数理解析研究所講究録 1269 巻 2002 年 42-47

Remarks on several theorems related to finiteness and the linearization problem on entire functions

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1 Introduction

Structurally finite entire functions are constructed from finitely many quadratic blocks and exponential blocks by (Klein and) Maskit surgeries, which shall be defined in Section 3, connecting two functions. For examples, every polynomial of degree $d+1$ is constructed from d quadratic blocks and the *complex error function* $a\int_0^z \exp t^2 dt + b$ is constructed from two exponential blocks so they are all structurally finite.

We shall study:

Question. We suppose that a structurally finite entire function has a cycle whose multiplier is $\lambda = e^{2\pi i\alpha}, \alpha \in \mathbb{R}\setminus\mathbb{Q}$ (then this cycle is said to be *irrationally indifferent*). If it is a Siegel cycle, then does α satisfies the *Brjuno condition*?

The Brjuno condition is defined by

$$\sum_{n>0} \frac{\log q_{n+1}}{q_n} < \infty,$$

where $\{q_n\}$ is the sequence of denominators of the rational numbers approximating $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ defined by its continued fraction expansion. An irrationally indifferent cycle of an entire function f of period n with multiplier $e^{2\pi i\alpha}$ is called a *Siegel* cycle if every element of this cycle has a neighborhood where the n times iteration f^n is conformally conjugate to the $2\pi\alpha$-rotation around the origin on a disk, otherwise a *Cremer* cycle. The converse of Question is true from the Brjuno Theorem which says that if α satisfies the Brjuno condition, then every holomorphic germ fixing the origin with multiplier $e^{2\pi i\alpha}$ has a neighborhood of the origin where it is conformally conjugate to the $2\pi\alpha$-rotation around the origin on a disk.

Yoccoz gave a beautiful alternative proof of the Brjuno theorem in [10] and also showed that if a *quadratic* polynomial has a Siegel *fixed point* whose multiplier is λ, then α satisfies the Brjuno condition. Pérez-Marco proved it for *structurally stable* polynomials with Siegel *fixed points.* In [7], we have proved it for a subclass of *n-subhyperbolic* polynomials, which shall be defined below, with Siegel *cycles*, and, in particular, we have that Question is true for Siegel cycles of quadratic polynomials.

In this article, we shall state Main Theorem on Question for structurally finite entire functions in Section 3. For this purpose, we also survey several useful theorems on transcendental entire or meromorphic functions with some kinds of finiteness which are sometimes stated only for polynomials or rational functions.

2 Several theorems related to finiteness

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function which is neither constant nor a linear transformation. The *Fatou set* $F(f)$ is the set of all points each of which has a neighborhood U where f^n is defined for all $n \in \mathbb{N}$ and $\{f^n|U\}_{n\in\mathbb{N}}$ is normal. The *Julia set* $J(f)$ is the compliment of $F(f)$ in $\hat{\mathbb{C}}$.

A component of $F(f)$ (a Fatou component, in abbreviation) is said to be *cyclic* if f^n maps U into itself for some $n \in \mathbb{N}$, otherwise *wandering.* For the classification of cyclic Fatou components into one of (super)attractive basins, parabolic basins, Siegel disks, Arnord-Herman rings and Baker domains, see, for examples, [6].

First, we assume that the number of singular values of f^{-1} is *finite.* We say such f to be of the *Speiser class.* Then it is known that every singular value is either a critical value or an asymptotic value.

The following theorem is essentially by Goldberg-Keen [4] and Eremenko-Lyubich [2], who proved it for entire functions.

Theorem 1 (No wandering domain theorem). *If a meremorphic function is of the Speiser class, then it has no wandering Fatou components.*

Eremenko-Lyubich also proved the following for entire functions. For how to generalize it to meromorphic functions, see [1], p172.

Theorem 2 (No Baker domain). *If a meremorphic function is of the Speis class, then it has no Baker domains.*

Second, we assume that the number of critical *points* and asymptotic values of f is finite. Since the number of critical values is less than that of critical points, f is then of the Speiser class. However, the converse does not hold (for example, consider $\sin z$).

It is known that for every meromorphic function, each of its attractive basins and parabolic basins contains at least one of critical points and asymptotic values. To describe a relation between other cyclic Fatou components or Cremer cycles and singular values, we prepare:

Definition (Omega limit set and recurrence). Let g be a meromorphic function. For $c \in \mathbb{C}$, the *omega limit set* $\omega(c) = \omega_g(c)$ is the set of all $z \in \hat{\mathbb{C}}$ such that $\lim_{i\to\infty} g^{n_i}(c) = z$ for some $\{n_i\} \subset \mathbb{N}$.

A point c is *recurrent* if $\omega(c) \ni c$.

The following theorem is essentially due to Mañé [5]. For an alternative proof, see [8].

Theorem 3 (Mañé). 1. *Let Λ be a compact subset of $J(f)$ which is forward invariant, that is, $f(\Lambda) \subset \Lambda$. If it contains none of parabolic periodic points and critical points and satisfies*

$$\Lambda \cap \left(\bigcup_{s;\text{a recurrent critical point or an asymptotic value}} \omega(s) \right) = \emptyset,$$

then $f|\Lambda$ is expanding, that is, there exists $N > 0$ such that for all $n > N$, $\min_{z\in\Lambda} \|(f^n)'(z)\|_{\sigma_{\hat{\mathbb{C}}}} > 1$, where $\sigma_{\hat{\mathbb{C}}}$ is the spherical metric on $\hat{\mathbb{C}}$.

2. *Let $\Gamma(\subset J(f))$ be any one of a Cremer cycle, a union of boundary components of a Siegel disk and that of an Arnold-Herman ring. Then there exists a point $s \in J(f)$ which is either a recurrent critical point or an asymptotic value such that $\omega(s) \supset \Gamma$.*

Definition (corresponding). Let Γ be any one of a Cremer cycle, a union of boundary components of a Siegel disk and that of an Arnold-Herman ring. A recurrent critical point or asymptotic value s *corresponds to* Γ if it satisfies $\omega(s) \supset \Gamma$.

Finally, we assume that the number of critical *points* of f and *transcendental singularities of* f^{-1} is finite. We fix the definition of the latter, which are ideal points, and see that the number of transcendental singularities of f^{-1} is less than that of asymptotic values of f.

Definition. For $a \in \hat{\mathbb{C}}$, let $A := \{A(r)\}_{r>0}$ be a family of domains in $\mathbb{C}$ such that for $r > 0$, $A(r)$ is a component of $f^{-1}(\mathbb{D}_r(a))$ and if $0 < r_1 < r_2$, then $A(r_1) \subset A(r_2)$.

Then $\bigcap_{r>0} \overline{A(r)}^{\hat{\mathbb{C}}}$ contains at most one point. If it is the infinity, the A is called a *transcendental singularity of* f^{-1} over a. Note that then the a is an asymptotic value of f. We say that the A *corresponds to* Γ if so does a.

Now we define the *n-subhyperbolicity* of such f.

Definition (n-subhyperbolicity [7]). For a non-negative integer n, f is *n-subhyperbolic* if

(i) there exist exactly n recurrent critical points of f or transcendental singularities of f^{-1} corresponding to irrationally indifferent cycles,

(ii) every critical point in $J(f)$ other than such ones as (i) and asymptotic values in $J(f)$ over which there is no such transcendental singularities of f^{-1} as (i) is eventually periodic, and

(iii) no orbits of singular values in $F(f)$ accumulate to $J(f)$.

An n-subhyperbolic f is *n-hyperbolic* if it has no such ones as (ii).

Remark. The 0-(sub)hyperbolicity agrees with just a (sub) hyperbolicity.

3 The structurally finite entire functions

First we explain two kinds of *building blocks.* Ones are *quadratic blocks*

$$az^2 + bz + c : \mathbb{C} \to \mathbb{C} \quad (a \neq 0),$$

and the others are *exponential blocks* (exp-blocks)

$$a \exp bz + c : \mathbb{C} \to \mathbb{C} \quad (ab \neq 0).$$

Definition (Maskit surgery [9]). Let $\pi_i : R_j \to \mathbb{C}$ $(j = 1, 2)$ be a possibly incomplete and branched holomorphic covering of $\mathbb{C}$ by a simply connected Riemann surface R_j, and let A_j be the set of all singular values of π_j for each $j = 1, 2$. Assume that there is a cross-cut L in $\mathbb{C}$, i.e., the image of a proper continuous injection of the real line into $\mathbb{C}$ such that

1. $L \cap A_1$ equals to $L \cap A_2$, and either is empty or consists of only one point z_0, which is an isolated point of each A_j,

2. $\mathbb{C} \setminus L$ consists of two connected components D_1 and D_2, where D_j contains $A_j \setminus \{z_0\}$ for each $j = 1, 2$, and

3. if $L \cap A_1 (= L \cap A_2) = \{z_0\}$, then z_0 is a critical value of each π_j, i.e., for a small disk U with center z_0 satisfying $U \cap A_j = \{z_0\}$, there exists a component W_j of $\pi_j^{-1}(U)$ which is relatively compact in $\mathbb{C}$ and contains a critical point of π_j for each $j = 1, 2$.

Under the above assumption, suppose that the projection π of a (possibly incomplete and branched) holomorphic covering of $\mathbb{C}$ by a simply connected Riemann surface R satisfies the following conditions: There exist

1. a component $\tilde{D}_1$ of $\pi_1^{-1}(D_2)$ and a component $\tilde{D}_2$ of $\pi_2^{-1}(D_1)$ such that for each $j = 1, 2$, $\pi_j : \tilde{D}_j \to D_{3-j}$ is a holomorphic surjection and $\tilde{D}_j \cap W_j \neq \emptyset$ if $L \cap A_j \neq \emptyset$,

2. a cross cut $\tilde{L}$ in $\mathbb{C}$ such that π gives a homeomorphism of $\tilde{L}$ onto L, and

3. conformal maps ϕ_j of $\mathbb{C} \setminus \tilde{D}_j$ onto U_j such that $\pi_j = \pi \circ \phi_j$ on $\mathbb{C} \setminus \tilde{D}_j$ for each $j = 1, 2$, where both U_1 and U_2 are the components of $\mathbb{C} \setminus \tilde{L}$.

Then we say that the holomorphic covering $\pi : R \to \mathbb{C}$ is constructed from the coverings $\pi_j : R_j \to \mathbb{C}$ $(j = 1, 2)$ by the Maskit surgery with respect to L and also, if $L \cap A_j \neq \emptyset$, to $\{W_j\}_{j=1,2}$. We also say that $\pi : R \to \mathbb{C}$ is constructed from $\pi_j : R_j \to \mathbb{C}$ by the Maskit surgery *attaching* $\pi_{3-j} : R_{3-j} \to \mathbb{C}$ with respect to L and possibly to $\{W_j\}_{j=1,2}$.

We especially call such a surgery a *Klein surgery* with respect to L if $L \cap A_j$ is empty for $j = 1, 2$.

Definition (structural finiteness). A *structurally finite entire function of type* (p, q) is an entire function constructed from p quadratic blocks and q exp-blocks.

Clearly, if f is a structurally finite entire function, then f has only finitely many critical points of f and transcendental singularities of f^{-1}. Conversely, that characterizes the structural finiteness of an entire function. For a combinatorial study of such entire functions, see Taniguchi [9].

Now we return our Question in Section 1 and state our Main Theorem:

Main Theorem. *If a* 1*-hyperbolic structurally finite entire function of type* $(p, q) \neq (0, 1)$ *has a Siegel fixed point whose multiplier is* $\lambda = e^{2\pi i \alpha}$*, then* α *satisfies the Brjuno condition.*

Remark. An example of such a function as the above is $\lambda \int_0^z (1 + t) e^t dt$. The above theorem for this function is first proved by Geyer [3].

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