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On the Regularity of the Power Language of a Regular Language

(*Extended abstract*)

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In 1996, *H. Calbrix* introduced the following notion. For an arbitrary language $L \subseteq X^*$, let us define the *power language* of L, in symbols $powlan(L)$, as follows:

$$powlan(L) := \{w^k : w \in L,\ k \in N\} = \bigcup_{w \in L} w^*$$

(where $w^0 = \lambda$, the *empty word*, $N = \{0, 1, 2, ...\}$).

Concerning this notion, Calbrix posed – and left open – the following problem.

Calbrix' Decision Problem (1996): Can we (algorithmically) decide for an arbitrary regular grammar G, whether $powlan(L(G))$ is regular, too?

This problem is *far from being trivial:* Let, e.g., $L := a^+b$ (regular), then

$$powlan(L) = \{(a^k b)^m : k \geq 1, m \geq 0\},$$

non-context-free. Furthermore, *even the case of a one-letter alphabet is nontrivial:* putting

$$L := \{a^{3+2n} : n \in N\}$$

(regular), we have

$$powlan(L) = \{a^{(3+2n)l} : n, l \in N\} = \{a^k : k \in N \setminus \{2^m : m \geq 1\}\},$$

again *non-context-free.*

In 2001, *T. Cachat* gave a *positive answer* to Calbrix' problem *in the one-letter case*, in his paper in the proceedings of the conference *DLT'2001 (Developments in Language Theory, 2001)* (in Vienna, Austria, July, 2001). In this (13-page) paper, even *Dirichlet's* famous, deep *theorem* (that if

$gcd(k,l) = 1$, then in the sequence $k, k+l, k+2l, \ldots$, there are infinitely many primes), is used.

In what follows, *we prove some starting results for the case* $|X| \geq 2$.

Proposition 1: The set of *linear grammars* G, for which $powlan(L(G))$ is *deterministic context-free* or *regular,* respectively, is *not recursively enumerable.*

For our next result, we recall the notion of the *primitive root* of a word x, in symbols, $root(x)$, which in case $x \neq \lambda$, equals the (uniquely existing) *primitive word* y for which $x \in y^+$, and in case $x = \lambda$ it equals λ. (A *primitive word* is a nonempty word which is no power of a shorter word.) The word function *root* is extended from words to languages in the usual way.

Proposition 2: It is *decidable* for an *arbitrary regular grammar* G, whether
(1) "$root(L(G))$ is *finite?*",
and, in the case of a positive answer to question (1), it is also *decidable,* whether
(2) "$powlan(L(G))$ is *regular?*"

Concerning the proof of Proposition 2 we mention that the decidability of (1) is proved in the following paper:

Horváth, S. and Ito, M.;
Decidable and Undecidable Problems of Primitive Words, Regular and Context-Free Languages, *JUCS (Journal of Universal Computer Science),* 5 (1999), pp. 532-541.

In this paper, in case of a "yes" to (1), even the elements of the (finite) $root(L(G))$ are constructed. Then, treating these primitive roots as single letters, we can, by applying Cachat's above mentioned result about the one-letter case, also obtain an effective answer to question (2).

In our last result we will use the notion of a *polyslender language*, recently intoduced by *P. Dömösi and M. Mateescu.* A language $L \subseteq X^*$ is called polyslender iff there is a polynomial p with coefficients from N and with positive main coefficient such that,

for every $n \in N, |L \cap X^n| \leq p(n)$.

Now we formulate our last result.

Proposition 3: Let $L \subseteq X^*$ be an *arbitrary infinite, polyslender language* (otherwise L even need not be recursively enumerable). Then L is *non-regular.*