数理解析研究所講究録 1269 巻 2002 年 24-33

Deformation of a structurally finite entire function

Masahiko Taniguchi

Graduate School of Science, Kyoto University

1 A survey on combinatorial models

Consider a possibly incomplete and branched holomorphic covering $\pi : R \to \mathbb{C}$ of $\mathbb{C}$ by a simply connected Riemann surface R. Then we can represent the projection π in a combinatorial manner. Nevanlinna ([2]) and others used so-called "line complexes". The origin of such combinatorial graphs would be a Klein diagram. But in the case of entire functions, we can use a simpler model (cf. [3]). First we recall the definition of such a combinatorial model.

Definition 1 (Configuration tree) A *configuration tree* is a planar tree with countably many vertices, one of which is marked as *the initial vertex* (and hence every edge has an orientation towards the initial vertex). A configuration tree is colored as follows:

1. There are two kind of vertices; white ones and black ones.
2. There are three kind of edges; white ones, black ones, and red ones.
3. Every connected component of the set of all white vertices and white edges can be identified with the tree $\mathbb{R}$ with vertices $\mathbb{Z}$, and hence is called a *$\mathbb{Z}$-unit.*
4. Every edge not in any $\mathbb{Z}$-unit is colored black or red, according as it starts from a black vertex or from a white vertex.

To recover the holomorphic covering, we associate a configuration tree with the *configuration data.*

1. *the singularity data*; a center locus is attached to every $\mathbb{Z}$-unit and a decoration locus is attached to every black edge, and
2. *a spider at* ∞, which assigns every distinct singularity datum a mutually disjoint path from ∞ to it.

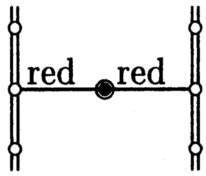

Figure 1: A configuration tree

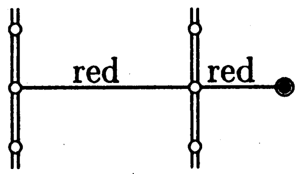

Figure 2: Another equivalent tree

The configuration data determine uniquely the holomorphic covering represented by the configuration tree. On the other hand, there are some ambiguity to determine a tree from a holomorphic covering. One of them is the choice of the initial vertex.

Definition 2 We call a pair of a red edge and its ending black vertex a *reduction pair*. And when we change the initial vertex to another one, we delete all reduction pairs whose red edges have the opposite orientation in the new tree, and attach a new pair to every white vertex such that a black edge now starts from it. We say that such a new configuration tree is obtained from the old one by a *change of the initial vertex*. Further, if a white vertex is the initial one, then we may attach a reduction pair and regard that the new black vertex is the initial one.

We say that two configuration trees are *equivalent*, if, after suitable changes of the initial vertices of both, they are identical including colors.

A typical example of a change of the initial vertex is the following

Example 3 The configuration trees in Figures 1 and 2 are equivalent. Actually, in Figure 1, change the initial vertex of the tree to the right white vertex. Then we should delete the right reduction pair. And we can add a new reduction pair to make the initial vertex black. This is the tree in Figure

2 Covering models and realizablility

To construct a covering model from a configuration tree, we impose some restriction on the singularity data.

Definition 4 The projection π is called a *function with a finite number of clusters of singularity* if the set of all singular values of π has only a finite number of accumulation points in $\mathbb{C}$. Further if the set of all singular values is bounded in $\mathbb{C}$, then we call such a π an *approximate Speiser function.* Finally, π is called a *Spieser function* as usual if it has only a finite number of singular values.

The following example shows the difference between these concepts.

Example 5 1. $z \sin z$ is a function with a finite number of (actually no) clusters of singurarity,

2. $\dfrac{\sin z}{z}$ is an approximate Speiser function,

3. $\sin z$ is a Speiser function.

Now the issue is whether the given tree (with some configuration data) can represent an entire function or not. This is a variant of the classical *type problem* of Riemann surfaces.

Definition 6 We say that a configuration tree T is *realizable* (with respect to some configuration data) if there is an entire function f which is represented by a tree equivalent to T under the following injunctions;

1. a black edge and its starting black vertex represent a Maskit surgery attaching a quadratic block

2. a red edge and its starting $\mathbb{Z}$-unit represent a Maskit surgery attaching an exp-block,

where corresponding cross-cuts intersect no legs of the spider. We call T a configuration tree of f. And in the case 1., we also say that a black edge and its starting black vertex represent a $\mathbb{C}$-*decoration* and the decorated $\mathbb{C}$-block (used in [1]), respectively.

Here, *quadratic blocks* are holomorphic covering given by

$$az^2 + bz + c : \mathbb{C} \to \mathbb{C} \qquad (a \neq 0),$$

exponential blocks (exp-blocks) are those by

$$a \exp bz + c : \mathbb{C} \to \mathbb{C} \qquad (ab \neq 0),$$

and $\mathbb{C}$-*blocks* are trivial coverings by

$$az + b : \mathbb{C} \to \mathbb{C} \qquad (a \neq 0).$$

And Maskit surgeries are defined as follows.

Definition 7 (Maskit surgery by connecting functions) Let $\pi_j : R_j \to \mathbb{C}$ $(j = 1, 2)$ be a possibly incomplete and branched holomorphic covering of $\mathbb{C}$ by a simply connected Riemann surface R_j and A_j be the set of all singular values of π_j for each j. Assume that there is a cross-cut L in $\mathbb{C}$, i.e. the image of a proper continuous injection of the real line into $\mathbb{C}$, such that

1. $L \cap A_1$ equals to $L \cap A_2$, and is either empty or consists of a single point z_0, which is an isolated point of each A_j,
2. $\mathbb{C} - L$ consists of two connected components D_1 and D_2, where D_j contains $A_j - \{z_0\}$ for each j, and
3. if $L \cap A_1 = L \cap A_2 = \{z_0\}$, then z_0 is a critical value of each π_j, i.e for a small disk U with center z_0 such that $U \cap A_j = \{z_0\}$, $\pi_j^{-1}(U)$ has a relatively compact component W_j which contains a critical point of π_j for each j.

Under the same circumstance as above, suppose that the projection π of a (possibly incomplete and branched) holomorphic covering $\pi : R \to \mathbb{C}$ by a simply connected Riemann surface R satisfies the following condition; there exist

1. a component $\tilde{D}_1$ of $\pi_1^{-1}(D_2)$ and a component $\tilde{D}_2$ of $\pi_2^{-1}(D_1)$ such that $\pi_j : \tilde{D}_j \to D_{3-j}$ is biholomorphic and $\tilde{D}_j \cap W_j \neq \emptyset$ if $L \cap A_j$ are non-empty,
2. a cross-cut $\tilde{L}$ in $\mathbb{C}$ such that π gives a homeomorphism of $\tilde{L}$ onto L, and
3. conformal maps ϕ_j of $\mathbb{C} - \tilde{D}_j$ onto U_j such that $\pi_j = \pi \circ \phi_j$ on $\mathbb{C} - \tilde{D}_j$ for each j, where U_1 and U_2 are components of $\mathbb{C} - \tilde{L}$.

Then we say that the projection π or the holomorphic covering $\pi : R \to \mathbb{C}$ is constructed from the coverings $\pi_j : R_j \to \mathbb{C}$ by the *Maskit surgery* with respect to L. We also say that $\pi : R \to \mathbb{C}$ is constructed from $\pi_k : R_k \to \mathbb{C}$ by the *Maskit surgery attaching* $\pi_{3-k} : R_{3-k} \to \mathbb{C}$ with respect to L for each $k = 1, 2$.

Example 8 1. The configuration trees in Figures 1 and 2 with two singularity data is realizable by a function

$$\mathrm{Cerf}(z) = a \int_0^z e^{t^2} dt + b.$$

2. The black $\mathbb{Z}$-unit with alternating singular data is realizable by a function $a \sin z + b$

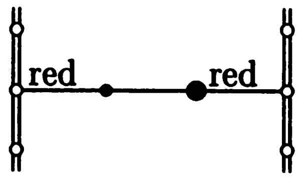

Figure 3: A configuration tree of $\exp z^2$

3. The black $\mathbb{Z}^+$-unit (the tree $\{x \geq 1\}$ with vertices $\mathbb{Z}^+$) with alternating singular data is realizable by a function

$$\Sigma^+(z) = a \int_0^z \prod_{k=1}^{\infty} \left(1 - \frac{t}{k^2}\right) dt + b.$$

Note that in each case, if the set of all singularity data consists of a single value, then the tree is not realizable.

Another typical example of non-realizable configuration tree with singularity data of Speiser type is the following

Example 9 Let T be a configuration tree such that every vertex is black and is an end point of exactly three black edges, and that every triple of edges connected at the same vertex is attached by the same triple of three distinct complex numbers. Then the second main theorem of Nevenlinna (cf. [2]) implies that T with this singularity data is not realizable.

Now we consider another entire Speiser function

$$f(z) = e^{z^2}.$$

Then from the covering structure, we can see that a configuration tree of f is

$$\exp + \text{quad} + \exp\,.$$

On the other hand, attaching a quadratic block to the covering induced by $\mathrm{Cerf}(z)$, we have a configuration tree

$$(\exp + \exp) + \text{quad}.$$

These two configuration trees are non-equivalent and also represent different functions. (If the center loci are the same, the latter is not realizable.) But the relation between these trees are similar to that between those in Figures 1 and 2. Thus as a basic deformation of trees, we can consider the following moves of edges.

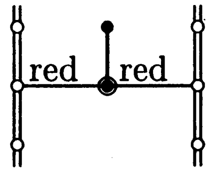

Figure 4: A configuration tree of a decorated Cerf(z)

Definition 10 (Simple move) Let T be a configuration tree. Fix a non-white edge L, and let v and v' be the ending and the starting vertex of L. Further Let L_1 be an edge starting from v, and w be the ending vertex of L_1. Then the *simple move of the edge L along L_1* is sliding the edge L along L_1 to a new edge L' from v' ending at w. We write as $T' = T(L; L_1)$ the new configuration tree obtained from T by this move. Also we call T the configuration tree obtained from T' by the the *inverse simple move* of L' along L_1, and denote T by $T'(L'; L_1^{-1})$. Here we assume that L and L' are attached the same singularity datum if they are black.

Example 11 (formal simple move) We call a simple move along an edge L_1 a *formal simple move* if L_1 is red. Figures 1 and 2 gives such a move.

A formal simple move induces no change of the realizing entire function.

In partucular, we can apply formal simple moves freely.

Proposition 12 *The projection π of a covering as above (with a finite number of clusters of singularity) can be represented by a configuration tree such that, for every starting white vertex w_0 of a red edge, there are no non-white edges ending at w_0.*

We say that such a tree as in the above proposition is *completely reduced.* In the sequel, we consider only completely reduced trees.

3 Geometirc deformation

Now to clearify the relation between realizing functions of the trees in Figures 3 and 4, for instance, with the same configuration data, we need to find a geometric representation of simple moves, which can be achieved by the following geometric deformation of the realizing function.

Here, we consider the projection π of a covering by a simply connected Riemann surface R with mutually distinct real and positive singurality data, and we attach to every tree of π the *canonical spider at* ∞, which is a spider whose legs are parallel to the imaginary axis and come from ∞ to the data.

Then a fundamental geometric deformation of such a function is the following one.

Definition 13 (Transposition of the spider's legs) Let T be a configuration tree which represents $\pi : R \to \mathbb{C}$ as above. Let $\{\alpha, \beta\}$ be the pair of distinct singularity data. Assume that $\alpha < \beta$ and that there are no singularity data in the open interval (α, β).

We say that g is obtained from π by a *transposition of the spider's legs* for the pair $\{\alpha, \beta\}$ if $g = \phi_1 \circ \pi \circ \phi_2$ with quasiconformal self-maps ϕ_1 of $\mathbb{C}$ and ϕ_2 of R such that

1. ϕ_1 restricted on the set of all singularity data gives the transposition of the singularity data α and β,
2. the leg ℓ_γ of the canonical spider ending at γ is mapped by ϕ_1 to the leg ending at $\phi_1(\gamma)$ for every singularity datum γ, except for the leg ℓ_β, and
3. the image $\phi_1(\ell_\beta)$ and the leg ℓ_α ending at $\alpha = \phi_1(\beta)$ in the canonical spider form a cross-cut which separates β from all singularity data other than $\{\alpha, \beta\}$.

Proposition 14 (Transposition lemma) *Let T be a completely reduced configuration tree which represents $\pi : R \to \mathbb{C}$ as in Definition 13, and α_1 and α_2 be two singularity data of T such that $\alpha_1 < \alpha_2$ and there are no singularity data in the open interval (α_1, α_2). Let α_j be attached to $\hat{L}_{\alpha_j}$ for each j, where $\hat{L}_{\alpha_j}$ is either a black edge or a $\mathbb{Z}$-unit of T.*

Now suppose that the shortest simple path connecting $\hat{L}_{\alpha_1}$ and $\hat{L}_{\alpha_2}$ in T contains a non-red edge. Then we can find a holomorphic function g which is quasiconformally equivalent to π, is represented by the same configuration tree T (with respect to the canonical spider), and gives the same set of singularity data as that of π, but the singularity data α_j is now attached to $\hat{L}_{\alpha_{3-j}}$ for each $j = 1, 2$.

Next, we consider the case that either $\hat{L}_{\alpha_1} \cup \hat{L}_{\alpha_2}$ is connected or the shortest simple path Γ connecting $\hat{L}_{\alpha_1}$ and $\hat{L}_{\alpha_2}$ contains no non-red edges. In the sequel, we consider the case that the *upper* white edge, i.e. the orientation of a white edge corresponds to the counter-clockwise rotation around the corresponding singularity datum. The case of the lower white edge can be treated similarly, and hence omitted. (Actually, this assumption depends on the choice of a covering model, and hence is not essential.)

Here, if $\hat{L}_{\alpha_j}$ is a black edge, then we set $L_j = \hat{L}_{\alpha_j}$, and if $\hat{L}_{\alpha_j}$ is a $\mathbb{Z}$-unit, then we take as L_j the white edge ending at or starting from $\hat{L}_{\alpha_1} \cap \hat{L}_{\alpha_2}$, or

$\hat{L}_{\alpha_j} \cap \Gamma$ if Γ is non-degenerate, for each j. Also let L_j^* be the red edge in Γ starting from $\hat{L}_{\alpha_j}$ if exists. Further, we assume that white L_j are upper. And we change the intial vertex to the common vertex of $\{L_j \cup L_j^*\}_{j=1}^2$. (Also we attach a reduction pair and apply a formal simple move if necessary to make the initial vertex black). Then we obtain the following

Proposition 15 *Under the same circumstances as in Proposition 14, suppose that either $\hat{L}_{\alpha_1} \cup \hat{L}_{\alpha_2}$ is connected or the shortest simple path Γ connecting $\hat{L}_{\alpha_1}$ and $\hat{L}_{\alpha_2}$ contains no non-red edges.*

Then the transposition of the spider's legs for $\{\alpha_1, \alpha_2\}$ gives a holomorphic function g quasiconformally equivalent to π whose configuration tree T_g is as follows:

1. *Suppose that both of L_j end at the same (initial) vertex. (Since the initial vertex is black, both of L_j are black.) Then the configuration tree T_g is obtained from T by the inverse simple move of L_1 along L_2, i.e.*
$$T_g = T(L_1; L_2^{-1}).$$

2. *Suppose that L_1 is white and that L_1^* exists and ends at the ending black vertex of L_2. Then the configuration tree T_g is obtained from T by the inverse simple move of L_1^* along L_2, i.e.*
$$T_g = T(L_1^*; L_2^{-1}).$$

3. *Suppose that L_2 is white and that L_2^* exsits and ends at the ending black vertex of L_1. Then the configuration tree T_g is obtained from T by the inverse simple move of L_1 along L_2 in the tree obtained by the inverse simple move of L_1 along L_2^*, i.e.*
$$T_g = T^*(L_1; L_2^{-1}),$$
where $T^* = T(L_1; (L_2^*)^{-1})$.

4. *Suppose that both of L_j are white and that L_j^* exist and end at the same (initial) vertex. Then the configuration tree T_g is obtained from T by the inverse simple move of L_1^* along L_2 in the tree obtained by the inverse simple move of L_1^* along L_2^*, i.e.*
$$T_g = T^*(L_1^*; L_2^{-1})$$
where $T^* = T(L_1^*; (L_2^*)^{-1})$.

In every case, α_j is attached to $\hat{L}_{\alpha_{3-j}}$ in T_g for each j.

Remark 16 *General cases can be treated by changing the initial vertex to the original one. As an example, we consider here the case that L_1 is white and L_1^* ends at the starting black vertex w of a black edge L_2. First change the initial vertex to w, the orientation of L_2 is reversed, which we denote by L_2^r. And by Proposition 15, we have the tree $T(L_1^*; (L_2^r)^{-1})$. Next change the initial vertex to the original one. Then the orientation of L_2 is again reversed, and hence we have the tree $T(L_1^*; L_2)$ with respect to the original initial vertex.*

4 Structurally finite entire functions

Now we cosider the simplest case.

Definition 17 We say that an entire function is *structurally finite* if it is constructed from a finite number of quadratic blocks and/or exp-blocks by Maskit surgeries. A structurally finite function is, by definition, *of type* (p, q) if it is constructed from p quadratic blocks and q exp-blocks.

In this case, we can find the "simplest" tree.

Definition 18 Let R be a $\mathbb{C}$-block attached p quadratic blocks with decoration loci $\{1, \cdots, p\}$ and q exp-blocks with center loci $\{p+1, \cdots, p+q\}$. We call the projection π the *standard simple function $f_{p,q}$ of type* (p, q). The corresponding tree $T_{p,q}$ is called the *standard tree of type* (p, q).

Now, take a structurally finite entire function f of type (p, q) which has $p + q$ singularity data. Then by using geometric deformations as above, we can show the following

Proposition 19 *The standard simple function $f_{p,q}$ of type (p, q) is quasiconformally equivalent to f. In particular, the tree $T_{p,q}$ is actually realizable.*

More generaly, we have the following

Theorem 20 *Let π be the projection of a covering by a Riemann surface R, which is constructed from p quadratic blocks and q exp-blocks by Maskit surgeries with respect to mutually distinct singularity loci. Then π is quasiconformally equivalent to f.*

For proofs, see [5].

Also structurally finite entire functions of type (p, q) admits explicit representations of the form

$$\int^{z} P(t)e^{Q(t)}dt$$

with polynomials $P(z)$ and $Q(z)$ of degree exactly p and q ([4]). We denote by $F_{p,q}$ the totality of such indefinite integrals.

Proposition 21 (cf. [1] Theorem 3.3.1) *The family $F_{p,q}$ is topologically complete.*

References

[1] S. Morosawa, Y. Nishimura, M. Taniguchi, and T. Ueda, *Holomorphic Dynamics,* Cambridge Univ. Press, 1999.

[2] R. Nevanlinna, *Analytic Functions,* Springer, 1970.

[3] M. Taniguchi, *Maskit surgery of entire functions,* RIMS kokyuroku **2001**, (2001) 7-16.

[4] M. Taniguchi, *Explicit representation of structurally finite entire functions,* Proc. Japan Acad. **77**, (2001), 68-70.

[5] M. Taniguchi, *Synthetic deformation spaces of an entire function,* to appear.