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The sets of non-escaping points of generalized Chebyshev mappings

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1 Introduction

Let $G_c$ be the polynomial self-mapping of $\mathbb{C}^2$ defined by

$$G_c(x, y) = (x^2 - cy, y^2 - cx).$$

It admits an invariant line $\{x = y\}$ on which it acts as the quadratic polynomial

$$f_c(z) = z^2 - cz.$$

If $c$ is real, the map $G_c$ admits an invariant plane $\{x = \overline{y}\}$, on which it acts as

$$F_c(z) = z^2 - cz\overline{z}.$$

The purpose of this paper is to understand the dynamics of $F_c$ as a self-map of $\mathbb{C}$. The mapping $G_c$ can be extended as a holomorphic self-map of $\mathbb{CP}^2$

$$g_c([x : y : z]) = [x^2 - cyz : y^2 - cxz : z^2].$$

Ueda [1999] shows that any holomorphic map on $\mathbb{CP}^2$ of degree 2 is equivalent to one of the following maps:

1. $U_1([x : y : z]) = [x^2 : y^2 : z^2]$,
2. $U_2([x : y : z]) = [x^2 + yz : y^2 : z^2]$,
3. $U_3([x : y : z]) = [x^2 + yz : y^2 + xz : z^2]$,
4. $U_4([x : y : z]) = [x^2 + \lambda xy + y^2 : z^2 + xy : yz]$.

Note that $g_c$ is equivalent to $U_3$. 
The map
\[ F_c(z) = z^2 - cz \]
has a connection with a physical model when \( c = 2 \). It is Chebyshev map
\[ F_2(z) = z^2 - 2z. \]

A. Lopes [1990,1992] considered the dynamics of \( F_2 \) as a special kind of Potts model and showed that triple point phase transition (three equilibrium states) exists. He conjectured that if \( c > 2 \), a Cantor set with expanding dynamics exists. It is known that for expanding systems equilibrium states are unique. We explain triple point phase transition. We consider the pressure
\[ P(t) = \sup_{\nu \in M(f)} \{ h(\nu) - \frac{t}{2} \int \log |\det(Df(z))| d\nu(z) \}. \]

\( M(f) \) denotes the set of invariant probabilities and \( h(\nu) \) is the entropy of \( \nu \). For each \( t \), if the measure \( \mu(t) \) is the solution of the variation problem, \( \mu(t) \) is called the equilibrium measure. Multiple equilibrium measures of \( F_2(z) = z^2 - 2z \) are stated as follows:

1. if \( -\frac{4}{3} < t \) then \( \mu(t) = \mu = \frac{1}{2} \delta_{p_2} + \frac{1}{2} \delta_{p_3} \)
2. if \( t = -\frac{4}{3} \) then there exist triple point phase transition \( \mu(t) : \mu \) (not magnetic), \( \frac{1}{2} \delta_{p_2} + \frac{1}{2} \delta_{p_3} \) (magnetic), \( \delta_{p_1} \) (anti-ferromagnetic),
3. if \( t < -\frac{4}{3} \) then there exist two equilibrium states \( \mu(t) : \frac{1}{2} \delta_{p_2} + \frac{1}{2} \delta_{p_3}, \delta_{p_1} \).

We give an affirmative answer to Lopes's conjecture. More generally, we show an analogue of the result which are well known for quadratic polynomials. In the paper we assume that \( c \) is real.
2 Dynamics of $G_c(x, y)$ and $F_c(z)$

We show the following two theorems. Let $K(g) = \{ z \in \mathbb{C} \mid g^n(z) : n = 0, 1, 2, ..., \text{is bounded} \}$.

**Theorem 1.** $K(F_c)$ is connected with the simply connected complement in $\mathbb{C}P^1$ if and only if $-4 \leq c \leq 2$.

**Theorem 2.** If $c > 2$, then

1. $K(F_c)$ is a Cantor set;
2. the two-dimensional Lebesgue measure of $K(F_c)$ is 0;
3. $F_c$ restricted to $K(F_c)$ is topological conjugate to the shift on 4 symbols;
4. the measure of maximal entropy of $G_c(x, y)$ is supported in the real plane $\{ x = \bar{y} \}$.

We see the analogue as follows.

Let $f_c(z) = z^2 + c$ and $F_c(z) = z^2 - cz$.

(a) $K(f_c)$ is connected with the simply connected complement if and only if $-2 \leq c \leq \frac{1}{4}$.

(A) $K(F_c)$ is connected with the simply connected complement if and only if $-4 \leq c \leq 2$.

Note that $f_c(x)$ on $[-2, \frac{1}{4}]$ and $F_c(x)$ on $[-4, 2]$ are topological conjugate.

(b) If $c < -2$ then,

1. $K(f_c)$ is a Cantor set;
2. the one dimensional Lebesgue measure of $K(f_c)$ is 0;
3. $\{K(f_c), f_c\}$ and $\{\Sigma_2, \sigma\}$ are equivalent;
4. Julia set of $f_c$ is included in the set $[-q, q]$.

(B) If $c > 2$ then,
(1) $K(F_c)$ is a Cantor set;
(2) the two dimensional Lebesgue measure of $K(F_c)$ is 0;
(3) $\{K(F_c), F_c\}$ and $\{\Sigma_4, \sigma\}$ are equivalent;
(4) the smallest Julia set of $G_c$ is included in the set $\{x = \bar{y}\}$.

To prove the assertion (1) of Theorem 2, we show the following result for non-conformal maps $F_c$. If $c > 2$, for any connected component $K(i_1, \ldots, i_n)$ in $F_c^{-n}(D)$, the diameter $[K(i_1, \ldots, i_n)]$ approaches 0 as $n \to \infty$. To prove this we introduce a Riemannian metric

$$\frac{1}{\mu}\{(\overline{z}^2 - 3z)dz^2 + (9 - z\overline{z})dzd\overline{z} + (z^2 - 3\overline{z})d\overline{z}^2\},$$

where $\mu = -z^2\overline{z}^2 + 4(z^3 + \overline{z}^3) - 18z\overline{z} + 27$.

This metric goes to $\infty$ on the boundary $\partial S$. This is a generalization of the invariant measure

$$\frac{1}{\pi \sqrt{x(1-x)}} \quad \text{for} \quad f(x) = 4x(1-x).$$

3 Proofs

We show only the proof of the assertion (4) of Theorem 2 in this paper. Proofs of the other assertions of Theorem 2 and that of Theorem 1 are stated in Uchimura [2001] and so are omitted. In this paper we use the same definitions and notations as are used in Uchimura [2001].

Lemma 1. The number of periodic points of order $n$ of $g_c([x : y : z]) = [x^2 - cyz : y^2 - cxz : z^2]$ with $z \neq 0$ is $4^n$.

Proof. From Corollary 3.2 of [Fornaess and Sibony, 1994], this lemma follows immediately. \qed

Lemma 2. If $c > 2$, the number of periodic points of order $n$ of the function $F_c(z) = z^2 - c\overline{z}$ is $4^n$. 

Proof. From the proof of Theorem 4.1 of [Uchimura, 2001], we see that there exists a positive integer $n$ such that

$$(F_{c})^{-n}(D_{c}) \subset \frac{c}{2}S.$$ 

Let $N$ be the smallest integer satisfying the above property. Let

$$\Gamma = (F_{c})^{-N}(int(D_{c})).$$

Then it can be proved that $\Gamma$ is an open connected set. From Proposition 2.2 of [Uchimura, 2001], we know that there exist homeomorphisms $\varphi_{k}, \ (k = 0, 1, 2, 3)$, from $\frac{c^{2}}{4}S$ to $S_{k}$ with $S_{k} \subset \frac{c}{2}S$ such that the composition $F_{c} \circ \varphi_{k}$ is an identity map. From Proposition 3.1 of [Uchimura, 2001], we have

$$(F_{c})^{-1}(\Gamma) \subset \Gamma.$$

Hence

$$\bigcup_{k=0}^{3} \varphi_{k}(\Gamma) \subset \Gamma$$

and so

$$\varphi_{k}(\Gamma) \subset \Gamma.$$ 

Applying Fixed Point Theorem to $\varphi_{k}$, we get a fixed point $p_{k}$ in $\Gamma$ such that $\varphi_{k}(p_{k}) = p_{k}$. Hence we have 4 fixed points of $F_{c}$.

By the similar argument, we can prove this lemma when $n > 1$. 

$\square$

Combining Lemma 1 and Lemma 2, we have the following proposition.

Proposition 3. If $c > 2$, then any periodic point of $G_{c}(x, y)$ lies in the plane $\{(x, \bar{x}) | x \in C\}$.

Let $H = \{(x, \bar{x}) | x \in C\}$. We denote the Jacobian matrix of the map $G_{c}(x, y)$ at the point $(u, v)$ by $DG_{c}(u, v)$. $G_{c}$ restricted on $H$ is the map $F_{c}(z)$. The map $F_{c}(z)$ may be viewed as a map from $\mathbb{R}^{2}$ to $\mathbb{R}^{2}$. We denote the Jacobian matrix of the map $F_{c}$ at $(u_{1}, u_{2})$ by $DF_{c}(u)$ where $u = u_{1} + iu_{2}, \ u_{1}, u_{2} \in \mathbb{R}$. 
Lemma 4. We consider the map $G_c(x, y)$ when $c$ is real. Let $(u, v)$ be a periodic point. Suppose the periodic point $(u, v)$ lies in $H$. Then the set of eigenvalues of $DG_c(u, \bar{u})$ are identical with that of $DF_c(u)$.

Proof. Clearly,

$$DG_c(x, y) = \begin{pmatrix} 2x & -c \\ -c & 2y \end{pmatrix}. $$

Then

$$DG_c(u, \bar{u}) = \begin{pmatrix} 2(u_1 + iu_2) & -c \\ -c & 2(u_1 - iu_2) \end{pmatrix}. $$

On the other hand,

$$DF_c(u) = \begin{pmatrix} 2u_1 - c & -2u_2 \\ 2u_2 & 2u_1 + c \end{pmatrix}. $$

Set

$$U = \frac{1}{2} \begin{pmatrix} 1 + i & -1 + i \\ 1 + i & 1 - i \end{pmatrix}. $$

Clearly $U$ is an unitary matrix. Then we can easily prove that

$$U^{-1}DG_c(u, \bar{u})U = DF_c(u). \quad \square $$

In Proposition 3, we show that if $c > 2$, all periodic points of $G_c(x, y)$ lie in $H$. Next we show they are all repelling.

Proposition 5. If $c > 2$, then any periodic point of $G_c(x, y)$ is repelling.

Proof. From Lemma 4, we see that to prove this proposition it suffices to show that any periodic point of $F_c(z)$ is repelling. This follows from the fact that for any connected component $K(i_1, \ldots, i_n)$ in $F_c^{-n}(D)$, the diameter $[K(i_1, \ldots, i_n)]$ approaches to 0 as $n \to \infty. \quad \square$

Combining Proposition 5 and Corollary V.2.1. in [Briend, 1997], we can prove the assertion (4) of Theorem 2. \quad \square
References


