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<tr>
<td>Author(s)</td>
<td>Uchimura, Keisuke</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2002), 1269: 103-109</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42159">http://hdl.handle.net/2433/42159</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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The sets of non-escaping points of generalized Chebyshev mappings

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1 Introduction

Let $G_c$ be the polynomial self-mapping of $\mathbb{C}^2$ defined by

$$G_c(x, y) = (x^2 - cy, y^2 - cx).$$

It admits an invariant line $\{x = y\}$ on which it acts as the quadratic polynomial

$$f_c(z) = z^2 - cz.$$

If $c$ is real, the map $G_c$ admits an invariant plane $\{x = \Psi\}$, on which it acts as

$$F_c(z) = z^2 - cz.$$

The purpose of this paper is to understand the dynamics of $F_c$ as a self-map of $\mathbb{C}$. The mapping $G_c$ can be extended as a holomorphic self-map of $\mathbb{C}P^2$

$$g_c([x : y : z]) = [x^2 - cyz : y^2 - cxz : z^2].$$

Ueda [1999] shows that any holomorphic map on $\mathbb{C}P^2$ of degree 2 is equivalent to one of the following maps:

1. $U_1([x : y : z]) = [x^2 : y^2 : z^2],$
2. $U_2([x : y : z]) = [x^2 + yz : y^2 : z^2],$
3. $U_3([x : y : z]) = [x^2 + yz : y^2 + xz : z^2],$
4. $U_4([x : y : z]) = [x^2 + \lambda xy + y^2 : z^2 + xy : yz].$

Note that $g_c$ is equivalent to $U_3$. 
The map

$$F_c(z) = z^2 - cz$$

has a connection with a physical model when $c = 2$. It is Chebyshev map

$$F_2(z) = z^2 - 2z.$$

A. Lopes [1990, 1992] considered the dynamics of $F_2$ as a special kind of Potts model and showed that triple point phase transition (three equilibrium states) exists. He conjectured that if $c > 2$, a Cantor set with expanding dynamics exists. It is known that for expanding systems equilibrium states are unique. We explain triple point phase transition. We consider the pressure

$$P(t) = \sup_{\nu \in M(f)} \{ h(\nu) - \frac{t}{2} \int \log |\det(Df(z))|d\nu(z) \}.$$  

$M(f)$ denotes the set of invariant probabilities and $h(\nu)$ is the entropy of $\nu$. For each $t$, if the measure $\mu(t)$ is the solution of the variation problem, $\mu(t)$ is called the equilibrium measure. Multiple equilibrium measures of $F_2(z) = z^2 - 2z$ are stated as follows:

1. If $-\frac{4}{3} < t$ then $\mu(t) = \mu(= \frac{3}{\pi^2} \frac{1}{\sqrt{-(z^2)^2 + 4(z^3 + z^3) - 18z^2 + 27}} dx dy),$

2. If $t = -\frac{4}{3}$ then there exist triple point phase transition $\mu(t)$:

$$\mu \text{ (not magnetic), } \frac{1}{2} \delta_{p_2} + \frac{1}{2} \delta_{p_3} \text{ (magnetic), } \delta_{p_1} \text{ (anti-ferromagnetic),}$$

3. If $t < -\frac{4}{3}$ then there exist two equilibrium states $\mu(t)$:

$$\frac{1}{2} \delta_{p_2} + \frac{1}{2} \delta_{p_3}, \delta_{p_1}.$$

We give an affirmative answer to Lopes's conjecture. More generally, we show an analogue of the result which are well known for quadratic polynomials. In the paper we assume that $c$ is real.
2 Dynamics of $G_c(x, y)$ and $F_c(z)$

We show the following two theorems. Let $K(g) = \{ z \in \mathbb{C} \mid g^n(z) : n = 0, 1, 2, \ldots, \text{is bounded} \}$.

**Theorem 1.** $K(F_c)$ is connected with the simply connected complement in $CP^1$ if and only if $-4 \leq c \leq 2$.

**Theorem 2.** If $c > 2$, then

1. $K(F_c)$ is a Cantor set;
2. the two-dimensional Lebesgue measure of $K(F_c)$ is 0;
3. $F_c$ restricted to $K(F_c)$ is topological conjugate to the shift on 4 symbols;
4. the measure of maximal entropy of $G_c(x, y)$ is supported in the real plane $\{ x = \overline{y} \}$.

We see the analogue as follows.

Let $f_c(z) = z^2 + c$ and $F_c(z) = z^2 - cz$.

(a) $K(f_c)$ is connected with the simply connected complement if and only if $-2 \leq c \leq \frac{1}{4}$.

(A) $K(F_c)$ is connected with the simply connected complement if and only if $-4 \leq c \leq 2$.

Note that $f_c(x)$ on $[-2, \frac{1}{4}]$ and $F_c(x)$ on $[-4, 2]$ are topological conjugate.

(b) If $c < -2$ then,

1. $K(f_c)$ is a Cantor set;
2. the one dimensional Lebesgue measure of $K(f_c)$ is 0;
3. $\{K(f_c), f_c\}$ and $\{\Sigma_2, \sigma\}$ are equivalent;
4. Julia set of $f_c$ is included in the set $[-q, q]$.

(B) If $c > 2$ then,
(1) $K(F_c)$ is a Cantor set;
(2) the two dimensional Lebesgue measure of $K(F_c)$ is 0;
(3) $\{K(F_c), F_c\}$ and $\{\Sigma_4, \sigma\}$ are equivalent;
(4) the smallest Julia set of $G_c$ is included in the set $\{x = \bar{y}\}$.

To prove the assertion (1) of Theorem 2, we show the following result for non-conformal maps $F_c$. If $c > 2$, for any connected component $K(i_1, ..., i_n)$ in $F_c^{-n}(D)$, the diameter $[K(i_1, ..., i_n)]$ approaches 0 as $n \to \infty$. To prove this we introduce a Riemannian metric

$$
\frac{1}{\mu}\{(z^2 - 3z)dz^2 + (9 - z\bar{z})dzd\bar{z} + (z^2 - 3\bar{z})d\bar{z}^2\},
$$

where $\mu = -z^2\bar{z}^2 + 4(z^3 + \bar{z}^3) - 18z\bar{z} + 27$.

This metric goes to $\infty$ on the boundary $\partial S$. This is a generalization of the invariant measure

$$
\frac{1}{\pi \sqrt{x(1-x)}} \quad \text{for} \quad f(x) = 4x(1-x).
$$

3 Proofs

We show only the proof of the assertion (4) of Theorem 2 in this paper. Proofs of the other assertions of Theorem 2 and that of Theorem 1 are stated in Uchimura [2001] and so are omitted. In this paper we use the same definitions and notations as are used in Uchimura [2001].

**Lemma 1.** The number of periodic points of order $n$ of $g_c([x : y : z]) = [x^2 - cyz : y^2 - cxz : z^2]$ with $z \neq 0$ is $4^n$.

**Proof.** From Corollary 3.2 of [Fornaess and Sibony, 1994], this lemma follows immediately. \(\square\)

**Lemma 2.** If $c > 2$, the number of periodic points of order $n$ of the function $F_c(z) = z^2 - c\bar{z}$ is $4^n$.
Proof. From the proof of Theorem 4.1 of [Uchimura, 2001], we see that there exists a positive integer $n$ such that

$$(F_c)^{-n}(D_c) \subset \frac{c}{2}S.$$ 

Let $N$ be the smallest integer satisfying the above property. Let

$$\Gamma = (F_c)^{-N}(int(D_c)).$$

Then it can be proved that $\Gamma$ is an open connected set. From Proposition 2.2 of [Uchimura, 2001], we know that there exist homeomorphisms $\varphi_k$, $(k = 0, 1, 2, 3)$, from $\frac{c^2}{4}S$ to $S_k$ with $S_k \subset \frac{c}{2}S$ such that the composition $F_c \circ \varphi_k$ is an identity map. From Proposition 3.1 of [Uchimura, 2001], we have

$$(F_c)^{-1}(\Gamma) \subset \Gamma.$$

Hence

$$\bigcup_{k=0}^{3} \varphi_k(\Gamma) \subset \Gamma$$

and so

$$\varphi_k(\Gamma) \subset \Gamma.$$ 

Applying Fixed Point Theorem to $\varphi_k$, we get a fixed point $p_k$ in $\Gamma$ such that $\varphi_k(p_k) = p_k$. Hence we have 4 fixed points of $F_c$.

By the similar argument, we can prove this lemma when $n > 1$.

Combining Lemma 1 and Lemma 2, we have the following proposition.

**Proposition 3.** If $c > 2$, then any periodic point of $G_c(x, y)$ lies in the plane $\{(x, \overline{x})|x \in \mathbb{C}\}$.

Let $H = \{(x, \overline{x})|x \in \mathbb{C}\}$. We denote the Jacobian matrix of the map $G_c(x, y)$ at the point $(u, v)$ by $DG_c(u, v)$. $G_c$ restricted on $H$ is the map $F_c(z)$. The map $F_c(z)$ may be viewed as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. We denote the Jacobian matrix of the map $F_c$ at $(u_1, u_2)$ by $DF_c(u)$ where $u = u_1 + iu_2, u_1, u_2 \in \mathbb{R}$. 
Lemma 4. We consider the map $G_c(x, y)$ when $c$ is real. Let $(u, v)$ be a periodic point. Suppose the periodic point $(u, v)$ lies in $H$. Then the set of eigenvalues of $DG_c(u, \bar{u})$ are identical with that of $DF_c(u)$.

Proof. Clearly,

$$DG_c(x, y) = \begin{pmatrix} 2x & -c \\ -c & 2y \end{pmatrix}.$$ 

Then

$$DG_c(u, \bar{u}) = \begin{pmatrix} 2(u_1 + iu_2) & -c \\ -c & 2(u_1 - iu_2) \end{pmatrix}.$$ 

On the other hand,

$$DF_c(u) = \begin{pmatrix} 2u_1 - c & -2u_2 \\ 2u_2 & 2u_1 + c \end{pmatrix}.$$ 

Set

$$U = \frac{1}{2} \begin{pmatrix} 1 + i & -1 + i \\ 1 + i & 1 - i \end{pmatrix}.$$ 

Clearly $U$ is an unitary matrix. Then we can easily prove that

$$U^{-1}DG_c(u, \bar{u})U = DF_c(u).$$ 

\[\square\]

In Proposition 3, we show that if $c > 2$, all periodic points of $G_c(x, y)$ lie in $H$. Next we show they are all repelling.

Proposition 5. If $c > 2$, then any periodic point of $G_c(x, y)$ is repelling.

Proof. From Lemma 4, we see that to prove this proposition it suffices to show that any periodic point of $F_c(z)$ is repelling. This follows from the fact that for any connected component $K(i_1, \ldots, i_n)$ in $F_c^{-n}(D)$, the diameter $[K(i_1, \ldots, i_n)]$ approaches to 0 as $n \to \infty$.

\[\square\]

Combining Proposition 5 and Corollary V.2.1. in [Briend, 1997], we can prove the assertion (4) of Theorem 2.

\[\square\]
References


