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Citation: 数理解析研究所講究録 (2002), 1268: 151-154

Issue Date: 2002-06

URL: http://hdl.handle.net/2433/42146

Type: Departmental Bulletin Paper

Textversion: publisher

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Model-robustness of equilibrium in game for modal logics *

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Abstract. We present the notion of robust models for a mixed strategy Nash equilibrium of a strategic form game and to introduce a group structure on the class of these models. From the algebraic point of view we give a characterization of the class of robust models by the class of epistemic models with common-knowledge of conjectures about the other players' actions.

Keywords: Robust model, Bayesian model, Nash equilibrium, Non-cooperative game, Common-knowledge, Conjectures, Rationality, Semigroup.

1. Introduction

This paper investigates the class of robust models for a mixed strategy Nash equilibrium of a finite strategic form game $G$.

The concept of Nash equilibrium has become central in game theory, economics and its related fields. R.J. Aumann and A. Brandenburger (1995) gives epistemic conditions for Nash equilibrium in the modal logic S5. However it is still not yet clear just what classes of models leading to a Nash equilibrium in the epistemic point of view.

The purposes of this paper are two points: First to to introduce a group structure on the class of robust models for a mixed strategy Nash equilibrium of the game as models for the modal logic S4, and secondly to characterize the class $R^{S4}(G)$ of the robust models by the class $E^{S4}(G)$ of all the models satisfying with common-knowledge of conjectures about the other players' actions. We show:

Main Theorem. The class $E^{S4}(G)$ is a non-empty subclass of $R^{S4}(G)$. Furthermore $E^{S4}(G)$ almost coincides with $R^{S4}(G)$.

2. Knowledge structure

Let $\Omega$ be a non-empty finite set called a state-space, $N$ a set of finitely many players $\{1, 2, \ldots, i, \ldots n\}$ at least two ($n \geq 2$), and let $\mu$ be a probability measure on $\Omega$ which is common for all players. Each member of $2^\Omega$ is called an event and each element of $\Omega$ called a state.

Definition 1. By a knowledge structure we mean a pair $<\Omega, (K_i)_{i \in N}>$, in which $\Omega$ be a non-empty set called a state-space, $K_i$ is a mapping of $2^\Omega$ into itself called player $i$'s knowledge operator. A common-knowledge structure is a quadruple $<\Omega, (K_i)_{i \in N}, K_E, K_C>$ in which $K_E$ is the mutual knowledge operator on $2^\Omega$ defined

* This is an extended abstract and the final form will be published elsewhere.
by \( K_E E = \cap_{i \in N} K_i E \). and \( K_C \) is a common-knowledge operator on \( 2^\Omega \) satisfying the fixed point property:

\[
K_C F \subseteq K_E (F \cap K_CF)
\]

for every \( F \) of \( 2^\Omega \).

The event \( K_i E \) is interpreted as the set of states of nature for which \( i \) knows \( E \) to be possible. The event \( K_E E \) is interpreted as that all players know \( E \).

The \textit{iterated} common-knowledge operator \( K_C \) is defined in the following way: Construct the descending chain \( \{K^m\} \) such that

\[
K^0 F := K_E F; \quad \bar{K}^{m-1} F := K_E (F \cap K^{m-1} F);
\]

\[
K^m F := \bar{K}^{m-1} F \cap K^{m-1} F.
\]

The operator \( K_C \) is given by the infimum of the chain:

\[
K_C E = \bigcap_{m=0,1,2,\ldots} K^m E.
\]

It is plainly observed that \( K_C \) satisfies Axiom \( \text{FP} \). The event \( K_C E \) is interpreted as that 'all players know that all players know that \( \ldots \) that all players knows \( E \).'

\textbf{Definition 2.} A knowledge structure (or a common-knowledge structure) is called a \( K \) knowledge (or a \( K \) common-knowledge) structure if it satisfies the properties:

1. For any \( E, F \in 2^\Omega \),
   \[
   N \quad K_i \Omega = \Omega \quad \text{and} \quad K_i \emptyset = \emptyset ;
   \]
   \[
   K \quad K_i (E \cap F) = K_i E \cap K_i F.
   \]

2. It is called a \( T \) knowledge (or a \( T \) common-knowledge) structure if it satisfies in addition
   \[
   T \quad K_i (E) \subseteq E \quad \text{for every} \quad E \in 2^\Omega.
   \]

3. It is called an \( S_4 \) knowledge (or an \( S_4 \) common-knowledge) structure if it satisfies in addition
   \[
   4 \quad K_i (E) \subseteq K_i (K_i (E)) \quad \text{for every} \quad E \in 2^\Omega.
   \]

Finally it is called an \( S_4 \) knowledge (or an \( S_4 \) common-knowledge) structure if it satisfies in addition

\[
5 \quad \Omega \setminus K_i E \subseteq K_i (\Omega \setminus K_i E) \quad \text{for every} \quad E \in 2^\Omega.
\]

\textbf{3. Associated information structure}

An information structure is a pair \( \langle \Omega, (Q_i)_{i \in N} \rangle \) in which \( \Omega \) is a non-empty state-space and \( Q_i \) is a mapping on \( \Omega \) into \( 2^\Omega \). It is called an RT-information structure if each \( Q_i \) satisfies the two conditions: For each \( \omega \) of \( \Omega \),

\[
\text{Ref} \quad \omega \in Q_i (\omega) ; \quad \text{and}
\]

\[
\text{Trn} \quad \xi \in Q_i (\omega) \implies Q_i (\xi) \subseteq Q_i (\omega).
\]
Definition 3. The associated information structure \((P_i)_{i \in N}\) with a knowledge structure \((\Omega, (K_i)_{i \in N})\) is the information structure \((\Omega, (P_i)_{i \in N})\) defined by \(P_i(\omega) = \bigcap_T\{T \in 2^\Omega \mid \omega \in K_i T\}\).

Remark 1. For each \(L = K, T, S4\) or \(S5\), an \(L\) knowledge structure \((\Omega, (K_i)_{i \in N})\) is uniquely determined by the associated information structure \((\Omega, (P_i)_{i \in N})\) as follows: \(K_i E = \{\omega \in \Omega \mid P_i(\omega) \subseteq E\}\).

Let \(K_i P_i : \Omega \to 2^\Omega\) be the mapping defined by \(K_i P_i(\omega) = K_i(P_i(\omega))\). It is plainly observed that \((K_i P_i)_{i \in N}\) is an RT-information structure for any \(L\) knowledge structure \((\Omega, (K_i)_{i \in N})\). It is noted that the associated information structure of an \(S4\) knowledge structure is an RT-information structure, because \(K_i P_i = P_i\). Then \(P_i(\omega)\) is interpreted as the set of all the states of nature that \(i\) knows to be possible at \(\omega\).

4. Robust models

Let \(\Omega\) be a non-empty finite state-space and \(\mu\) a probability measure on \(\Omega\) which is common for all players. Let \(G = \langle N, (A_i)_{i \in N}, (g_i)_{i \in N}\rangle\) be a finite strategic form game: \(A_i\) is a finite set of \(i\)'s actions and \(g_i\) is \(i\)'s payoff-function of \(A := \times_{i \in N} A_i\) into \(R\). \(i\)'s overall conjecture \(\phi_i\) is a probability distribution on \(A_{-i}\). For each player \(j\) other than \(i\), this induces the marginal on \(j\)'s actions called \(i\)'s individual conjecture about \(j\). Let \(a = (a_1, \ldots, a_n)\) be a random variable of \(A\). If \(x\) is a such function and \(x\) is a value of it, we denote by \([x = x]\) (or simply by \([x]\)) the set \(\{\omega \in \Omega \mid x(\omega) = x\}\).

Accordingly \(P_i\) with \(\mu\) yields the \(i\)'s overall conjecture defined by the marginal \(\phi_i(a_{-i}, \omega) = \mu([a_{-i} = a_{-i}]|K_i P_i(\omega))\) viewed as a random variable of \(i\)'s conjecture \(\phi_i\). Denote \(\phi_i = \cap_{a_{-i} \in A_{-i}}[\phi_i(a_{-i}, \cdot) = \phi(a_{-i})]\) and \([\phi] = \cap_{i \in N} [\phi_i]\). We assume here that \([a_i] \subseteq K_i([a_i])\) for every \(a_i\) of \(A_i\). An player \(i\) is said to be rational at \(\omega\) if each \(i\)'s actual action \(a_i\) maximizes the expectation of his actually played pay-off function \(g_i\) at \(\omega\) when the other players actions are distributed according to his conjecture \(\phi_i(\omega)\).

Formally, letting \(a_i = a_i(\omega), \exp(g_i(a_i, a_{-i}); \omega) \geq \exp(g_i(b_i, a_{-i}); \omega)\) for every \(b_i\) in \(A_i, \) where \(\exp\) is defined by \(\exp(g_i(b_i, a_{-i}); \omega) := \sum_{a_{-i} \in A_{-i}} g_i(b_i, a_{-i}) \phi_i(a_{-i}, \omega)\).

Let \(R_i\) be the set of all the states at which an player \(i\) is rational, and denote \(R := \cap_{i \in N} R_i\).

For a profile of conjectures \(\phi = (\phi_i)_{i \in N}\), we denote by \(B_{\phi}^{S4}(G)\) the class of all the epistemic models \(B = \langle \Omega, \mu, (K_i)_{i \in N}, (A_i)_{i \in N}\rangle\) on the game \(G\) for the logic \(S4\), and denote \(B_{\phi}^{S4}(G) = \cup_{\phi \in x_{i \in N} \Delta(A_{-i})} B_{\phi}^{S4}(G)\). We call \(B_{\phi}^{S4}(G)\) the class of all Bayesian models on \(G\). Let \(E_{\phi}^{S4}(G)\) be the subclass consisting of all the models \(B \in B_{\phi}^{S4}(G)\) with \(K_C(\phi) \cap K_E R \neq \emptyset\), and denote \(E_{\phi}^{S4}(G) = \cup_{\phi \in x_{i \in N} \Delta(A_{-i})} E_{\phi}^{S4}(G)\), called the class of models with epistemic conditions for \(G\). Let \(NE(G)\) be the set of all mixed strategy Nash equilibria and let \(\sigma = (\sigma_i)_{i \in N} \in NE(G)\). We set by \(R_{\phi}^{S4}(G)\) the class of all the models \(B\) such that there exists a profile of conjectures \(\phi \in x_{i \in N} \Delta(A_{-i})\) with the property that for every \(i, j \in N, i \neq j\) and for all \(a_i \in A_i, \phi_j(a_i) = \sigma_i(a_i)\). We will call \(R_{\phi}^{S4}(G)\) the class of robust models for a Nash equilibrium \(\sigma\) of \(G\). Denote \(R^{S4}(G) = \cup_{\sigma \in NE(G)} R^{S4}(G)\).
5. Proof of Main Theorem

For the first part that $B^S_4(G) \subseteq R^S_4(G)$: It can be plainly observed that for
\( \sigma \in \text{NE}(G) \), $B^S_4(G) \neq \emptyset$ and so $B^S_4(G) \neq \emptyset$. Let $B = \langle \Omega, \mu, (K_i)_{i \in N}, (a_i)_{i \in N} \rangle \in E^S_4(G)$ and $B = \langle \phi \rangle \in E^S_4(G)$. Take $\omega \in K_G([\phi]) \cap K_E$. Set $\sigma_i(a_i) := \phi_j(a_i)$ with $j \neq i$ and $\sigma = (\sigma_i)_{i \in N}$. We can observe that for every $a \in \prod_{i \in N} \text{Supp}(\sigma_i) \setminus \sigma_i(a_i) = \sigma_1(a_1) \cdots \sigma_{i-1}(a_{i-1})\sigma_i(a_i) \cdots \sigma_n(a_n)$, and thus we obtain that each action $a_i$ appearing with positive probability in $\sigma_i$ at $\omega$ maximizes $g_i$ against $\phi_i(a_{-i})$. This implies that $\sigma = (\sigma_i)_{i \in N} \in \text{NE}(G)$, and so $B \in R^S_4(G)$.

For the second part: We can introduce the operation $\oplus : B^S_4(G) \times B^S_4(G) \to B^S_4(G)$ such that the structure $<B^S_4(G), \oplus>$ is an abelian semigroup. Now we can explicitly state the second part of the main theorem as follows:

**Theorem 1.** Let the notation be the same as above. Then the following statements are true:

(i) Both structures $<B^S_4(G), \oplus>$ and $<R^S_4(G), \oplus>$ are sub-semigroups of $<B^S_4(G), \oplus>$. Furthermore, $E^S_4(G)$ is a sub-semigroup of $R^S_4(G)$.

(ii) The quotient group $G(E^S_4(G))$ of $E^S_4(G)$ is isomorphic to the quotient group $G(R^S_4(G))$ of $R^S_4(G)$.

**Proof.** For (i): It is easy to verify (i). For (ii): First we note that the cancellation law holds in the abelian semigroups $<E^S_4(G), \oplus>$ and $<R^S_4(G), \oplus>$ respectively. This implies that the quotient groups $G(E^S_4(G))$ and $G(R^S_4(G))$ of $<E^S_4(G), \oplus>$ and $<R^S_4(G), \oplus>$ respectively are constructed, and it can be verified that $G(E^S_4(G)) \cong G(R^S_4(G))$.

6. Concluding remarks

1. A model $B$ for the $S_4$ logic is called a model for the logic $S_5$ provided that for any $E \in 2^\Omega$,
   \[ \Omega \setminus K_i E \subseteq K_i(\Omega \setminus K_i E). \]
   Or equivalently, the associated information structure $(P_i)_{i \in N}$ makes an information partition; that is, for each $\omega \in \Omega$,
   \[ \text{Sym} \ \xi \in P_i(\omega) \text{ implies } P_i(\xi) = P_i(\omega). \]
   In view of R.J. Aumann and A. Brandenburger (1995) it can be shown that $E^{S_5}(G) \subseteq R^S_4(G)$ in the models for the logic $S_5$.

2. Furthermore, the main theorem is still true in the class of models for each modal logics $K, KT, S_4 = KT_4$ and $S_5 = KT_{45}$.

References
