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\textbf{\(\lambda\)-Calculus with Lazy Lists}  
\textit{– Extended Abstract –}

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\begin{abstract}
\begin{quote}
Into \(\lambda\)-calculus we introduce lazy lists \(\vec{a}\) whose naïve meaning is an infinite list consisting of variables, \((a_0, a_1, a_2, \ldots)\). It is shown that there exist maps which form a Galois connection from Parigot’s \(\lambda\mu\)-calculus to the \(\lambda\)-calculus with lazy list. The translations form not only an equational correspondence but also a reduction correspondence between the two calculi.
\end{quote}
\end{abstract}

1 Introduction

We introduce lazy lists into \(\lambda\)-calculus. The introduction of infinite lists is motivated by a study on denotational semantics of type-free \(\lambda\mu\)-calculus [Pari92, Pari97, BHF99, BHF01].

Given domains \(U \times U \simeq U \simeq [U \rightarrow U]\) such as in Lambek-Scott [LS86], we have established a continuation denotational semantics of type-free \(\lambda\mu\)-calculus [Fuji02], which formally coincides with the CPS-translation [HS97, SR98, Fuji01] followed by the direct denotational semantics of the \(\lambda\)-calculus [Scot72, Stoy77]. See also the literature [HS97, SR98, Sel01] for continuation semantics of \(\lambda\mu\)-calculus.

This article shows that there exists a one-to-one correspondence between the \(\lambda\mu\)-calculus and the \(\lambda\)-calculus with lazy lists.

2 \(\vec{\lambda}\)-calculus

We have two kinds of variables, the traditional variables in the \(\lambda\)-calculus denoted by \(x\) and variables for lazy lists denoted by \(\vec{a}\). Our intended meaning is that \(\vec{a}\) denotes an infinite list of variables, \(\vec{a} \simeq (a_0, a_1, a_2, \ldots)\). The denotational meaning of \(\vec{a}\) would be given by elements of domain \(E^\omega\) which is a solution of the domain equation \(D \simeq D \times D\). From this intension, the expression \(M\vec{a}\) says that \(M\) is a function which can accept infinite inputs, and \(\lambda\vec{a}.M\) is a function characterized by \(D^D \simeq D^{D \times D} \simeq D^D\), that is, \(\lambda\vec{a} \cdots M\vec{a} \cdots\) can behaves like \(\lambda x\lambda\vec{a} \cdots Mx\vec{a} \cdots\). Under this informal meaning, potentially infinite applications of \(\beta\)-reduction should be performed as follows:
Following this intended meaning, we define the λ-calculus with infinite lists as follows. A term in the form of $\overline{a}$ is called a lazy list.

**Definition 1 (λ-calculus)**

$$\overline{\lambda} \ni M ::= x \mid \overline{a} \mid \lambda x.M \mid \lambda\overline{a}.M \mid MM$$

$$(\beta) (\lambda x.M_1)M_2 = M_1[x := M_2]$$

$$(\eta) \lambda x.Mx = M \text{ if } x \notin FV(M)$$

$$(\tilde{\beta}) (\lambda\overline{a}.M_1)M_2 = \begin{cases} M_1[\overline{a} := M_2] & \text{if } M_2 \text{ is in the form of a lazy list} \\ \lambda\overline{a}.M_1[\overline{a} := M_2] & \text{otherwise} \end{cases}$$

$$(\tilde{\eta}) \lambda\overline{a}.M\overline{a} = M \text{ if } \overline{a} \notin FV(M)$$

The term $M_1[x := M_2]$ denotes the usual capture-free substitution of $M_2$ for $x$ in $M_1$. The term $M_1[\overline{a} := M_2]$ indicates the capture-free substitution defined in the following:

(i) $x[\overline{a} := M] = x$

(ii) $\overline{b}[\overline{a} := M] = \begin{cases} M & \text{if } \overline{b} \equiv \overline{a} \text{ and } M \text{ is a lazy list} \\ M\overline{b} & \text{if } \overline{b} \equiv \overline{a} \text{ and } M \text{ is not a lazy list} \\ \overline{b} & \text{otherwise} \end{cases}$

(iii) $(\lambda x.M_1)[\overline{a} := M] = \lambda x.M_1[\overline{a} := M]$

(iv) $(\lambda\overline{b}.M_1)[\overline{a} := M] = \lambda\overline{b}.M_1[\overline{a} := M]$

(v) $(M_1M_2)[\overline{a} := M] = \begin{cases} ((M_1[\overline{a} := M])M_2) & \text{if } M_2 \equiv \overline{a} \text{ and } M \text{ is not a lazy list} \\ (M_1[\overline{a} := M])(M_2[\overline{a} := M]) & \text{otherwise} \end{cases}$

The axiom $(\tilde{\beta})$ says that a function which can accept an infinite list has taken an infinite list in the case where $M_2$ is in the form of a lazy list. In the case where $M_2$ is not in the form of a lazy list, $(\tilde{\beta})$ means that a function which can accept an infinite list has taken only a finite input, so that we still have $\lambda\overline{a}$ even after this. $(\tilde{\eta})$ says that $\lambda\overline{a}.M\overline{a}$ is an infinite $\eta$-expansion of $M$.

We write $\overline{\Lambda} \vdash M_1 = M_2$ or $((\beta, \eta, \tilde{\beta}, \tilde{\eta}) \vdash M_1 = M_2$ if $M_1 = M_2$ is derived from the axioms $(\beta)$, $(\eta)$, $(\tilde{\beta})$, or $(\tilde{\eta})$. As an abbreviation, we may write $M_1 =_{\overline{\Lambda}} M_2$ for this. We adopt a rewriting theory of $\overline{\Lambda}$ by rewriting the left-hand side of each axiom to the corresponding right-hand side. The binary relation $\rightarrow$, $\rightarrow^+$, or $\rightarrow^*$ denotes the one-step rewriting, the transitive closure of $\rightarrow$, or the reflexive and transitive closure of $\rightarrow$, respectively.

**Proposition 1 (1)** $\overline{\Lambda} \vdash \lambda x.x = \lambda\overline{a}.\overline{a}$
$\overline{\Lambda} \vdash \lambda x.\lambda \tilde{a}.M[\tilde{a} := x] = \lambda \tilde{a}.M$

Proof. (1) $\lambda \tilde{a}.\tilde{a}$ can be regarded as an infinite $\eta$-expansions of $\lambda x.x$:
$$\lambda \tilde{a}.\tilde{a} =_\eta \lambda x.(\lambda \tilde{a}.\tilde{a})x =_\tilde{\beta} \lambda x.\lambda \tilde{a}.x\tilde{a} =_\eta \lambda x.x$$

(2) The abstraction by $\lambda x$ can be absorbed in the infinite $\lambda$-abstraction by $\lambda \tilde{a}$:
Let $x \notin FV(M)$.
$$\lambda \tilde{a}.M =_\eta \lambda x.(\lambda \tilde{a}.M)x =_\tilde{\beta} \lambda x.\lambda \tilde{a}.M[\tilde{a} := x] \quad \square$$

3 Relationship between $\lambda\mu$-calculus and $\overline{\lambda}$-calculus

We show that the $\overline{\lambda}$-calculus is a conservative extension over Parigot's $\lambda\mu$-calculus [Pari92, Pari97].

Definition 2 ($\lambda\mu$-calculus)
$$\Lambda\mu \ni M ::= x \mid \lambda x.M \mid MM \mid \mu\alpha.M \mid [\alpha]M$$

($\beta$) $(\lambda x.M_1)M_2 = M_1[x := M_2]$

($\eta$) $\lambda x.Mx = M$ if $x \notin FV(M)$

($\mu$) $(\mu\alpha.M_1)M_2 = \mu\alpha.M_1[\alpha \Leftarrow M_2]$

($\mu\beta$) $[\alpha](\mu\beta.M) = M[\beta := \alpha]$

($\mu\eta$) $\mu\alpha.[\alpha]M = M$ if $\alpha \notin FV(M)$

The $\lambda\mu$-term $M_1[\alpha \Leftarrow M_2]$ denotes a term obtained by replacing each subterm of the form $[\alpha]M$ in $M_1$ with $[\alpha](MM_2)$. This operation is inductively defined as follows:

1. $x[\alpha \Leftarrow M] = x$
2. $(\lambda x.M_1)[\alpha \Leftarrow M] = \lambda x.M_1[\alpha \Leftarrow M]$  
3. $(M_1M_2)[\alpha \Leftarrow M] = (M_1[\alpha \Leftarrow M])(M_2[\alpha \Leftarrow M])$
4. $(\mu\beta.N)[\alpha \Leftarrow M] = \mu\beta.N[\alpha \Leftarrow M]$
5. $([\beta]M_1)[\alpha \Leftarrow M] = \begin{cases} [\beta]((M_1[\alpha \Leftarrow M])M), & \text{for } \alpha \equiv \beta \\ [\beta](M_1[\alpha \Leftarrow M]), & \text{otherwise} \end{cases}$

Definition 3 Translation $[\ ] : \Lambda\mu \rightarrow \overline{\Lambda}$

1. $[x] = x$
2. $[\lambda x.M] = \lambda x.[M]$
3. $[M_1M_2] = [M_1][M_2]$
4. $[\mu x.M] = \lambda \vec{a}. [M]$

5. $[[\alpha]M] = [M] \vec{a}$

Lemma 1 Let $M, N \in \Lambda_{\mu}$.

$[M[\alpha \Leftarrow N]] = [M][\vec{a} := [N]]$


Proposition 2 If $M_{1} \rightarrow_{\Lambda_{\mu}} M_{2}$, then $[M_{1}] \rightarrow_{\overline{\lambda}} [M_{2}]$.

Proof. By induction on the derivation of $M_{1} \rightarrow_{\Lambda_{\mu}} M_{2}$. We show some of the base cases.

Case of $(\mu)$:
$[(\mu x.M)N] = (\lambda \vec{a}. [M])[N]$
$\rightarrow_{\beta} \lambda \vec{a}. [M][\vec{a} := [N]]$ since $[N]$ is not a lazy list
$= \lambda \vec{a}. [M[\alpha \Leftarrow N]] = [\mu x.M[\alpha \Leftarrow N]]$

Case of $(\beta)$:
$[(\lambda x.M)N] = (\lambda x. [M])[N]$
$\rightarrow_{\beta} [M][x := [N]] = [M[x := N]]$

Definition 4 Translation $\lfloor \cdot \rfloor : \overline{\Lambda} \rightarrow \Lambda_{\mu}$

(i) $\lfloor x \rfloor = x$

(ii) $\lfloor \vec{a} \rfloor = [a](\lambda x.x)$

(iii) $\lfloor \lambda x.M \rfloor = \lambda x. [M]$

(iv) $\lfloor \lambda \vec{a}.M \rfloor = \mu a. [M]$

(v) $\lfloor M_{1} M_{2} \rfloor = \begin{cases} [a][M_{1}] & \text{if } M_{2} \equiv \vec{a} \text{ for some } \vec{a} \\ [M_{1}][M_{2}] & \text{otherwise} \end{cases}$

Lemma 2 (i) Let $M \in \overline{\Lambda}$.

$[M][a := b] = [M[a := \vec{b}]]$

(ii) Let $M, N \in \overline{\Lambda}$ where $N$ is not a lazy list.

$[M][a \Leftarrow [N]] \rightarrow^{*}_{\beta} [M[a := [N]]]$

Proof. By induction on the structure of $M$. We show some cases for (ii).

Case of $M \equiv \vec{a}$:
$[\vec{a}][a \Leftarrow [N]] = ([a](\lambda x.x))[a \Leftarrow [N]] = [a]((\lambda x.x)[N])$
$\rightarrow_{\beta} [a][N] = [N\vec{a}] = [a[\vec{a} := N]]$ since $N$ is not a lazy list.

Case of $M \equiv M_{1} M_{2}$:
We show the subcase $M_{2}$ of $\vec{a}$ here.

$[M_{1} M_{2}][a \Leftarrow [N]] = [M_{1} \vec{a}][a \Leftarrow [N]]$
Proposition 3 Let $M_1, M_2 \in \vec{\Lambda}$.
If $M_1 \rightarrow_{\vec{\Lambda}} M_2$ then $[M_1] \rightarrow_{\Lambda\mu}^* [M_2]$.

Proof. By induction on the derivation of $M_1 \rightarrow_{\vec{\Lambda}} M_2$.
Case of $(\vec{\beta})$ where $N$ is not a lazy list:
\[[(\lambda \vec{a}. M) N] = ([\lambda a. [M]][N)]\]
\[\rightarrow_{\mu} [\mu a. [M][a \leftarrow [N]]]\]
\[\rightarrow_{*}^{\beta} [\mu a. [M][a \leftarrow [N]]]\] by Lemma 2
\[= [\lambda \vec{a}. M][a \leftarrow [N]]\]

Case of $(\vec{\beta})$ where $N$ is a lazy list:
\[[(\lambda \vec{a}. M) \vec{b}] = [\vec{b}][\mu a. [M]]\]
\[\rightarrow_{\mu \rho} [\mu a. [M][a := \vec{b}]]\]

Case of $(\beta)$:
\[[(\lambda x. M) N] = (\lambda x. [M])[N]\]
\[\rightarrow_{\beta} [M][x := [N]] = [M[x := N]]\] by Lemma 2
\[\square\]

Proposition 4 The maps $[.] : \Lambda\mu \rightarrow \vec{\Lambda}$ and $[.] : \vec{\Lambda} \rightarrow \Lambda\mu$ establish a one-to-one correspondence between $\Lambda\mu$ and $\vec{\Lambda}$:

(i) For any $M \in \Lambda\mu$, $M = [M]$.

(ii) For any $M \in \vec{\Lambda}$, $[M] \rightarrow_{*}^{\beta} M$.

Proof. By induction on the structure of $M$. For (ii) we show one of the base cases.
Case of $M \equiv \vec{a}$:
\[[(\vec{a})] = [(\lambda x. x)] = (\lambda x. x) \rightarrow_{\beta} \vec{a}\]

Definition 5 (Sabry-Walder [SW96])
The maps $[.] : \Lambda\mu \rightarrow \vec{\Lambda}$ and $[.] : \vec{\Lambda} \rightarrow \Lambda\mu$ form a Galois connection from $\Lambda\mu$ to $\vec{\Lambda}$ whenever $M \rightarrow_{\Lambda\mu}^* [P]$ if and only if $[M] \rightarrow_{\hat{\Lambda}}^* P$.

It can be confirmed that the maps $[.] : \Lambda\mu \rightarrow \vec{\Lambda}$ and $[.] : \vec{\Lambda} \rightarrow \Lambda\mu$ form a Galois connection by Propositions 2, 3, 4, and the following proposition:
Proposition 5 (Sabry-Wadler [SW96])

The maps \([\ ]\) and \([\ ]\) form a Galois connection from \(\Lambda\mu\) to \(\overline{\Lambda}\) if and only if conditions hold:

(i) \(M \rightarrow_{\Lambda\mu} [M]\),

(ii) \([P] \rightarrow_{\overline{\Lambda}} P\),

(iii) \(M_1 \rightarrow_{\Lambda\mu} M_2\) implies \([M_1] \rightarrow_{\overline{\Lambda}} [M_2]\), and

(iv) \(P_1 \rightarrow_{\overline{\Lambda}} P_2\) implies \([P_1] \rightarrow_{\Lambda\mu} [P_2]\).

See also the following diagrams:

(i) \[
\begin{array}{ccc}
M & \in & \Lambda\mu \\
\lceil \cdot \rceil_{\Lambda\mu} & \Downarrow & \overline{\Lambda} \\
\lceil [M] \rceil & \in & \Lambda\mu
\end{array}
\]

(ii) \[
\begin{array}{ccc}
[P] & \in & \Lambda\mu \\
\lceil \cdot \rceil & \Downarrow & \overline{\Lambda} \\
[P] & \in & \Lambda\mu
\end{array}
\]

(iii) \[
\begin{array}{ccc}
M_1 & \in & \Lambda\mu \\
\lceil \cdot \rceil_{\Lambda\mu} & \Downarrow & \overline{\Lambda} \\
M_2 & \in & \Lambda\mu
\end{array}
\]

(iv) \[
\begin{array}{ccc}
[P_1] & \in & \Lambda\mu \\
\lceil \cdot \rceil_{\Lambda\mu} & \Downarrow & \overline{\Lambda} \\
[P_2] & \in & \Lambda\mu
\end{array}
\]

Proposition 6 (i) (Conservative extension) Let \(M_1, M_2 \in \Lambda\mu\).

If we have \([M_1] =_{\overline{\Lambda}} [M_2]\), then \(M_1 =_{\Lambda\mu} M_2\).

(ii) (Galois connection)

The maps \([\ ] : \Lambda\mu \rightarrow \overline{\Lambda}\) and \([\ ] : \overline{\Lambda} \rightarrow \Lambda\mu\) form a Galois connection between \(\lambda\mu\)-calculus to the \(\overline{\Lambda}\)-calculus.
(iii) (Equational/reduction correspondence)

1. $M_1 =_{\Lambda\mu} M_2$ if and only if $[M_1] =_{\vec{\Lambda}} [M_2]$.
   In particular, $M_1 \to_{\Lambda\mu}^+ M_2$ if and only if $[M_1] \to_{\vec{\Lambda}}^+ [M_2]$.

2. $P_1 =_{\overline{\Lambda}} P_2$ if and only if $[P_1] =_{\Lambda\mu} [P_2]$.

Proof. (i) From Proposition 4.
(ii) From Propositions 2, 3, 4, and 5.
(iii) From (ii).

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References


