow$(a) in (2) if $d \geq 4$. (a)$\Rightarrow$(b) is essentially proved in [Te-Hi$_1$]. In the case $d = 3$, to show (c)$\Rightarrow$(a), we need some combinatorial argument using the induction theorem of Brücker-Eberhard. See §3 for the detailed proof.

In §4, we consider a class of simplicial complexes which are pure and strongly connected. For this class the following theorem holds:

THEOREM 0.2. *Let Δ be a $(d-1)$-dimensional pure and strongly connected simplicial complex with n vertices. Then:*
(1) *We have*

$$c_i(\Delta) \leq \frac{(i-1)\binom{n-d+1}{i}}{\binom{n}{i}} + 1, \; i = 1, 2, \ldots, n.$$

(2) *The following conditions are equivalent:*
(a) *Δ is a $(d-1)$-tree.*

(b)$c_i(P) = \frac{(i-1)\binom{n-d+1}{i}}{\binom{n}{i}} + 1$ *for all i with* $2 \le i \le n-d+1$.

(c)$c_i(P) = \frac{(i-1)\binom{n-d+1}{i}}{\binom{n}{i}} + 1$ *for some i with* $2 \le i \le n-d+1$.

§1. Preliminaries

We first give the definition according to [Br-He], [Hi], [Ho], and/or [St$_2$]. See those references for detailed information.

We first fix notation. Let $\mathbf{N}$(resp.$\mathbf{Z}$) denote the set of nonnegative integers (resp. integers).

A *simplicial complex* Δ on the *vertex set* $V = \{x_1, x_2, \ldots, x_n\}$ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \le i \le n$ and (ii) $F \in \Delta$, $G \subset F \Rightarrow G \in \Delta$. The vertex set of Δ is denoted by $V(\Delta)$. Each element F of Δ is called a *face* of Δ. We call $F \in \Delta$ an *i-face* if $\sharp(F) = i+1$ and we call a maximal face a *facet*. Let F be a face but not a facet. We call F *free* if there is a unique facet G such that $F \subset G$. We define $\partial\Delta = \bigcup_{F:\ \text{a free face of} \Delta} 2^F$ and call it the *boundary complex* of Δ. We define the *dimension* of $F \in \Delta$ to be $\dim F = \sharp(F) - 1$ and the *dimension* of Δ to be $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. We say that Δ is *pure* if every facet has the same dimension. In a $(d-1)$-dimensional pure complex Δ, we call $(d-2)$-face a *subfacet*. We say that a pure complex Δ is *strongly connected* if for any two facets F and G, there exists a sequence of facets

$$F = F_0, F_1, \ldots, F_m = G$$

such that $F_{i-1} \cap F_i$ is a subfacet for $i = 1, 2, \ldots, m$. We put $\Delta(m) = 2^{[m]}$.

Let Δ_i be a $(d-1)$-dimensional pure simplicial complex for $i = 1, 2$. If $\Delta_1 \cap \Delta_2 = 2^F$ for some F with $\dim F = d-2$, we denote $\Delta_1 \cup_F \Delta_2$ for $\Delta_1 \cup \Delta_2$. We sometimes denote $\Delta_1 \cup_* \Delta_2$ for $\Delta_1 \cup_F \Delta_2$ if we do not need to express F explicitely.

We define a $(d-1)$-tree inductively as follows.

(1)$\Delta(d)$ is a $(d-1)$-tree.

(2)If Υ is a $(d-1)$-tree, then so is $\Upsilon \cup_* \Delta(d)$.

If $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m$ are $(d-1)$-trees, we abbreviate $\Delta \cup_* \Upsilon_1 \cup_* \Upsilon_2 \cup_* \cdots \cup_* \Upsilon_m$ as $\Delta \bigcup((d-1)\text{-branches})$.

Let $f_i = f_i(\Delta)$, $0 \le i \le d-1$, denote the number of i-faces in Δ. We define $f_{-1} = 1$. We call $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ the *f-vector* of Δ. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

For a simplicial polytope P, we define $f(P) = f(\Delta(P))$ and $h(P) = h(\Delta(P))$.

A *stacked polytope* is a simplicial polytope which is obtained from a simplex by successive addition of pyramids over facets. For a d-dimensional stacked polytope P, there exists a d-tree Δ such that $\Delta(P) = \Delta$.

THEOREM 1.1 (LOWER BOUND THEOREM) (see [Br, Corollary 19.6] for the f-vector version). *Let P be a d-dimensional simplicial polytope with n vertices. Put $h(P) = (h_0, h_1, \ldots, h_d)$. Then:*
(1) *We have $h_i \geq n - d$ for $1 \leq i \leq d-1$.*
(2) *Moreover, we assume $d \geq 4$. Then the following three conditions are equivalent:*
(a) *P is a stacked polytope.*
(b) *$h_i = n - d$ for all i with $1 \leq i \leq d-1$.*
(c) *$h_i = n - d$ for some i with $2 \leq i \leq d-2$.*

Let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n-variables over a field k. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, with $\{i_1, i_2, \ldots, i_r\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k.

Next we summarize basic facts on the Hilbert series. Let k be a field and R a homogeneous k-algebra. We means a *homogeneous k-algebra R* by a noetherian graded ring $R = \bigoplus_{i\geq 0} R_i$ generated by R_1 with $R_0 = k$. In this case R can be written as a quotient algebra $k[x_1, x_2, \ldots, x_n]/I$, where $\deg x_i = 1$. In this article we always use the representatation A/I with $A = k[x_1, x_2, \ldots, x_n]$ a polynomial ring and with $I_1 = (0)$.

Let M be a graded R-module with $\dim_k M_i < \infty$ for all $i \in \mathbf{Z}$, where $\dim_k M_i$ denotes the dimension of M_i as a k-vector space.

The *Hilbert series* of M is defined by

$$F(M,t) = \sum_{i\in\mathbf{Z}} (\dim_k M_i)t^i.$$

It is well known that the Hilbert series $F(R,t)$ of R can be written in the form

$$F(R,t) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^{\dim R}},$$

where $h_0(=1), h_1, \ldots, h_s$ are integers with $e(R) := h_0 + h_1 + \cdots + h_s \geq 1$. The vector $h(R) = (h_0, h_1, \ldots, h_s)$ is called the *h-vector* of R.

We consider $k[\Delta]$ as the graded algebra $k[\Delta] = \bigoplus_{i\geq 0} k[\Delta]_i$ with $\deg x_j = 1$ for $1 \leq j \leq n$. The Hilbert series $F(k[\Delta], t)$ of a Stanley-Reisner ring $k[\Delta]$ can be written as follows:

$$\begin{aligned} F(k[\Delta],t) &= 1 + \sum_{i=1}^{d} \frac{f_{i-1}t^i}{(1-t)^i} \\ &= \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}, \end{aligned}$$

where $\dim \Delta = d-1$, $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$, and $h(\Delta) = (h_0, h_1, \ldots, h_d)$.

Let A be the polynomial ring $k[x_1, x_2, \ldots, x_n]$ over a field k. Let M be a finitely generated graded A-module and let

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h,j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0$$

be a graded minimal free resolution of M over A. We call $\beta_{i,j}(M)$ the (i,j)-Betti number of M over A. We define a *Castelnuovo-Mumford regularity* $\operatorname{reg} M$ of M by

$$\operatorname{reg} M = \max\ \{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

If a homogeneous k-algebra R is Cohen-Macaulay, we have

$$\operatorname{reg} R = \max\ \{s \mid h_s \neq 0\}.$$

The Betti numbers of the Stanley-Reisner ring can be expressed in terms of the reduced homology of some subcomplexes:

THEOREM 1.2 (Hochster's formula [Ho, Theorem 5.1]).

$$\beta_{i,j}(k[\Delta]) = \sum_{F\subset V,\ \sharp(F)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_F; k),$$

where

$$\Delta_F = \{G \in \Delta \mid G \subset F\}.$$

§2. Betti numbers of 2-linear part of free resolutions of homogeneous algebras

In this section, we consider upper bounds for Betti numbers of 2-linear part of minimal free resolutions of homogeneous k-algebras. First we consider the Cohen-Macaulay case. More or less, it seems to be known, but we include it for convenience of readers. (see e.g., [Ei-Go]).

PROPOSITION 2.1. *Let k be a field, and let R be a Cohen-Macaulay homogeneous k-algebra with codimension c (≥ 1). Then:*
(1) *We have*

$$\beta_{i,i+1}(R) \leq i\binom{c+1}{i+1}, \quad i = 1, 2, \ldots, c.$$

(2) *The following four conditions are equivalent:*
(a) *The h-vector of R is $(1, c)$.*
(b) *R has a 2-linear resolution.*
(c) $\beta_{i,i+1}(R) = i\binom{c+1}{i+1}$ *for all i with $1 \leq i \leq c$.*
(d) $\beta_{i,i+1}(R) = i\binom{c+1}{i+1}$ *for some i with $1 \leq i \leq c$.*

Proof. (1)We may assume that k is an infinite field, and R is artinian with codimension c. Put $R = A/I$ with $I_1 = 0$. We have $\beta_{i-1,i+1}(I) \leq \beta_{i-1,i+1}(\mathrm{gin} I)$, where $\mathrm{gin} I$ is a generic initial ideal of I with respect to a reverse lexicograghic order. Put $J := \mathrm{gin} I = (x^{m_1}, \ldots, x^{m_\mu})$, where $x^{m_j} = x_1^{m_{j1}} x_2^{m_{j2}} \cdots x_c^{m_{jc}}$ and $\{x^{m_1}, \ldots, x^{m_\mu}\}$ is minimal generators of J. Since J is Borel fixed, we have

$$\beta_{i-1,i+1}(J) = \dim \mathrm{Tor}\,_{i-1}^{A}(J, k)_{i+1} = \sum_{t=1}^{c} d_t \binom{t-1}{i-1}$$

where

$$d_t := \sharp\{j; \mid m_j \mid = 2, \ \max m_j = t\},$$

with $\mid m_j \mid := m_{j1} + m_{j2} + \cdots + m_{j\mu}$ and $\max m_j := \max\{i; m_{ji} \neq 0\}$(see [Gr ,Cor 1.32]).

Since $d_t \leq t$,

$$\beta_{i,i+1}(R) \leq \beta_{i-1,i+1}(J) \leq \sum_{t=1}^{c} t\binom{t-1}{i-1} = i\binom{c+1}{i+1}.$$

(2)(a)$\Rightarrow$ (b). Since h-vector of R is $(1, c)$, we have reg $R = 1$. Hence R has a 2-linear resolution.

(b) $\Rightarrow$ (a) also holds.

((a) and (b)) $\Rightarrow$ (c) follows from a simple calculation.

(c)$\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (a). We prove that if $h_2 > 0$, then $\beta_{i,i+1}(R) < i\binom{c+1}{i+1}$ for all i with $1 \le i \le c$, where $(h_0, h_1, h_2, \dots, h_s)$ is the h-vector of R. Under the same notation of the proof of (1), we have $d_c < c$, since $h_2 > 0$ and J is Borel fixed. Hence,

$$\beta_{i,i+1}(R) \le \beta_{i-1,i+1}(J) < \sum_{t=1}^{c} t\binom{t-1}{i-1} = i\binom{c+1}{i+1}.$$

Q.E.D.

Next we consider the Gorenstein case. The next proposion is just a corollary of the Migliore-Nagel theorem [Mi-Na, Theorem 8.13].

PROPOSITION 2.2. *Let k be a field of characteristic 0. Let R be a Gorenstein homogeneous k-algebra over k with codimension $c(\ge 2)$ and* reg $R \ge 3$. *Suppose its Artinian reduction has the weak Lefschetz property. Then we have*

$$\beta_{i,i+1}(R) \le i\binom{c}{i+1}, \quad i = 1, 2, \dots, c-1.$$

Furthermore, we assume that reg $R \ge 4$. *Then the following three conditions are equivalent:*
(a) *The h-vector of R is $(1, c, c, \dots, c, 1)$.*
(b) $\beta_{i,i+1}(R) = i\binom{c}{i+1}$, *for all i with $1 \le i \le c-1$.*
(c) $\beta_{i,i+1}(R) = i\binom{c}{i+1}$, *for some i with $1 \le i \le c-1$.*

Proof. Case (i). Suppose the h-vector of R is $h(R) = (1, c, c, \dots, c, 1)$. By [Mi-Na, Theorem 8.13] and Proposition 2.1 (2), we have

$$\beta_{i,i+1}(R) \le \beta_{i,i+1}(A/L) = i\binom{c}{i+1},$$

if reg $R \ge 3$, where L is the the lex-segment ideal with $h(A/L) = (1, c-1)$. Now we assume reg $R \ge 4$. By [Mi-Na, Corollary 8.14], we have

$$\beta_{i,i+1}(R) = i\binom{c}{i+1}.$$

Case (ii). Suppose the h-vector of R is $h(R) = (1, h_1, h_2, \dots, h_s)$ and that $h_1 < h_2$. Then we have

$$\beta_{i,i+1}(R) \le \beta_{i,i+1}(A/L) < i\binom{c}{i+1}.$$

by [Mi-Na, Theorem 8.13] and Proposition 2.1(2), where L is the the lex-segment ideal with $h(A/L) = (1, c-1, h_2 - h_1, \dots h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1}) \neq (1, c-1)$. Q.E.D.

§3. Proof of Theorem 0.1

In this section we fix a field k of characteristic 0. Let P be a d-dimensional simplicial polytope with n vertices. Since $k[\Delta(P)]$ is a Gorenstein homogeneous k-algebra which has an Artinian reduction with the weak Lefschetz property, we apply Proposition 2.2. Then we obtain (1). If $d \geq 4$, (c)⇒(a) is obtained by Proposition 2.2 and the Lower Bound Theorem. (a)⇒(b) in (2) is essentially proved in [Te-Hi$_1$]. To show (c)⇒(a) in the case of $d = 3$, since the boundary complex of a 3-dimensional simplicial polytope is nothing but a triangulation of a sphere, we have only to prove the following:

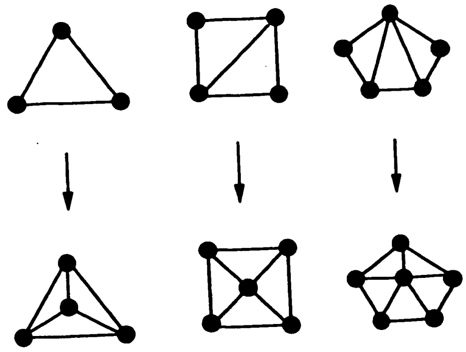
THEOREM 3.1. *Let Δ be a triangulation of S^2 with n (≥ 6) vertices. Suppose Δ is not isomorphic to the boundary complex of a stacked polytope. Then we have*

$$\beta_{i,i+1}(k[\Delta]) < i\binom{n-3}{i+1},$$

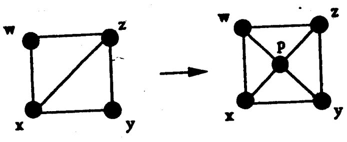
for $2 \leq i \leq n-4$.

To prove the theorem, we use:

THEOREM 3.2 (THE INDUCTION THEOREM OF BRÜCKER-EBERHARD) (cf. [Oda, p190]). *Suppose a finite triangulation Δ of S^2 is given. We get a triangulation Δ'* of S^2 *with one more vertex, if a vertex of Δ is " split into two " by one of the three steps (A), (B), (C) shown in the figures below. We can obtain any given finite triangulation of S^2 from the tetrahedral triangulation by splitting vertices finitely many times.*



LEMMA 3.3. *Let Δ be a triangulation of $\mathbf{S}^2$ on a vertex set V with n vertices. And let Δ' be a triangulation obtained from Δ by (B) in the Induction Theorem, which is indicated as below.*



Put $V' := V \cup \{p\}$ and $W := W' \setminus \{p\}$ for $W' \subset V'$.
(1) *We have $|\dim_k \tilde{H}_0(\Delta'_{W'}; k) - \dim_k \tilde{H}_0(\Delta_W; k)| \leq 1$ for $W' \subset V'$.*
(2) *$\dim_k \tilde{H}_0(\Delta'_{W'}; k) = \dim_k \tilde{H}_0(\Delta_W; k) + 1$ holds if and only if W' is one of following cases;*
(a) *$p \in W'$, $w, x, y, z \notin W'$, and $\sharp(W') \geq 2$.*
(b) *$x, z \in W'$, $p, w, y \notin W'$, and x and z are disconnected in $\Delta'_{W'}$.*
(3) *Let $n(a)_j$ (resp. $n(b)_j$) be the number of j-element subsets W' of V' which satisfy the condition (a) (resp. (b)). Then we have $n(a)_j = \binom{n-4}{j-1}$ and $n(b)_j \leq \binom{n-4}{j-2}$ for $j \geq 2$.*
(4) *Furthermore, we assume that Δ is isomorphic to the boundary complex of a stacked polytope, and that Δ' obtained by (B) is not isomorphic to the boundary complex of a stacked polytope. Then we have $n(b)_j < \binom{v-4}{j-2}$ for $j \geq 3$.*

Proof. (1) and (2) can be proved by one by one checking.

(3) As j-element subset W' satisfying (a) we can freely choose $(j-1)$ elements from $V - \{w, x, y, z\}$, which has just $(n-4)$ elements. We use similar argument for (b).

(4)Since Δ is isomorphic to the boundary complex of a stacked polytope, there exists a 3-tree Γ on the vertex set $V(\Delta)$ with $\partial\Gamma = \Delta$.

First we prove $\{w, y\} \notin \Gamma$. Assume that $\{w, y\} \in \Gamma$. Since Γ is a 3-tree, we have for all $W \subset V(\Delta)$, $\tilde{H}_i(\Gamma_W; k) = 0$ for $i \geq 1$. Hence $\{w, x, y\}, \{w, y, z\} \in \Gamma$ and $\{w, x, y, z\} \in \Gamma$. Therefore Γ can be expressed as

$$\Gamma = 2^{\{w,x,y,z\}} \cup_{\{w,x,y\}} \Gamma_1 \cup_{\{w,y,z\}} \Gamma_2,$$

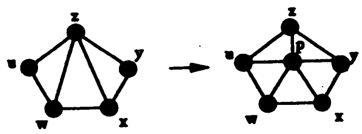
where Γ_1 and Γ_2 are 3-trees or $\{\emptyset\}$.

Put

$$\Gamma' := [2^{\{p,w,x,y\}} \cup_{\{w,x,y\}} \Gamma_1] \cup_{\{p,w,y\}} [2^{\{p,w,y,z\}} \cup_{\{w,y,z\}} \Gamma_2],$$

which is also a 3-tree. Then we have $\partial\Gamma' = \Delta'$, and Δ' is also isomorphic to the boundary complex of a stacked polytope, which is contadiction to the assumption. Hence $\{w, y\} \notin \Gamma$. There exists $q \in V(\Delta)$ such that $\{q, w, x, z\} \in \Gamma$. Hence $\{q, x\}, \{q, z\} \in \Gamma^{(1)} = \Delta^{(1)}$. For $3 \leq j \leq n-2$, choose j-elment subset $W' \subset V(\Delta')$ such that $q, x, z \in W'$ and $p, w, y \notin W'$. Then x and z are connected in $\Delta'_{W'}$ and W' does not satisfy the condition (b). Hence $n(b)_j < \binom{n-4}{j-2}$ for $j \geq 3$. Q.E.D

LEMMA 3.4. *Let Δ be a triangulation of S^2 on a vertex set V with n vertices. And let Δ' be a triangulation obtained from Δ by (C) in the Induction Theorem, which is indicated as below.*



Put $V' := V \cup \{p\}$ and $W := W' \setminus \{p\}$ for $W' \subset V'$.
(1) *We have* $|\dim_k \tilde{H}_0(\Delta'_{W'}; k) - \dim_k \tilde{H}_0(\Delta_W; k)| \leq 1$ *for* $W' \subset V'$
(2) $\dim_k \tilde{H}_0(\Delta'_{W'}; k) = \dim_k \tilde{H}_0(\Delta_W; k) + 1$ *holds if and only if W' is one of following cases;*

(a_1)$p \in W'$, $u, w, x, y, z \notin W'$, *and* $\sharp(W') \geq 2$.
(a_2)$w, z \in W'$, $p, u, x, y, \notin W'$, *and w and z are disconnected in* $\Delta'_{W'}$.
(a_3)$x, z \in W'$, $p, u, w, y \notin W'$ *and x and z are disconnected in* $\Delta'_{W'}$.
(a_4)$u, x, z \in W'$, $p, w, y \notin W'$ *and u and x are disconnected in* $\Delta'_{W'}$.
(a_5)$w, x, z \in W'$, $p, u, y \notin W'$ *and w and z are disconnected in* $\Delta'_{W'}$.
(a_6)$w, y, z \in W'$, $p, u, x \notin W'$ *and w and y are disconnected in* $\Delta'_{W'}$.

(3)*If $W \in V$ satisfies one of the following (b_1) or (b_2), then* $\dim_k \tilde{H}_0(\Delta'_{W'}; k) = \dim_k \tilde{H}_0(\Delta_W; k) - 1$ *holds;*

(b_1)$p, u, x \in W'$, $w, y, z \notin W'$ *and u and x are disconnected in* $\Delta'_{W'}$.
(b_2) $p, w, y \in W'$, $u, x, z \notin W'$ *and w and y are disconnected in* $\Delta'_{W'}$.

(4)*Let $n(a_i)_j$, $1 \leq i \leq 8$ (resp. $n(b_i)_j$, $1 \leq i \leq 2$) be the number of j-element subsets W' of V' which satisfy the condition (a_i) (resp. (b_i)). Then we have* $n(a_1)_j = \binom{n-5}{j-1}$, $n(a_2)_j \leq \binom{n-5}{j-2}$, $n(a_3)_j \leq \binom{n-5}{j-2}$, $n(a_5)_j \leq \binom{n-5}{j-3}$, $n(a_4)_j \leq n(b_1)_j$ *and* $n(a_6)_j \leq n(b_2)_j$ *for* $j \geq 3$.
(5)*Furthermore, we assume that Δ is isomorphic to the boundary complex of a stacked polytope. Then we have* $n(a_2)_j < \binom{n-5}{j-2}$ *or* $n(a_3)_j < \binom{n-5}{j-2}$.

Proof. (1),(2), and (3) follow from one by one checking.

(4)For $n(a_1)_j$, $n(a_2)_j$, $n(a_3)_j$, and $n(a_5)_j$ we can see the assertion as in Lemma 3.3 (3).

Let $A_{i,j}$, $1 \leq i \leq 6$ (resp. $B_{i,j}$, $1 \leq i \leq 2$), be the set of all j-element subsets W' of V' which satisfy the condition (a_i) (resp. (b_i)). We define the map $A_{4,j} \to B_{1,j}$ ($W' \mapsto W' \cup \{p\} \setminus \{z\}$), which is easily seen to be well-defined and injective. Then we have $n(a_4)_j \leq n(b_1)_j$ for $j \geq 3$. We can prove $n(a_6)_j \leq n(b_2)_j$ for $j \geq 3$ in the same way.

(5)There exists a 3-tree Γ on the vertex set $V(\Delta)$ with $\partial\Gamma = \Delta$. We have $\{u, w, x, z\} \notin \Gamma$ or $\{w, x, y, z\} \notin \Gamma$. As in the proof of Lemma 2.2(4), we have $\{u, x\} \notin \Gamma$ or $\{w, y\} \notin \Gamma$. We assume $\{u, x\} \notin \Gamma$. Then there exists $q \in V(\Delta)$ such that $\{q, u, w, z\} \in \Gamma$. Hence $\{q, w\}, \{q, z\} \in \Gamma^{(1)} = \Delta^{(1)}$. For $3 \leq j \leq n-2$, choose j-element subset $W' \subset V(\Delta')$ such that $q, w, z \in W'$ and $p, u, w, y \notin W'$. Then w and z are connected in $\Delta'_{W'}$ and W' does not satisfy the condition (a_2). Hence $n(a_2)_j < \binom{n-5}{j-2}$ for $j \geq 3$. Similarly, if $\{w, y\} \notin \Gamma$, then we have $n(a_3)_j < \binom{n-5}{j-2}$ for $j \geq 3$. Q.E.D

LEMMA 3.5. *Let Δ be a triangulation of $\mathbf{S}^2$ with n vertices. And let Δ' be a triangulation obtained from Δ by (A),(B), or (C) in the Induction Theorem above. Then:*

(1) *We have for $i \geq 1$,*

$$\beta_{i,i+1}(k[\Delta']) \leq \beta_{i,i+1}(k[\Delta]) + \beta_{i-1,i}(k[\Delta]) + \binom{n-3}{i}.$$

(2)*Furthermore, we assume that Δ is isomorphic to the boundary complex of a stacked polytope, and that Δ' obtained by (B)or (C) is not isomorphic to the boundary complex of a stacked polytope. Then we have for $i \geq 1$,*

$$\beta_{i,i+1}(k[\Delta']) < \beta_{i,i+1}(k[\Delta]) + \beta_{i-1,i}(k[\Delta]) + \binom{n-3}{i}.$$

Proof. (1)In the case of (A), the assertion is proved in [Te-Hi$_1$, Lemma 2.3.1] with equality. By Hochster's formula we have

$$\begin{aligned}
\beta(k[\Delta'])_{i,i+1} &= \sum_{W' \subset V',\ \sharp(W')=i+1} \dim_k \tilde{H}_0(\Delta'_{W'}; k) \\
&= \sum_{v \notin W' \subset V',\ \sharp(W')=i+1} \dim_k \tilde{H}_0(\Delta'_{W'}; k) \\
&\quad + \sum_{v \in W' \subset V',\ \sharp(W')=i+1} \dim_k \tilde{H}_0(\Delta'_{W'}; k).
\end{aligned}$$

Hence, for the case (B) by Lemma 3.3(3) we have

$$\begin{aligned}\beta_{i,i+1}(k[\Delta']) &\leq \sum_{W\subset V,\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta_W;k) \\ &\quad + \sum_{W\subset V,\ \sharp(W)=i} \dim_k \tilde{H}_0(\Delta_W;k) + \binom{n-4}{i} + \binom{n-4}{i-1} \\ &= \beta_{i,i+1}(k[\Delta]) + \beta_{i-1,i}(k[\Delta]) + \binom{n-3}{i}\end{aligned}$$

as desired.

For the case (C), similarly, by Lemma 3.4(4) we have

$$\begin{aligned}\beta_{i,i+1}(k[\Delta']) &\leq \sum_{W\subset V,\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta_W;k) \\ &\quad + \sum_{W\subset V,\ \sharp(W)=i} \dim_k \tilde{H}_0(\Delta_W;k) + \binom{n-5}{i} + 2\binom{n-5}{i-1} + \binom{n-5}{i-2} \\ &= \beta_{i,i+1}(k[\Delta]) + \beta_{i-1,i}(k[\Delta]) + \binom{n-3}{i}.\end{aligned}$$

(2)Apply Lemmas 3.3(4) and 3.4(5) instead of Lemmas 3.3(3) and 3.4(4) in the above proof. Q. E. D.

Proof of Theorem 3.1. We give a proof by induction n. Thanks to Lemma 3.5, we have

$$\begin{aligned}\beta_{i,i+1}(k[\Delta]) &< i\binom{n-4}{i+1} + (i-1)\binom{n-4}{i} + \binom{n-4}{i} \\ &= i(\binom{n-4}{i+1} + \binom{n-4}{i}) \\ &= i\binom{n-3}{i+1}\end{aligned}$$

as required. Q. E. D.

§4. Proof of Theorem 0.2

In this section we consider upper bounds for the Betti numbers of minimal free resolutions of the Stanley-Reisner rings of pure and strongly connected simplicial complexes.

In the case of the Stanley-Reisner rings, we can take a class of pure and strongly connected complexes, which is a wider class than one of Cohen-Macaulay complexes, to obtain the same upper bounds. Compare the following Thorem 4.1 with Propositon 2.1.

We know that every $(d-1)$-dimensional pure and strongly connected simplicial complex can be constructed from the $(d-1)$-dimensional elementary simplex $\Delta(d)$ by a succession

$$\Delta(d) = \Delta_1 \to \Delta_2 \to \cdots \to \Delta_{f_{d-1}}$$

of one of the following two operations :
(1)$\Delta_{i+1} = \Delta_i \cup_{F'} 2^F$, where $x \notin V(\Delta_i)$, F' is a subfacet of Δ_i and $F = F' \cup \{x\}$.
(2)$\Delta_{i+1} = (\Delta_i \cup_{F'} 2^F)(x \to y)$, where $x \notin V(\Delta_i)$, F' is a subfacet of Δ_i and $y \in V(\Delta_i)$ such that x and y are separated and $F = F' \cup \{x\}$(cf.[Te]).

Now we prove the main result in this section.

THEOREM 4.1. *Let Δ be a $(d-1)$-dimensional pure and strongly connected simplicial complex with n vertices. Suppose Δ is not a simplex. Then:*
(1) *We have*

$$\beta_{i,i+1}(k[\Delta]) \le i\binom{n-d+1}{i+1}.$$

(2) *The following four conditions are equivalent:*
(a)*Δ is a $(d-1)$-tree.*
(b)*I_Δ has a 2-linear resolution.*
(c) $\beta_{i,i+1}(k[\Delta]) = i\binom{n-d+1}{i+1}$ *for all i with $1 \le i \le n-d$.*
(d) $\beta_{i,i+1}(k[\Delta]) = i\binom{n-d+1}{i+1}$ *for some i with $1 \le i \le n-d$.*

Proof. (1) Let V be the vertex set of Δ. We prove the theorem by induction on the number f_{d-1} of facets in Δ.

First if $f_{d-1} = 2$, then $k[\Delta]$ is a hypersurface of degree 2. In this case the theorem is clear.

Suppose $f_{d-1} \ge 3$. Then there exists a facet $F \in \Delta$ such that

$$\Delta' := \{H \in \Delta \mid H \subset G \text{ for some facet } G(\ne F) \in \Delta\}$$

is pure and strongly connected. Denote by V' the vertex set of Δ' and by f'_{d-1} the number of facets in Δ'. There are two cases (cf.[Te]).

Case(i) $V \neq V'$. Put $V \setminus V' = \{x\}$. Then Δ can be expressed as $\Delta = \Delta' \cup_{F'} 2^F$, where F' is a subfacet of Δ and $F = F' \cup \{x\}$. Let W be a subset of V with $\sharp(W) \geq 2$. Put $W' = W \setminus \{x\}$. If $x \in W$ and $W \cap F' = \emptyset$, then

$$\dim_k \tilde{H}_0(\Delta_W; k) = \dim_k \tilde{H}_0(\Delta'_{W'}; k) + 1.$$

Otherwise,

$$\dim_k \tilde{H}_0(\Delta_W; k) = \dim_k \tilde{H}_0(\Delta'_{W'}; k).$$

By Hochster's formula, we have

$$\begin{aligned}
\beta_{i,i+1}(k[\Delta]) &= \sum_{x \notin W \subset V,\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta_W; k) \\
&\quad + \sum_{x \in W \subset V,\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta_W; k) \\
&= \sum_{W' \subset V',\ \sharp(W')=i+1} \dim_k \tilde{H}_0(\Delta'_{W'}; k) \\
&\quad + \sum_{W' \subset V',\ \sharp(W')=i} \dim_k \tilde{H}_0(\Delta'_{W'}; k) + \binom{n-d}{i} \\
&= \beta_{i,i+1}(k[\Delta']) + \beta_{i-1,i}(k[\Delta']) + \binom{n-d}{i} \\
&\leq i\binom{n-d}{i+1} + (i-1)\binom{n-d}{i} + \binom{n-d}{i} \\
&= i\{\binom{n-d}{i+1} + \binom{n-d}{i}\} \\
&= i\binom{n-d+1}{i+1}.
\end{aligned}$$

Case(ii) $V = V'$. In this case Δ can be expressed as

$$\Delta = (\Delta' \cup_{F'} 2^F)(x \to y),$$

where $x \notin V'$, and F' is a subfacet of Δ' and $y \in V'$ such that x and y are separated, and that $F = F' \cup \{x\}$. Since $\Delta' \subset \Delta$ we have $\Delta'_W \subset \Delta_W$ for all $W \subset V$. Then we have $\dim \tilde{H}_0(\Delta_W; k) \leq \dim \tilde{H}_0(\Delta'_W; k)$. Then we have

$$\beta_{i,i+1}(k[\Delta]) \leq \beta_{i,i+1}(k[\Delta']) \leq i\binom{n-d+1}{i+1}.$$

(2)(a) $\Rightarrow$ (b)is proved in [Fr].

(b)⇒ (c). Since Δ is pure and strongly connected and $(d-1)$-dimensional, it is $(d-1)$-connected. Hence $\beta_{n-d+1,n-d+2}(k[\Delta]) = 0$. Since I_Δ has a 2-linear resolution, $k[\Delta]$ is Cohen-Macaulay. When $k[\Delta]$ is Cohen-Macaulay and that $k[\Delta]$ has a 2-linear resolution, we know $\beta_{i,i+1}(k[\Delta]) \leq i\binom{n-d+1}{i+1}$ for all i with $1 \leq i \leq n-d$ by Proposition 2.1.

(c) ⇒ (d) is obvious.

(d)⇒ (a). We prove that if Δ is not a $(d-1)$-tree, then $\beta_{i,i+1}(k[\Delta]) < i\binom{n-d+1}{i+1}$ for all i with $1 \leq i \leq n-d$.

We may assume that Δ' is a $(d-1)$-tree by argument in the proof of (1), where Δ' is defined in the proof of (1). Since Δ is not a $(d-1)$-tree, Δ can be expressed as

$$\Delta = (\Delta' \cup_{F'} 2^F)(x \to y),$$

as in the proof of (1) case(ii). There exists a sequence of facets of $\Delta' \cup_{F'} 2^F$,

$$y \in F_1, F_2, \ldots, F_m = F$$

such that $F_p \neq F_q$ for $1 \leq p < q \leq m$ and $y \notin F_2$ and $F_j \cap F_{j+1}$ are subfacets for $1 \leq j \leq m-1$ and $F' = F_{m-1} \cap F_m$. Put $G = F_1 \cap F_2$. Since x and y are separated, then $m \geq 3$, hence, $G \neq F'$. Fix $z \in F' \backslash G$. For $2 \leq j \leq n-d+1$, choose $W \subset V$ such that $y, z \in W$, $W \cap G = \emptyset$, and $\sharp(W) = j$. Hence y and z are disconnected in Δ'_W, but connected in Δ_W . Therefore, we have $\dim \tilde{H}_0(\Delta_W; k) < \dim \tilde{H}_0(\Delta'_W; k)$. By Hochster's formula we have

$$\beta_{i,i+1}(k[\Delta]) < \beta_{i,i+1}(k[\Delta']) = i\binom{n-d+1}{i+1}.$$

Q.E.D.

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