On the structure of weak interlaced bilattice $\mathcal{K}(L)$

島根大学総合理工学部 近藤通朗 (Michiro Kondo)

Department of Mathematics and Computer Science
Shimane University

Abstract

We study fundamental properties of weak interlaced bilattices $\mathcal{K}(L)$ and show that for any weak interlaced bilattice $\mathcal{W}$ there exists a lattice $L$ such that $\mathcal{W}$ can be embedded into a weak interlaced bilattice $\mathcal{K}(L)$. Hence, any interlaced bilattice can be embedded into the weak interlaced bilattice $\mathcal{K}(L)$ for some lattice $L$.

1 Introduction

It is well-known that the Kleene's 3-valued logic plays an important role in the field of multiple-valued logics. The logic has three values $\text{false}$, $\text{true}$, and $\perp$ (unknown) as truth values. These values have two informal orderings concerning "amount of knowledge" and "degree of truth". For example, if we think of a certain proposition such as Goldbach's conjecture assigned $\perp$ as truth value, then it is possible that we can conclude the truth value of the proposition as $\text{true}$ or $\text{false}$ with increasing knowledge. Thus in the ordering of knowledge, $\perp$ is smaller than $\text{true}$ and $\text{false}$. A sentence with $\perp$ is between $\text{false}$ and $\text{true}$ in the ordering of degree of truth. In this way it can be considered that the three valued logic has two orderings. Belnap ([2]), Ginsberg ([5]), and others proposed concept of a bilattice which has two orderings and proved some fundamental results ([1, 3, 4]). It is shown by Fitting ([3]) that bilattices can give a uniform semantics for many lanuages of logic programming. Since then the theory of bilattices is a hot reserach field.

On the other hand, as in Fuzzy logics, a truth value can be taken as a closed interval $[a, b]$. Let $L$ be a lattice and $\mathcal{K}(L)$ be the set of all closed intervals of $L$. In this case we also define two orderings. For $[a, b], [c, d] \in \mathcal{K}(L)$, if $[a, b] \subseteq [c, d]$ then the knowledge in $[a, b]$ is greater than that in $[c, d]$. Thus we set $[a, b] \subseteq_k [c, d]$ if $[a, b] \subseteq [c, d]$. Likewise we also define $[a, b] \subseteq_t [c, d]$ if $a \leq c$ and $b \leq d$, because $[c, d]$ is greater than $[a, b]$ in the ordering degree of truth. The structure $\mathcal{K}(L) = < \mathcal{K}(L), \subseteq_t, \subseteq_k >$ which precise definition is given below has the property of weak interlaced bilattice.

In [3, 4], Fitting, Font and Moussavi have investigated the structrure of $\mathcal{K}(L)$ and proved that if $L$ is a bounded lattice, then $\mathcal{K}(L)$ is a weak
interlaced bilattice ([4]). Now does the converse hold?, that is, is there a lattice \( L \) such that \( \mathcal{W} \cong K(L) \) for every weak interlaced bilattice \( \mathcal{W} \)?

Clearly we answer ”No”. Because we have a simple counterexample. Let \( B \) be a weak interlaced bilattice with 5 elements, for example, a set \( \{0, p, \perp, q, 1\} \) with \( 0 \leq_t p \leq_t \perp \leq_t q \leq_t 1 \), \( \perp \leq_k p \leq_k 0 \) and \( \perp \leq_k q \leq_k 1 \).

It is obvious that \( B \) is a weak interlaced bilattice. Suppose that there is a lattice \( L \) such that \( B \cong K(L) \).

If \( |L| \geq 3 \), then there exists an element \( a \in L \) such that \( 0 < a < 1 \). For that element we have \([0, 0], [0, a], [0, 1], [a, 1], [a, a], [1, 1] \in K(L) \) and \(|K(L)| \geq 6 \). Since \(|B| = 5 \), it must be \(|L| \leq 2 \). But, in this case, we have \(|K(L)| \leq 3 \). This means that there is no lattice \( L \) such that \( B \cong K(L) \).

Now we settle a more general question.

**Question**: For every weak interlaced bilattice \( \mathcal{W} \), is there a lattice \( L \) such that \( \mathcal{W} \) can be embedded to \( K(L) \)?

In this note we study properties of \( K(L) \) and answer the question.

## 2 Definition of \( K(L) \)

We define a structure \( K(L) \) for any lattice \( L \). Let \( L = (L, \leq) \) be a lattice and \( K(L) \) be the set of all closed intervals of \( L \), that is,

\[
K(L) = \{ [a, b] | a \leq b, a, b \in L \}
\]

\([a, b] = \{ x | a \leq x \leq b \} \).

For any \([a, b], [c, d] \in K(L)\), we define two orderings \( \subseteq_t, \subseteq_k \) on \( K(L) \) as follows:

\[
[a, b] \subseteq_t [c, d] \iff a \leq c, b \leq d
\]

\[
[a, b] \subseteq_k [c, d] \iff a \leq c, b \geq d
\]

We set \( K(L) = \langle K(L), \subseteq_t, \subseteq_k \rangle \). It is obvious from definition that \([0, 0] ([1, 1]) \) is the minimum (maximum) element with respect to \( \subseteq_t \). On the other hand, while \([0, 1] \) is the minimum element, there is no maximum element with respect to the ordering \( \subseteq_k \). This means that \( K(L) \) is a lattice with respect to \( \subseteq_t \) and is a semi-lattice concering \( \subseteq_k \). Four operators \( \cap_t, \cup_t, \cap_k, \cup_k \) are
defined by

\[ \inf_{\leq_t} \{a, b\} = a \land_t b \]
\[ \sup_{\leq_t} \{a, b\} = a \lor_t b \]
\[ \inf_{\leq_k} \{a, b\} = a \land_k b \]
\[ \sup_{\leq_k} \{a, b\} = a \lor_k b \]

(if it is defined)

Next we give definitions of an interlaced bilattice and of a weak interlaced bilattice. A relational system \( < B, \leq_t, \leq_k > \) is called an interlaced bilattice if it satisfies

1. \( B \) is a non-empty set
2. \( < B, \leq_t >, < B, \leq_k > \) are bounded lattices and satisfy
   
   (a) \( x \leq_t y \Rightarrow x \otimes z \leq_t y \otimes z, x \oplus z \leq_t y \oplus z \)
   
   (b) \( x \leq_k y \Rightarrow x \land z \leq_k y \land z, x \lor z \leq_k y \lor z \)

where four operators are defined by

\[ \inf_{\leq_t} \{x, y\} = x \land y \]
\[ \sup_{\leq_t} \{x, y\} = x \lor y \]
\[ \inf_{\leq_k} \{x, y\} = x \otimes y \]
\[ \sup_{\leq_k} \{x, y\} = x \oplus y \]

By \( 0(1) \), we mean the minimum (maximum) element with respect to the ordering \( \leq_t \). We also denote by \( \perp (\top) \) the minimum (maximum) element concerning to \( \leq_k \).

A map \( \neg \) from \( B \) into itself is called a negation if

\[ x \leq_t y \Rightarrow \neg y \leq_t \neg x \]
\[ x \leq_k y \Rightarrow \neg x \leq_k \neg y \]
\[ \neg \neg x = x. \]

For lattices \( L_1 =< L_1, \land_1, \lor_1 > \) and \( L_2 =< L_2, \land_2, \lor_2 > \), we define operations \( \land, \lor, \otimes, \oplus \) on the product \( L_1 \times L_2 \): For \( (a, b), (c, d) \in L_1 \times L_2 \),

\[ (a, b) \land (c, d) = (a \land_1 c, b \lor_2 d) \]
\[ (a, b) \lor (c, d) = (a \lor_1 c, b \land_2 d) \]
\[ (a, b) \otimes (c, d) = (a \land_1 c, b \land_2 d) \]
\[ (a, b) \oplus (c, d) = (a \lor_1 c, b \lor_2 d). \]
The structure $L_1 \odot L_2 = \langle L_1 \times L_2, \land, \lor, \otimes, \oplus >$ is called a Ginsberg product. There are some fundamental results about the structure:

**Proposition 1 (Fitting).** If $L_1, L_2$ are bounded lattices then the Ginsberg product $L_1 \odot L_2 = \langle L_1 \times L_2, \land, \lor, \otimes, \oplus >$ is an interlaced bilattice. Especially, $L \odot L$ is an interlaced bilattice with negation $\neg$, where $\neg$ is defined by $\neg(a, b) = (b, a)$.

It is proved that the converse holds by Avron ([1]).

**Proposition 2 (Avron).** For any interlaced bilattice $B$, there are bounded lattices $L_1, L_2$ such that $B \cong L_1 \odot L_2$. In particular, for any interlaced bilattice $B$ with negation, there is a bounded lattice $L$ such that $B \cong L \odot L$.

It is clear from definition that orderings $\sqsubseteq_t, \sqsubseteq_k$ on $\mathcal{K}(L)$ are the same as $\leq_t, \leq_k$ on Ginsberg product $L \odot L$, respectively:

\[
\begin{align*}
\sqsubseteq_t \text{ in } \mathcal{K}(L) & \iff \leq_t \text{ in } L \odot L \\
\sqsubseteq_k \text{ in } \mathcal{K}(L) & \iff \leq_k \text{ in } L \odot L
\end{align*}
\]

Hence in the following we use the same symbols $\land, \lor, \otimes, \oplus$ in $\mathcal{K}(L)$ and in $L \odot L$.

Next we give a definition of a weak interlaced bilattice according to Font ([4]). A structure $W = \langle W, \leq_t, \leq_k >$ is called a weak interlaced bilattice if

1. $\langle W, \leq_t >$ : lattice
2. $\langle W, \leq_k >$ : meet semilattice
3. $a \leq_k b, c \leq_k d \implies a \land c \leq_k b \land d, a \lor c \leq_k b \lor d$
4. $a \leq_t b, c \leq_t d \implies a \otimes c \leq_t b \otimes d,$
5. $a \leq_t b, c \leq_t d \implies a \oplus c \leq_t b \oplus d$ if $a \oplus c$ and $b \oplus d$ exist.

3 Properties of weak interlaced bilattices

For any weak interlaced bilattice $W$, if we define

\[
\begin{align*}
L_1 &= \{ x \in W \mid x \leq_k 0 \} = [\bot, 0]_k \\
L_2 &= \{ x \in W \mid x \leq_k 1 \} = [\bot, 1]_k
\end{align*}
\]

then we have
Proposition 3.

\[ L_1 = [\bot, 0]_k = [0, \bot]_t \]
\[ L_2 = [\bot, 1]_k = [\bot, 1]_t \]

Proof. Let \( x \in [\bot, 0]_k \). Since \( \bot \leq_k x \leq_k 0 \), we have \( \bot \vee \bot \leq_k x \vee \bot \leq_k 0 \vee \bot \) by definition of weak interlaced bilattice. From \( \bot \vee \bot = 0 \vee \bot = \bot \), it follows that \( x \vee \bot = \bot \) and hence that \( x \leq_t \bot \). This means \( [\bot, 0]_k \subseteq [0, \bot]_t \).

Conversely, suppose \( x \in [0, \bot]_t \). If we put \( u = 0 \otimes x \), then it is clear that \( u \leq_k 0 \) and \( u \leq_k x \). Since \( 0 \leq_t x \), we have \( 0 \otimes x \leq_t x \otimes x = x \) and hence \( u \leq_t x \). It follows from \( \bot \leq_k u \) that \( x \wedge \bot \leq_k x \wedge u \). Since \( x \leq_t \bot \), we also have \( x \wedge \bot = x \). On the other hand, since \( u \leq_t x \), we get \( u \wedge x = u \). Theses imply that \( x \leq_k u \) and hence that \( x = u \). Thus we have \( x \leq_k 0 \). Namely, we have \( [0, \bot]_t \subseteq [\bot, 0]_k \).

The second equation can be proved similarly. \( \square \)

The result implies that \( L_1 \) and \( L_2 \) are lattices with ordering \( \leq_1 \) and \( \leq_2 \) in \( B \), respectively, where \( \leq_1 \) and \( \leq_2 \) are defined by

\[ \leq_1 = \leq_t = \geq_k \]
\[ \leq_2 = \leq_t = \leq_k \]

Thus we can consider the Ginsberg product \( L_1 \odot L_2 \), which becomes an interlaced bilattice. Moreover we can prove

Proposition 4. Let \( \mathcal{W} \) be any weak interlaced bilattice. For any \( x \in \mathcal{W} \), we have

\[ x = (x \otimes 0) \oplus (x \otimes 1) = (x \wedge \bot) \vee (x \vee \bot) \]

Proof. See Avron [1] Cor.3.8 \( \square \)

Now we investigate a relation between a weak interlaced bilattice \( \mathcal{W} \) and an interlaced bilattice \( L_1 \odot L_2 \) constructed by \( \mathcal{W} \).

Lemma 1. A map \( \xi : \mathcal{W} \to L_1 \times L_2 \) defined by \( \xi(x) = (x \otimes 1, x \otimes 0) = (x \vee \bot, x \wedge \bot) \) is an embedding.

This means that

Theorem 1. Any weak interlaced bilattice can be embedded into an interlaced bilattice.
4 Answer to the question

In this section we give a positive answer to the question above. Since any weak interlaced bilattice $\mathcal{W}$ can be embedded to an interlaced bilattice, it suffices to show that any interlaced bilattice of a form $L_1 \circ L_2$ is embeddable into a weak interlaced bilattice $\mathcal{K}(L)$ for some lattice $L$. Because, from proposition 2, every interlaced bilattice has a form of $L_1 \circ L_2$ for some lattices $L_1, L_2$. Let $L_1 \circ L_2$ be any interlaced bilattice and $L$ be a set $(L_1 \times \{0\}) \cup (L_2 \times \{1\})$. We define an order $\sqsubseteq$ on $L$. For any element $(a, i), (b, j) \in L$, we define

$$(a, i) \sqsubseteq (b, j) \iff i < j \text{ or } i = j \text{ and } a \leq b$$

It is easy to show that the relation $\sqsubseteq$ is a partially order on $L$ and that

$$(a, i) \land (b, j) = \inf\{(a, i), (b, j)\} = \begin{cases} (a \wedge b, i) & \text{if } i = j \\ (a, i) & \text{if } i < j \\ (b, j) & \text{if } i > j \end{cases}$$

$$(a, i) \lor (b, j) = \sup\{(a, i), (b, j)\} = \begin{cases} (a \vee b, i) & \text{if } i = j \\ (b, j) & \text{if } i < j \\ (a, i) & \text{if } i > j \end{cases}$$

Hence $L$ is a lattice with this order. Let $\mathcal{K}(L)$ be the set of all elements $[(a, i), (b, j)]$ such that $(a, i) \sqsubseteq (b, j)$ for $(a, i), (b, j) \in L$. In this case, four operators $\land, \lor, \otimes, \oplus$ on $\mathcal{K}(L)$ are defined as follows:

$[[a, i), (b, j)] \land [(a', i'), (b', j')] = [(a, i) \land (a', i'), (b, j) \land (b', j')]$

$[[a, i), (b, j)] \lor [(a', i'), (b', j')] = [(a, i) \lor (a', i'), (b, j) \lor (b', j')]$

$[[a, i), (b, j)] \otimes [(a', i'), (b', j')] = [(a, i) \land (a', i'), (b, j) \lor (b', j')]$

$[[a, i), (b, j)] \oplus [(a', i'), (b', j')] = [(a, i) \lor (a', i'), (b, j) \land (b', j')]$

Of course, the last equation is defined when $(a, i) \lor (a', i') \leq (b, j) \land (b', j')$. Now we define a map $\xi : L_1 \circ L_2 \to \mathcal{K}(L)$ by

$$\xi(a, b) = [(a, 0), (b, 1)]$$

It is obvious that $\xi$ is well-defined and injective. We only show that $\xi$ is a homomorphism. We only think of two cases. For the case of $(a, b) \land (a', b')$, we have
\[ \xi((a, b) \land (a', b')) = \xi(a \land a', b \lor b') \]
\[ = [(a \land a', 0), (b \lor b', 1)] \]
\[ = [[(a, 0) \land (a', 0), (b, 1) \lor (b', 1)] \]
\[ = [(a, 0), (b, 1)] \otimes [(a', 0), (b', 1)] \]
\[ = \xi(a, b) \otimes \xi(a', b') \]

For another case of \((a, b) \oplus (a', b')\), we also have

\[ \xi((a, b) \oplus (a', b')) = \xi(a \lor a', b \lor b') \]
\[ = [(a \lor a', 0), (b \lor b', 1)] \]
\[ = [[(a, 0) \lor (a', 0), (b, 1) \lor (b', 1)] \]
\[ = [(a, 0), (b, 1)] \lor [(a', 0), (b', 1)] \]
\[ = \xi(a, b) \lor \xi(a', b') \]

Hence the map \(\xi : L_1 \odot L_2 \rightarrow K(L)\) is an embedding, that is,

**Theorem 2.** For every interlaced bilattice \(L_1 \odot L_2\), there exists a lattice \(L\) such that it is embedded into a weak interlaced bilattice \(K(L)\).

From these results, we have have a main theorem.

**Theorem 3.** Every interlaced bilattice \(\mathcal{W}\) can be embedded into a weak interlaced bilattice \(K(L)\) for some lattice \(L\).

**References**


Michiro Kondo
Current Address:
e-mail: kondo@sie.dendai.ac.jp
School of Information Environment
Tokyo Denki University, Inzai, 270-1382
JAPAN