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AHLFORS THEORY AND COMPLEX DYNAMICS: PERIODIC POINTS OF ENTIRE FUNCTIONS

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ABSTRACT. We give a new proof of the result that transcendental entire functions have infinitely many periodic points of all periods greater than one, and we discuss the main tool used there: the Ahlfors theory of covering surfaces.

1. INTRODUCTION

This paper is an extended version of a series of two talks given at the Research Institute of Mathematical Sciences in Kyoto about the Ahlfors theory of covering surfaces, and the applications it has found in complex dynamics. Some applications of one of the principal results of the Ahlfors theory – the five islands theorem – to various questions in complex dynamics have been surveyed in [11]. This includes topics such as the Hausdorff dimension of Julia sets or the existence of singleton components of Julia sets. In the first part of this paper (§§2–3) we discuss Ahlfors' "Scheibensatz," which contains the five islands theorem as a special case. Then we describe in some detail how the Ahlfors theory can be used to prove the existence of periodic points of a given period, a topic treated rather briefly in [11]. Thus the present paper complements the survey [11] in some sense.

Based on ideas introduced by Essén and Wu [18, 19], and extended in [5], we present a reasonably self-contained proof of the result (Theorem E in §5) that a transcendental entire function has infinitely many repelling periodic points of all periods greater than one.

2. THE PRINCIPAL RESULTS OF THE AHLFORS THEORY

Let $D \subset \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be a domain and let $f : D \to \widehat{\mathbb{C}}$ be a meromorphic function. Let $V \subset \widehat{\mathbb{C}}$ be a Jordan domain. A simply-connected component U of $f^{-1}(V)$ with $\overline{U} \subset D$ is called an *island* of f over V. Note that then $f|_U : U \to V$ is a proper map. The degree of this proper map is called the *multiplicity* of the island U. An island of multiplicity 1 is called a *simple island*.

Let now $q \in \mathbb{N}$ and $\mu_1, \ldots, \mu_q \in \mathbb{N}$, and let $D_1, \ldots, D_q \subset \widehat{\mathbb{C}}$ be Jordan domains with pairwise disjoint closures. By $\mathcal{F}(D, \{(D_j, \mu_j)\}_{j=1}^{q})$ we denote the family of all functions meromorphic in D which have no island of multiplicity less than μ_j over D_j, for all

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$j \in \{1, \dots, q\}$. We shall always suppose that

$$\sum_{j=1}^{q} \left(1 - \frac{1}{\mu_j}\right) > 2. \tag{1}$$

One of the main results of the Ahlfors theory (called "Scheibensatz" by Ahlfors [1, p. 190]) can be stated as follows.

Theorem A.1. *$\mathcal{F}\left(D, \{(D_j, \mu_j)\}_{j=1}^{q}\right)$ is normal.*

A closely related statement is as follows.

Theorem A.2. *$\mathcal{F}\left(\mathbb{C}, \{(D_j, \mu_j)\}_{j=1}^{q}\right)$ contains only the constant functions.*

In the above results, we may put $\mu_j = \infty$, meaning that $1/\mu_j = 0$ in (1) and that the functions in $\mathcal{F}$ have no islands at all over D_j.

We discuss some special cases of Theorem A.1.

(i) $q = 5$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 2$. Then Theorem A.1 says that a family of meromorphic functions is normal, if the functions in the family do not have a simple island over any of five given Jordan domains with disjoint closures. This is the celebrated Ahlfors five islands theorem.

(ii) $q = 4$, $\mu_1 = \mu_2 = \mu_3 = 2$, $\mu_4 = \infty$. We note that if f is holomorphic and $\infty \in D_4$, then f has no island over D_4. Theorem A.1 thus implies that a family of holomorphic functions is normal, if the functions in the family do not have a simple island over any of three given plane Jordan domains with pairwise disjoint closures.

(iii) $q = 3$, $\mu_1 = \mu_2 = 3$, $\mu_3 = \infty$. With $\infty \in D_3$ we now deduce from Theorem A.1 that a family of holomorphic functions is normal, if the functions in the family do not have an island of multiplicity less than three over any of two given plane Jordan domains with disjoint closures.

As already mentioned, Theorems A.1 and A.2 can be considered as the main results of the Ahlfors theory of covering surfaces. Besides Ahlfors's original paper [1], we refer to [21, Chapter 5], [28, Chapter XIII] or [34, Chapter VI] for an account of the Ahlfors theory. A new proof of Theorems A.1 and A.2 was given in [10]. (Actually [10] was mainly concerned with the Ahlfors five islands theorem, but it was pointed out in [10, §5.1] that the method used also yields the more general "Scheibensatz.") In the first part of the proof in [10] it was shown by a fairly simple and elementary argument that the conclusion of Theorems A.1 and A.2 holds if the D_j are sufficiently small disks. In the second part of the proof quasiconformal mappings were used to reduce the case of general Jordan domains D_j to the case of small disks.

Since the version where the D_j are small disks suffices for the applications considered in this paper (as well as for many other applications), and since its proof is considerably easier and more elementary than the proof of the general version, we state this simplified version formally. We use the notation $D(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$ for $a \in \mathbb{C}$ and $r > 0$. In the following, let $a_1, \dots, a_q \in \mathbb{C}$ be distinct and let $\mu_1, \dots, \mu_q \in \mathbb{N}$, and suppose that (1) is satisfied.

Theorem B.1. *There exists $\varepsilon > 0$ such that $\mathcal{F}\left(D, \{(D(a_j, \varepsilon), \mu_j)\}_{j=1}^q\right)$ is normal.*

Theorem B.2. *There exists $\varepsilon > 0$ such that $\mathcal{F}\left(\mathbb{C}, \{(D(a_j, \varepsilon), \mu_j)\}_{j=1}^q\right)$ contains only the constant functions.*

For completeness we include a proof of Theorems B.1 and B.2 in §3 below, following the arguments of [10, 11].

3. A proof of Theorems B.1 and B.2

We denote the spherical derivative of a meromorphic function f by $f^\#$.

Lemma 1. *Let $D \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of functions meromorphic in D. If $\mathcal{F}$ is not normal, then there exist a sequence (z_k) in D, a sequence (ρ_k) of positive real numbers, a sequence (f_k) in $\mathcal{F}$, a point $z_0 \in D$ and a non-constant meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ such that $z_k \to z_0$, $\rho_k \to 0$ and $f_k(z_k + \rho_k z) \to f(z)$ locally uniformly in $\mathbb{C}$. Moreover, f can be chosen such that $f^\#(z) \leq 1 = f^\#(0)$ for all $z \in \mathbb{C}$.*

This lemma is due to Zalcman [35]. For a survey of various applications of this lemma we refer to [36]. We shall also need the following result.

Lemma 2. *Let $q \in \mathbb{N}$, $a_1, \dots, a_q \in \widehat{\mathbb{C}}$ distinct and $\mu_1, \dots, \mu_q \in \mathbb{N}$. Suppose that* (1) *is satisfied. Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function. Suppose that the a_j-points of f have multiplicity at least μ_j, for all $j \in \{1, \dots, q\}$. Then f is constant.*

This result was proved by Nevanlinna using his theory on the distribution of values, see [27, p. 102] or [28, §X.3]. A different proof was given by Robinson [29]. For a proof of Lemma 2 based on Lemma 1 we refer to [10, §3].

It is clear that Lemma 2 follows from Theorem B.2. Using Lemma 1, however, we will see that Theorems B.1 and B.2 can in turn be deduced from Lemma 1.

*To deduce Theorem B.*1 *from Lemma* 1 we assume that Theorem B.1 is false. Applying Lemma 1 to the family $\mathcal{F}\left(\mathbb{C}, \{(D(a_j, \varepsilon), \mu_j)\}_{j=1}^5\right)$ we obtain a meromorphic function $f_\varepsilon : \mathbb{C} \to \widehat{\mathbb{C}}$ with $f_\varepsilon^\#(z) \leq 1 = f_\varepsilon^\#(0)$ for all $z \in \mathbb{C}$. It is easy to see that $f_\varepsilon \in \mathcal{F}\left(\mathbb{C}, \{(D(a_j, \varepsilon'), \mu_j)\}_{j=1}^5\right)$ if $\varepsilon' > \varepsilon$. By Marty's theorem, $\{f_\varepsilon\}_{\varepsilon > 0}$ is normal. Thus there exists a sequence (ε_k) tending to zero such that $f_{\varepsilon_k} \to f$ for some meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$. Since $f_\varepsilon^\#(0) = 1$ for all $\varepsilon > 0$ we have $f^\#(0) = 1$ so that f is non-constant. Moreover, we see that all a_j-points of f have multiplicity at least μ_j, for $j \in \{1, \dots, 5\}$, contradicting Lemma 2. $\square$

*To prove Theorem B.*2 we note that if $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is a non-constant meromorphic function, then $\{f(nz)\}_{n \in \mathbb{N}}$ is not normal at 0. Thus Theorem B.2 follows immediately from Theorem B.1. $\square$

We note that Lemma 1 can in turn be used to deduce Theorem B.1 from Theorem B.2.

4. Fixed points and periodic points

Let X, Y be sets and let $f : X \to Y$ be a function. We define the iterates $f^n : X_n \to Y$ by $X_1 := X, f^1 := f$ and $X_n := f^{-1}(X_{n-1} \cap Y), f^n := f^{n-1} \circ f$ for $n \in \mathbb{N}$, $n \geq 2$. Note that $X_2 = f^{-1}(X_1 \cap Y) \subset X = X_1$ and thus $X_{n+1} \subset X_n \subset X$ for all $n \in \mathbb{N}$.

A point $\xi \in X$ is called a *periodic point of period* n of f if $\xi \in X_n$ and $f^n(\xi) = \xi$, but $f^m(\xi) \neq \xi$ for $1 \leq m \leq n-1$. A periodic point of period 1 is called a *fixed point.* The periodic points of period n are thus the fixed points of f^n which are not fixed points of f^m for any m less than n.

We shall be concerned with periodic points of holomorphic functions. Let ξ be a periodic point of period n of a holomorphic function f. Then $\lambda := (f^n)'(\xi)$ is called the *multiplier* of ξ. We say that a periodic point is *repelling, indifferent* or *attracting* depending on whether the modulus of its multiplier is greater than, equal to or less than 1. An indifferent periodic point is called *rationally indifferent* if the multiplier is a root of unity and *irrationally indifferent* otherwise.

The following lemma indicates why the Ahlfors theory may be useful to prove the existence of fixed points and periodic points.

Lemma 3. *Let $D \subset \widehat{\mathbb{C}}$ be a domain, $f : D \to \widehat{\mathbb{C}}$ a meromorphic function and $V \subset \mathbb{C}$ a Jordan domain.*

(i) *If f has a simple island U over V such that $\overline{U} \subset V$, then f has a repelling fixed point in U.*
(ii) *If f has an island U over V such that $\overline{U} \subset V$, then f has a fixed point in U which is repelling or has multiplier* 1.

Here we shall need only (i), but for completeness we have also included (ii). We will, however, only sketch the proof of (ii).

To prove (i) we note that there exists a branch ϕ of the inverse function of f such that $\phi(V) = U$. It follows easily that $\phi^n \to u$ as $n \to \infty$ for some $u \in U$, locally uniformly in V. This implies that u is an attracting fixed point of ϕ, and thus a repelling fixed point of f. □

To prove (ii) we consider the set P of all images of critical points of $f|_U$ under iterates of $f|_U$; that is, P is the set of all $z \in V$ for which there exists $n \in \mathbb{N}$ and $c \in U$ such that $f'(c) = 0$, $f^n(c) = z$ and $f^m(c) \in U$ for $1 \leq m < n$. Then $P \cap V \setminus U$ is finite. Given $v \in V \setminus P$ there exists $u \in U \setminus P$ with $f(u) = v$ and we can connect u and v by a path $\gamma \subset V \setminus P$. Moreover, there exists a simply-connected domain W with $\gamma \subset W \subset V \setminus P$. Let now $\phi : W \to \mathbb{C}$ be the branch of the inverse function of f which satisfies $\phi(v) = u$. Then we find that all iterates of ϕ are defined on V, and it turns out that $\bigcup_{n=1}^{\infty} \phi^n(\gamma)$ defines a curve which ends at a point $a \in U$. Moreover, a is a fixed point of f which is repelling or has multiplier 1; see, e. g., [24, p. 154] or [33, p. 57] for more details of this argument. □

An alternative, less elementary proof of Lemma 3, (ii) is described after Lemma 4 in §6 below.

The periodic points play an important role in complex dynamics. Let f be an entire or rational function. The basic objects studied in complex dynamics are the *Julia set* of f which, by definition, is the set where the iterates of f fail to be normal, and its complement, the set of normality or *Fatou set* of f. One of the fundamental results of the theory is that the Julia set is equal to the closure of the set of repelling periodic points. While this result was obtained for rational functions already by Fatou [20, §30, p. 69] and Julia [23, p. 99, p. 118] in their memoirs that founded the theory, it was proved for entire functions much later by Baker [3]. Baker's proof is based on Theorem A.1 and Lemma 3, (i). Meanwhile, however, simpler proofs based on Lemma 1 are available [4, 12, 31]; see also [11, §6.2] for further discussion.

5. Existence of periodic points of a given period

We consider the polynomial $p(z) := -z + z^2$. We note that $p(z) - z = z(z-2)$ and $p^2(z) - z = z^3(z-2)$. Thus p and p^2 have the same fixed points so that p has no periodic point of period 2. The following result of Baker [2] shows that p is essentially the only polynomial of degree greater than one where periodic points of some period are missing.

Theorem C. *Let f be a polynomial of degree $d \geq 2$ and let $n \in \mathbb{N}$. Suppose that f has no periodic point of period n. Then $d = n = 2$. Moreover, there exists a linear transformation L such that $f(z) = L^{-1}(p(L(z)))$, with $p(z) = -z + z^2$.*

The following result was conjectured in [22, Problem 2.20] and proved in [7, Theorem 1] and [8, §1.6, Satz 2].

Theorem D. *Let f be a transcendental entire function and let $n \in \mathbb{N}$, $n \geq 2$. Then f has infinitely many periodic points of period n.*

Actually the following stronger result was proved in [7, 8].

Theorem E. *Let f be a transcendental entire function and let $n \in \mathbb{N}$, $n \geq 2$. Then f has infinitely many repelling periodic points of period n.*

Similarly to Theorem C one can also describe the cases where a polynomial fails to have repelling periodic points of some period [8, §1.4, Satz 1].

Theorem F. *Let f be a polynomial of degree $d \geq 2$ and let $n \in \mathbb{N}$. Suppose that f has no repelling periodic point of period n. Then one of the following cases holds:*

(i) $n = 1$, $d \geq 2$,
(ii) $n = 2$, $d = 2$,
(iii) $n = 2$, $d = 3$,
(iv) $n = 2$, $d = 4$,
(v) $n = 3$, $d = 2$.

Examples in [8, §1.4] show that each of the five exceptional cases listed in Theorem F does occur.

The proof of Theorem F actually gives the following result.

Theorem G. *Let f be a polynomial of degree $d \geq 2$ and let $n \in \mathbb{N}$. Let N be the number of repelling periodic points of period N. Then*

$$N \geq d^n - \sum_{k<n, k|n} d^k - 2n(d-1). \tag{2}$$

A new proof of Theorem D was given in [5, §4]. Here we show that the arguments developed there also lead to a new proof of Theorem E. As we will use Theorem G there, we will also sketch its proof.

6. Polynomial-like maps

Besides the results from the Ahlfors theory discussed in §2, we will need the concept of a polynomial-like map to prove Theorem D. By definition, if $U, V \subset \mathbb{C}$ are bounded, simply-connected domains with $\overline{U} \subset V$, and if $f : U \to V$ is a proper holomorphic map (of degree d), then the triple (f, U, V) is called a *polynomial-like map* (of *degree* d). We note that if f, U and V are as in Lemma 3, then (f, U, V) is a polynomial-like map. The fundamental result about polynomial-like maps is the following one (see [13, Theorem VI.1.1] or [16, Theorem 1]).

Lemma 4. *Let (f, U, V) be a polynomial-like map of degree d. Then there exists a polynomial p of degree d and a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ such that $f(z) = \phi^{-1}(p(\phi(z)))$ for all $z \in U$. Moreover, $\phi(U)$ contains the filled Julia set of p and thus, in particular, all periodic points of p.*

Here the *filled Julia set* of a polynomial p is defined as the set of all points that do not tend to ∞ under iteration of p.

We remark that it easy to see that polynomials have a fixed point which is repelling or has multiplier 1. Therefore Lemma 3 follows from Lemma 4. However, Lemma 4 can be considered as a fairly advanced result, and thus the proof of Lemma 3 sketched in §4 is considerably more elementary than the one via Lemma 4.

Using Lemma 4 one can generalize many results about the dynamics of polynomials to polynomial-like mappings. In particular this applies to Theorems C, F and G, as well as to Lemmas 8–10 in §9 below. Alternatively, we can prove these Theorems and Lemmas directly for polynomial-like maps, with essentially the same proofs that work for polynomials, and thus avoid the use of Lemma 4.

7. Preliminaries for the proof of Theorem E

As mentioned, we shall use arguments similar to those used in [5], which in turn were very much inspired by papers by Essén and Wu [18, 19].

Let $f : D \to \mathbb{C}$ be holomorphic and let $U, V \subset \mathbb{C}$ be Jordan domains. Similarly as in [5] we will use the notation $U \overset{f,m}{\rightsquigarrow} V$ if $f|_{D \cap U}$ has an island of multiplicity at most m over V. We write $U \overset{f}{\rightsquigarrow} V$ if $f|_{D \cap U}$ has an island over V; that is, if $U \overset{f,m}{\rightsquigarrow} V$ for some $m \in \mathbb{N}$. Note that if $U \overset{f,m}{\rightsquigarrow} V$ and $V \overset{g,n}{\rightsquigarrow} W$ then $U \overset{g \circ f, mn}{\rightsquigarrow} W$. Lemma 3 now takes the following form.

Lemma 5.

(i) *If $V \overset{f,1}{\rightsquigarrow} V$, then f has a repelling fixed point in V.*

(ii) *If $V \overset{f}{\rightsquigarrow} V$, then f has a fixed point in V which is repelling or has multiplier* 1.

As in [5], we shall use some elementary concepts from graph theory. For a set V and a set $E \subset V \times V$ we call the pair $G = (V, E)$ a *digraph*. The elements of V are called *vertices* and those of E are called *edges*. In contrast to usual terminology we allow edges e of the form $e = (v, v)$ with $v \in V$. (Such edges are called *loops*.)

Let $n \in \mathbb{N}$ and $w = (v_0, v_1, \dots, v_n) \in V^{n+1}$. Then w is called a *closed walk of length* n if $(v_{k-1}, v_k) \in E$ for $k \in \{1, \dots, n\}$ and $v_0 = v_n$. Note that we have not excluded the case that $v_j = v_k$ for $j, k \in \{1, \dots, n\}$, $j \neq k$. We call a closed walk $w = (v_0, v_1, \dots, v_n)$ *primitive* if there does not exist $p \in \mathbb{N}$, $1 \leq p < n$, such that $p|n$ and $v_j = v_k$ for all

$j, k \in \{1, \ldots, n\}$ satisfying $p|(j-k)$. A primitive closed walk is thus a closed walk which is not obtained by running through a closed walk of smaller length several times.

Given pairwise disjoint disks $D_1, \ldots, D_q \subset \mathbb{C}$ and a holomorphic function f we consider the digraphs $G(f, \{D_j\}_{j=1}^q) = (V, E)$ and $G_1(f, \{D_j\}_{j=1}^q) = (V, E_1)$, with vertex set $V := \{D_1, \ldots, D_q\}$, and edge sets given by $E := \{(D_j, D_k) \in V \times V : D_j \overset{f}{\rightsquigarrow} D_k\}$ and $E_1 := \{(D_j, D_k) \in V \times V : D_j \overset{f,1}{\rightsquigarrow} D_k\}$.

Combining Lemma 5 with the remarks preceding it we obtain the following result.

Lemma 6.
(i) *If $G_1(f, \{D_j\}_{j=1}^q) = (V, E_1)$ contains a primitive closed walk of length n, then f has a repelling periodic point of period n in each D_j belonging to the walk.*
(ii) *If $G(f, \{D_j\}_{j=1}^q) = (V, E)$ contains a primitive closed walk of length n, then f has a periodic point of period n in each D_j belonging to the walk.*

To give conditions where the hypothesis of Lemma 6 are satisfied, we recall that the *outdegree* of a vertex v in a graph (V, E) is defined to be the cardinality of the set of all $u \in V$ for which $(v, u) \in E$. We have the following elementary results [5, Lemmas 6 and 9].

Lemma 7. *Let $q, n \in \mathbb{N}$, $n \geq 2$, and let $G = (V, E)$ be a digraph with q vertices.*
(i) *If $q \geq 6$ and if the outdegree of each vertex is a least $q - 2$, then G contains a primitive closed walk of length n.*
(ii) *If $q \geq 4$ and if the outdegree of each vertex is a least $q - 1$, then G contains a primitive closed walk of length n.*

Finally we recall that a family $\mathcal{F}$ of functions holomorphic in a domain D is called *quasinormal* (cf. [14, 25, 30]) if for each sequence (f_k) in $\mathcal{F}$ there exists a subsequence (f_{k_j}) and a finite set $E \subset D$ such that (f_{k_j}) converges locally uniformly in $D \backslash E$. If the cardinality of the exceptional set E can be bounded independently of the sequence (f_k), and if q is the smallest such bound, then we say that $\mathcal{F}$ is *quasinormal of order q*.

Note that the maximum principle implies that if a sequence (f_k) of functions holomorphic in a domain D converges locally uniformly in $D \backslash E$ for some finite subset E of D, but not in D, then $f_k \to \infty$ in $D \backslash E$.

8. A proof of Theorem E

We choose a sequence (c_k) in $\mathbb{C}$ which tends to ∞ and define $f_k : \mathbb{C} \to \mathbb{C}$ by $f_k(z) = f(c_k z)/c_k$. It is easy to see that no subsequence of (f_k) is normal at 0. We note that if ξ is a periodic point of f_k, then $c_k \xi$ is a periodic point of f, with the same period and multiplier. Let $\mathcal{F} := \{f_k\}_{k \in \mathbb{N}}$. We consider two cases.

Case 1: *$\mathcal{F}$ is not quasinormal of order* 6. Then there exists a subsequence (f_{k_j}) and six distinct points $a_1, \ldots, a_6 \in \mathbb{C}$ such that no subsequence of (f_{k_j}) is normal at one of these six points. Without loss of generality we may assume that $a_1 = 0$ and that for each $\ell \in \{1, \ldots, 6\}$, no subsequence of (f_k) is normal at a_ℓ. We choose $\varepsilon > 0$ as in Theorem B.1. It follows from Theorem B.1 that if $\ell \in \{1, \ldots, 6\}$ and k is large enough, then $D(a_\ell, \varepsilon) \overset{f_k,1}{\rightsquigarrow} D(a_m, \varepsilon)$ for at least four values of $m \in \{1, \ldots, 6\}$. Thus each vertex of $G_1(f, \{D(a_j, \varepsilon)\}_{j=1}^6) = (V, E_1)$ has outdegree at least 4. Lemmas 6 and 7 now imply that if k is large enough, then f_k has a repelling periodic point ξ_k of period n. Moreover, we may assume that $\xi_k \in D(a_{\ell_k}, \varepsilon)$ with $\ell_k \neq 1$. This implies that $\zeta_k := c_k \xi_k \to \infty$. Since ζ_k is a repelling periodic point of f, we conclude that f has infinitely many repelling periodic points.

Case 2: *$\mathcal{F}$ is quasinormal of order* 6. Then there exists a subsequence (f_{k_j}) and six points $a_1, \dots, a_6 \in \mathbb{C}$ such that (f_{k_j}) converges locally uniformly in $\mathbb{C} \setminus \{a_1, \dots, a_6\}$. On the other hand, no subsequence of (f_k) is normal at 0 and thus we deduce that $0 \in \{a_1, \dots, a_6\}$ and that $f_{k_j} \to \infty$ in $\mathbb{C} \setminus \{a_1, \dots, a_6\}$. Without loss of generality we shall assume that $f_k \to \infty$ in $\mathbb{C} \setminus \{a_1, \dots, a_6\}$. Since $f_k(0) = f(0)/c_k \to 0$ we find that if k is large enough, then there exists a component U_k of $f_k^{-1}(D(0,1))$ such that $(f_k, U_k, D(0,1))$ is a polynomial-like map, say of degree d_k. It is easy to see that $d_k \to \infty$ as $k \to \infty$. Lemma 4 and Theorem G now imply that f_k and hence f have at least $N_k := d_k^n - \sum_{\ell<n, \ell|n} d_k^\ell - 2n(d_k - 1)$ repelling periodic points of period n. Since $d_k \to \infty$ we conclude that $N_k \to \infty$ as $k \to \infty$. □

Remark. If we only want to prove Theorem D, then we can do without Lemma 4 and Theorem G. Instead we choose c_k such that $f(c_k)$ remains bounded so that $f_k(1) = f(c_k)/c_k \to 0$. In Case 2 we then find that $1 \in \{a_1, \dots, a_6\}$ and that if k is sufficiently large, then $G(f_k, \{D(0, \frac{1}{4}), D(1, \frac{1}{4})\})$ is the complete digraph, and thus contains primitive closed walks of any length. This, together with Lemma 6, (ii) yields the conclusion.

9. Some results from complex dynamics

To prove Theorem G we shall need some classical results from complex dynamics; see, e. g., [6, 13, 24, 26, 33] for an introduction to the subject. Let f be a polynomial of degree at least two and let z_0 be a periodic point of period p of f. For $1 \leq j \leq p-1$ we define $z_j := f^j(z_0)$. We call $\{z_0, z_1, \dots, z_{p-1}\}$ a *cycle of periodic points.*

Suppose first that z_0 is attracting. Then the other z_j are also attracting. We also say that the cycle of periodic points is attracting. For $0 \leq j \leq p-1$ we denote by U_j the component of the Fatou set that contains z_j. Then $\bigcup_{j=0}^{p-1} U_j$ is called *a cycle of immediate attracting basins.*

Lemma 8. *Each cycle of immediate attracting basins contains a critical point.*

Suppose now that z_0 is rationally indifferent and let t be the smallest positive integer such that $(f^p)'(z_0)^t = 1$. Then f^{pt} has the form

$$f^{pt}(z) = z + a_{m+1}(z - z_0)^{m+1} + O\left((z - z_0)^{m+2}\right) \tag{3}$$

as $z \to z_0$, with $a_{m+1} \neq 0$. It turns out that m is of the form $m = \ell t$ for some $\ell \in \mathbb{N}$. Moreover, for $k \in \mathbb{N}$ we have

$$f^{kpt}(z) = z + k a_{m+1}(z - z_0)^{m+1} + O\left((z - z_0)^{m+2}\right)$$

as $z \to z_0$. Next, for $0 \leq j \leq p-1$ there are m components U_{ij} $(1 \leq i \leq m)$ of the Fatou set of f such that $z_j \in \partial U_{ij}$ and $f^{\nu p}|_{U_{ij}} \to z_j$ as $\nu \to \infty$. The U_{ij} are called *Leau domains.* The set of the $pm = p\ell t$ Leau domains D_{ij} falls into ℓ disjoint subsets called *cycles of Leau domains*, each of pt domains, the domains of each subset being permuted cyclically by f.

Lemma 9. *Each cycle of Leau domains contains a critical point.*

Finally we mention the following result of Douady [15].

Lemma 10. *A polynomial of degree d has at most $d-1$ non-repelling cycles of periodic points.*

The idea in the proof of Lemma 10 is to perturb f slightly so that the indifferent periodic cycles become attracting. More specifically, a perturbation of the form $z \mapsto f_\varepsilon(z) := f(z) + \varepsilon P(z)$ with a suitable polynomial P (of high degree) and sufficiently small $\varepsilon > 0$ yields the desired result. Note that the degree of f_ε as a polynomial may be larger than d. The point is that if ε is sufficiently small, then there exist domains U and V containing the filled Julia set of f such that (f_ε, U, V) is a polynomial-like map of degree d. The conclusion then follows from Lemmas 4 and 8.

However, as remarked earlier, one can also prove Lemma 10 without making reference to Lemma 4, by proving Lemma 8 directly for polynomial-like maps. Thus Lemma 10 has a fairly elementary proof. The proof of the corresponding result for rational functions, due to Shishikura [32], is much more involved; see also [17].

Finally we note that for our purposes a weaker bound for the number of non-repelling cycles of a polynomial of degree d would suffice, e. g., the bound $2d-2$ obtained with Fatou's method.

10. A proof of Theorems F and G

Proof of Theorem G. For $k \in \mathbb{N}$ we denote by F_k the number of fixed points of f^k counted according to multiplicity, and by $\overline{F}_k$ the corresponding number where multiplicities are ignored. Similarly, the number of periodic points of f of period k is denoted by P_k, if multiplicities are counted, and by $\overline{P}_k$ otherwise. Clearly we have $\overline{P}_k \le \overline{F}_k \le F_k = d^k$ and

$$\overline{F}_n - \overline{P}_n \le \sum_{k<n, k|n} \overline{P}_k \le \sum_{k<n, k|n} d^k.$$

We write $\overline{F}_n = F_n - (F_n - \overline{F}_n) = d^n - (F_n - \overline{F}_n)$ and obtain

$$\overline{P}_n \ge \overline{F}_n - \sum_{k<n, k|n} d^k = d^n - \sum_{k<n, k|n} d^k - (F_n - \overline{F}_n). \tag{4}$$

To estimate the term $F_n - \overline{F}_n$, let z_0 be a fixed point of f^n that contributes to it. Let m be the contribution of z_0 to the term $F_n - \overline{F}_n$; that is, z_0 is a fixed point of f^n of multiplicity $m+1$. Let p be the period of z_0. Then the periodic cycle $\{z_0, z_1, \dots, z_{p-1}\}$, with $z_j = f^j(z_0)$, contributes pm to the term $F_n - \overline{F}_n$. Let ℓ be the number of cycles of Leau domains associated to the periodic cycle $\{z_0, z_1, \dots, z_{p-1}\}$. We shall show that

$$pm \le n\ell. \tag{5}$$

In fact, let t be the smallest positive integer such that $(f^p)'(z_0)^t = 1$. Then $pt|n$, and f^{pt} has the form (3), with $m = \ell t$. Now (5) follows since $pm = p\ell t$ and $pt \le n$.

It follows from (5) that $F_n - \overline{F}_n \le nL$, where L is the number of cycles of Leau domains of f. Since $L \le d-1$ by Lemma 9, we obtain

$$F_n - \overline{F}_n \le n(d-1) \tag{6}$$

Finally we note that if Q denotes the number of non-repelling periodic points of period n, then $N = \overline{P}_n - Q$. Since the number of non-repelling periodic cycles of period n is at most $d-1$ by Lemma 10, we obtain $Q \leq n(d-1)$ and thus

$$N \geq \overline{P}_n - n(d-1). \tag{7}$$

Combining (4), (6) and (7) we obtain (2). □

Proof of Theorem F. As already mentioned, Theorem F follows easily from Theorem G. In fact, if $n = 2$, then $N \geq d^2 - d - 4(d-1) = (d-1)(d-4)$ by (2), and thus $N > 0$ if $d > 4$.

If $n = 3$, then (2) yields $N \geq d^3 - d - 6(d-1) = (d-1)(d-2)(d+3)$ so that $N > 0$ if $d > 2$.

Finally, if $n \geq 4$, let m be the largest integer less than n that divides n. Then $m \leq n-2$ since $n \geq 4$. From (2) we obtain

$$\begin{aligned} N &\geq d^n - \sum_{k \leq m} d^k - 2n(d-1) \\ &= d^n - \frac{d^{m+1} - d}{d-1} - 2n(d-1) \\ &\geq d^n - d^{m+1} + d - 2n(d-1) \\ &\geq d^n - d^{n-1} + d - 2n(d-1) \\ &= (d^{n-1} - 2n)(d-1) + d \\ &\geq (2^{n-1} - 2n) + d \\ &\geq d, \end{aligned}$$

and this completes the proof of Theorem F. □

Remark. It follows from (4) and (6) that

$$\overline{P}_n \geq d^n - \sum_{k<n, k|n} d^k - n(d-1).$$

This implies that $\overline{P}_n > 0$, except possibly if $n = d = 2$. A further investigation of the case $n = d = 2$ then leads to Theorem C. This is essentially the proof of Theorem C given by Baker [2].

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