On the structure of weak interlaced bilattice $\mathcal{K}(L)$ (Algorithms in Algebraic Systems and Computation Theory)

Author(s)
Kondo, Michiro

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On the structure of weak interlaced bilattice $\mathcal{K}(L)$

Department of Mathematics and Computer Science
Shimane University

Abstract

We study fundamental properties of weak interlaced bilattices $\mathcal{K}(L)$ and show that for any weak interlaced bilattice $\mathcal{W}$ there exists a lattice $L$ such that $\mathcal{W}$ can be embedded into a weak interlaced bilattice $\mathcal{K}(L)$. Hence, any interlaced bilattice can be embedded into the weak interlaced bilattice $\mathcal{K}(L)$ for some lattice $L$.

1 Introduction

It is well-known that the Kleene's 3-valued logic plays an important role in the field of multiple-valued logics. The logic has three values $false$, $true$, and $\bot$ (unknown) as truth values. These values have two informal orderings concerning "amount of knowledge" and "degree of truth". For example, if we think of a certain proposition such as Goldbach's conjecture assigned $\bot$ as truth value, then it is possible that we can conclude the truth value of the proposition as $true$ or $false$ with increasing knowledge. Thus in the ordering of knowledge, $\bot$ is smaller than $true$ and $false$. A sentence with $\bot$ is between $false$ and $true$ in the ordering of degree of truth. In this way it can be considered that the three valued logic has two orderings. Belnap ([2]), Ginsberg([5]), and others proposed concept of a bilattice which has two orderings and proved some fundamental results ([1, 3, 4]). It is shown by Fitting ([3]) that bilattices can give a uniform semantics for many languages of logic programming. Since then the theory of bilattices is a hot research field.

On the other hand, as in Fuzzy logics, a truth value can be taken as a closed interval $[a, b]$. Let $L$ be a lattice and $\mathcal{K}(L)$ be the set of all closed intervals of $L$. In this case we also define two orderings. For $[a, b], [c, d] \in \mathcal{K}(L)$, if $[a, b] \subseteq [c, d]$ then the knowledge in $[a, b]$ is greater than that in $[c, d]$. Thus we set $[a, b] \subseteq_k [c, d]$ if $[a, b] \subseteq [c, d]$. Likewise we also define $[a, b] \subseteq_t [c, d]$ if $a \leq c$ and $b \leq d$, because $[c, d]$ is greater than $[a, b]$ in the ordering degree of truth. The structure $\mathcal{K}(L) = < \mathcal{K}(L), \subseteq_t, \subseteq_k >$ which precise definition is given below has the property of weak interlaced bilattice.

In [3, 4], Fitting, Font and Moussavi have investigated the structure of $\mathcal{K}(L)$ and proved that if $L$ is a bounded lattice, then $\mathcal{K}(L)$ is a weak
interlaced bilattice ([4]). Now does the converse hold?, that is, is there a lattice $L$ such that $\mathcal{W} \cong \mathcal{K}(L)$ for every weak interlaced bilattice $\mathcal{W}$?

Clearly we answer "No". Because we have a simple counterexample. Let $B$ be a weak interlaced bilattice with 5 elements, for example, a set $\{0, p, \perp, q, 1\}$ with $0 \leq_t p \leq_t \perp \leq_t q \leq_t 1$, $\perp \leq_k p \leq_k 0$ and $\perp \leq_k q \leq_k 1$. It is obvious that $B$ is a weak interlaced bilattice. Suppose that there is a lattice $L$ such that $B \cong \mathcal{K}(L)$. If $|L| \geq 3$, then there exists an element $a \in L$ such that $0 < a < 1$. For that element we have $[0, 0], [0, a], [0, 1], [a, 1], [a, a], [1, 1] \in \mathcal{K}(L)$ and $|\mathcal{K}(L)| \geq 6$. Since $|B| = 5$, it must be $|L| \leq 2$. But, in this case, we have $|\mathcal{K}(L)| \leq 3$. This means that there is no lattice $L$ such that $B \cong \mathcal{K}(L)$.

Now we settle a more general question.

**Question**: For every weak interlaced bilattice $\mathcal{W}$, is there a lattice $L$ such that $\mathcal{W}$ can be embedded to $\mathcal{K}(L)$?

In this note we study properties of $\mathcal{K}(L)$ and answer the question.

## 2 Definition of $\mathcal{K}(L)$

We define a structure $\mathcal{K}(L)$ for any lattice $L$. Let $L = (L, \leq)$ be a lattice and $K(L)$ be the set of all closed intervals of $L$, that is,

$$K(L) = \{[a, b] | a \leq b, a, b \in L\}$$

$$[a, b] = \{x | a \leq x \leq b\}.$$  

For any $[a, b], [c, d] \in K(L)$, we define two orderings $\subseteq_t, \subseteq_k$ on $K(L)$ as follows:

$$[a, b] \subseteq_t [c, d] \iff a \leq c, b \leq d$$

$$[a, b] \subseteq_k [c, d] \iff a \leq c, b \geq d$$

We set $\mathcal{K}(L) = \langle K(L), \subseteq_t, \subseteq_k \rangle$. It is obvious from definition that $[0, 0]$ ([1, 1]) is the minimum (maximum) element with respect to $\subseteq_t$. On the other hand, while $[0, 1]$ is the minimum element, there is no maximum element with respect to the ordering $\subseteq_k$. This means that $\mathcal{K}(L)$ is a lattice with respect to $\subseteq_t$ and is a semi-lattice concerning $\subseteq_k$. Four operators $\cap_t, \cup_t, \cap_k, \cup_k$ are
defined by

\[
\begin{align*}
\inf_{\leq t}\{a, b\} &= a \land_t b \\
\sup_{\leq t}\{a, b\} &= a \lor_t b \\
\inf_{\leq k}\{a, b\} &= a \land_k b \\
\sup_{\leq k}\{a, b\} &= a \lor_k b
\end{align*}
\] (if it is defined)

Next we give definitions of an interlaced bilattice and of a weak interlaced bilattice. A relational system \(<B, \leq_t, \leq_k>\) is called an interlaced bilattice if it satisfies

1. \(B\) is a non-empty set
2. \(<B, \leq_t>, <B, \leq_k>\) are bounded lattices and satisfy

   (a) \(x \leq_t y \implies x \otimes z \leq_t y \otimes z, x \oplus z \leq_t y \oplus z\)

   (b) \(x \leq_k y \implies x \land z \leq_k y \land z, x \lor z \leq_k y \lor z\)

where four operators are defined by

\[
\begin{align*}
\inf_{\leq t}\{x, y\} &= x \land y \\
\sup_{\leq t}\{x, y\} &= x \lor y \\
\inf_{\leq k}\{x, y\} &= x \otimes y \\
\sup_{\leq k}\{x, y\} &= x \oplus y
\end{align*}
\]

By \(0(1)\), we mean the minimum (maximum) element with respect to the ordering \(\leq_t\). We also denote by \(\bot(\top)\) the minimum (maximum) element concerning to \(\leq_k\).

A map \(-\) from \(B\) into itself is called a negation if

\[
\begin{align*}
x \leq_t y &\implies -y \leq_t -x \\
x \leq_k y &\implies -x \leq_k -y \\
-(-x) &= x.
\end{align*}
\]

For lattices \(L_1 =<L_1, \land_1, \lor_1>\) and \(L_2 =<L_2, \land_2, \lor_2>\), we define operations \(\land, \lor, \otimes, \oplus\) on the product \(L_1 \times L_2\): For \((a, b), (c, d) \in L_1 \times L_2\),

\[
\begin{align*}
(a, b) \land (c, d) &= (a \land_1 c, b \lor_2 d) \\
(a, b) \lor (c, d) &= (a \lor_1 c, b \land_2 d) \\
(a, b) \otimes (c, d) &= (a \land_1 c, b \land_2 d) \\
(a, b) \oplus (c, d) &= (a \lor_1 c, b \lor_2 d).
\end{align*}
\]
The structure $L_1 \odot L_2 =< L_1 \times L_2, \wedge, \vee, \otimes, \oplus >$ is called a Ginsberg product. There are some fundamental results about the structure:

**Proposition 1 (Fitting).** If $L_1, L_2$ are bounded lattices then the Ginsberg product $L_1 \odot L_2 =< L_1 \times L_2, \wedge, \vee, \otimes, \oplus >$ is an interlaced bilattice. Especially, $L \odot L$ is an interlaced bilattice with negation $\neg$, where $\neg$ is defined by $\neg(a, b) = (b, a)$.

It is proved that the converse holds by Avron ([1]).

**Proposition 2 (Avron).** For any interlaced bilattice $B$, there are bounded lattices $L_1, L_2$ such that $B \cong L_1 \odot L_2$. In particular, for any interlaced bilattice $B$ with negation, there is a bounded lattice $L$ such that $B \cong L \odot L$.

It is clear from definition that orderings $\subseteq_t, \subseteq_k$ on $\mathcal{K}(L)$ are the same as $\leq_t, \leq_k$ on Ginsberg product $L \odot L$, respectively:

- $\subseteq_t$ in $\mathcal{K}(L) \iff \leq_t$ in $L \odot L$
- $\subseteq_k$ in $\mathcal{K}(L) \iff \leq_k$ in $L \odot L$

Hence in the following we use the same symbols $\wedge, \vee, \otimes, \oplus$ in $\mathcal{K}(L)$ and in $L \odot L$.

Next we give a definition of a weak interlaced bilattice according to Font ([4]). A structure $W =< W, \leq_t, \leq_k >$ is called a weak interlaced bilattice if

1. $< W, \leq_t >$ : lattice
2. $< W, \leq_k >$ : meet semilattice
3. $a \leq_t b, c \leq_k d \Rightarrow a \wedge c \leq_k b \wedge d, a \vee c \leq_k b \vee d$
4. $a \leq_t b, c \leq_t d \Rightarrow a \otimes c \leq_t b \otimes d$,
5. $a \leq_t b, c \leq_t d \Rightarrow a \oplus c \leq_t b \oplus d$ if $a \oplus c$ and $b \oplus d$ exist.

### 3 Properties of weak interlaced bilattices

For any weak interlaced bilattice $W$, if we define

- $L_1 = \{x \in W \mid x \leq_k 0\} = [\bot, 0]_k$
- $L_2 = \{x \in W \mid x \leq_k 1\} = [\bot, 1]_k$,

then we have
Proposition 3.

\[ L_1 = [\bot, 0]_k = [0, \bot]_t \]
\[ L_2 = [\bot, 1]_k = [\bot, 1]_t \]

**Proof.** Let \( x \in [\bot, 0]_k \). Since \( \bot \leq_k x \leq_k 0 \), we have \( \bot \vee \bot \leq_k x \vee \bot \leq_k 0 \vee \bot \) by definition of weak interlaced bilattice. From \( \bot \vee \bot = 0 \vee \bot = \bot \), it follows that \( x \vee \bot = \bot \) and hence that \( x \leq_t \bot \). This means \( [\bot, 0]_k \subseteq [0, \bot]_t \).

Conversely, suppose \( x \in [0, \bot]_t \). If we put \( u = 0 \otimes x \), then it is clear that \( u \leq_k 0 \) and \( u \leq_k x \). Since \( 0 \leq_t x \), we have \( 0 \otimes x \leq_t x \otimes x = x \) and hence \( u \leq_t x \). It follows from \( \bot \leq_k u \) that \( x \wedge \bot \leq_k x \wedge u \). Since \( x \leq_t \bot \), we also have \( x \wedge \bot = x \). On the other hand, since \( u \leq_t x \), we get \( u \wedge x = u \). Theses imply that \( x \leq_k u \) and hence that \( x = u \). Thus we have \( x \leq_k 0 \). Namely, we have \( [0, \bot]_t \subseteq [\bot, 0]_k \).

The second equation can be proved similarly.

\[ \square \]

The result implies that \( L_1 \) and \( L_2 \) are lattices with ordering \( \leq_1 \) and \( \leq_2 \) in \( B \), respectively, where \( \leq_1 \) and \( \leq_2 \) are defined by

\[ \leq_1 = \leq_t = \geq_k \]
\[ \leq_2 = \leq_t = \leq_k \]

Thus we can consider the Ginsberg product \( L_1 \odot L_2 \), which becomes an interlaced bilattice. Moreover we can prove

**Proposition 4.** Let \( \mathcal{W} \) be any weak interlaced bilattice. For any \( x \in \mathcal{W} \), we have

\[ x = (x \otimes 0) \oplus (x \otimes 1) = (x \wedge \bot) \vee (x \vee \bot) \]

**Proof.** See Avron [1] Cor.3.8

Now we investigate a relation between a weak interlaced bilattice \( \mathcal{W} \) and an interlaced bilattice \( L_1 \odot L_2 \) constructed by \( \mathcal{W} \).

**Lemma 1.** A map \( \xi : \mathcal{W} \rightarrow L_1 \times L_2 \) defined by \( \xi(x) = (x \otimes 1, x \otimes 0) = (x \vee \bot, x \wedge \bot) \) is an embedding.

This means that

**Theorem 1.** Any weak interlaced bilattice can be embedded into an interlaced bilattice.
4 Answer to the question

In this section we give a positive answer to the question above. Since any weak interlaced bilattice $\mathcal{W}$ can be embedded to an interlaced bilattice, it suffices to show that any interlaced bilattice of a form $L_1 \odot L_2$ is embeddable into a weak interlaced bilattice $\mathcal{K}(L)$ for some lattice $L$. Because, from proposition 2, every interlaced bilattice has a form of $L_1 \odot L_2$ for some lattices $L_1, L_2$. Let $L_1 \odot L_2$ be any interlaced bilattice and $L$ be a set $(L_1 \times \{0\}) \cup (L_2 \times \{1\})$. We define an order $\sqsubseteq$ on $L$. For any element $(a, i), (b, j) \in L$, we define

$$(a, i) \sqsubseteq (b, j) \iff i < j \text{ or } i = j \text{ and } a \leq b$$

It is easy to show that the relation $\sqsubseteq$ is a partially order on $L$ and that

$$(a, i) \land (b, j) = \inf\{(a, i), (b, j)\} = \begin{cases} (a \land b, i) & \text{if } i = j \\ (a, i) & \text{if } i < j \\ (b, j) & \text{if } i > j \end{cases}$$

$$(a, i) \lor (b, j) = \sup\{(a, i), (b, j)\} = \begin{cases} (a \lor b, i) & \text{if } i = j \\ (b, j) & \text{if } i < j \\ (a, i) & \text{if } i > j \end{cases}$$

Hence $L$ is a lattice with this order. Let $\mathcal{K}(L)$ be the set of all elements $[(a, i), (b, j)]$ such that $(a, i) \sqsubseteq (b, j)$ for $(a, i), (b, j) \in L$. In this case, four operators $\land, \lor, \otimes, \oplus$ on $\mathcal{K}(L)$ are defined as follows:

$$[(a, i), (b, j)] \land [(a', i'), (b', j')] = [(a, i) \land (a', i'), (b, j) \land (b', j')]$$

$$[(a, i), (b, j)] \lor [(a', i'), (b', j')] = [(a, i) \lor (a', i'), (b, j) \lor (b', j')]$$

$$[(a, i), (b, j)] \otimes [(a', i'), (b', j')] = [(a, i) \land (a', i'), (b, j) \lor (b', j')]$$

$$[(a, i), (b, j)] \oplus [(a', i'), (b', j')] = [(a, i) \lor (a', i'), (b, j) \land (b', j')]$$

Of course, the last equation is defined when $(a, i) \lor (a', i') \leq (b, j) \land (b', j')$. Now we define a map $\xi : L_1 \odot L_2 \to \mathcal{K}(L)$ by

$$\xi(a, b) = [(a, 0), (b, 1)]$$

It is obvious that $\xi$ is well-defined and injective. We only show that $\xi$ is a homomorphism. We only think of two cases. For the case of $(a, b) \land (a', b')$, we have
\[ \xi((a, b) \land (a', b')) = \xi(a \land a', b \lor b') \]
\[ = [(a \land a', 0), (b \lor b', 1)] \]
\[ = [(a, 0) \land (a', 0), (b, 1) \lor (b', 1)] \]
\[ = [(a, 0), (b, 1)] \otimes [(a', 0), (b', 1)] \]
\[ = \xi(a, b) \otimes \xi(a', b') \]

For another case of \((a, b) \oplus (a', b')\), we also have
\[ \xi((a, b) \oplus (a', b')) = \xi(a \lor a', b \lor b') \]
\[ = [(a \lor a', 0), (b \lor b', 1)] \]
\[ = [(a, 0) \lor (a', 0), (b, 1) \lor (b', 1)] \]
\[ = [(a, 0), (b, 1)] \lor [(a', 0), (b', 1)] \]
\[ = \xi(a, b) \lor \xi(a', b') \]

Hence the map \( \xi : L_1 \odot L_2 \rightarrow \mathcal{K}(L) \) is an embedding, that is,

**Theorem 2.** For every interlaced bilattice \( L_1 \odot L_2 \), there exists a lattice \( L \) such that it is embedded into a weak interlaced bilattice \( \mathcal{K}(L) \).

From these results, we have have a main theorem.

**Theorem 3.** Every interlaced bilattice \( \mathcal{W} \) can be embedded into a weak interlaced bilattice \( \mathcal{K}(L) \) for some lattice \( L \).

References


Michiro Kondo
Current Address:
e-mail: kondo@sie.dendai.ac.jp
School of Information Environment
Tokyo Denki University, Inzai, 270-1382
JAPAN