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The sets of non-escaping points of generalized Chebyshev mappings

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1 Introduction

Let G_c be the polynomial self-mapping of $\mathbf{C}^2$ defined by

$$G_c(x,y)=(x^2-cy,y^2-cx).$$

It admits an invariant line $\{x=y\}$ on which it acts as the quadratic polynomial

$$f_c(z)=z^2-cz.$$

If c is real, the map G_c admits an invariant plane $\{x=\overline{y}\}$, on which it acts as

$$F_c(z)=z^2-c\overline{z}.$$

The purpose of this paper is to understand the dynamics of F_c as a self-map of $\mathbf{C}$. The mapping G_c can be extended as a holomorphic self-map of $\mathbf{CP}^2$

$$g_c([x:y:z])=[x^2-cyz:y^2-cxz:z^2].$$

Ueda [1999] shows that any holomorphic map on $\mathbf{CP}^2$ of degree 2 is equivalent to one of the following maps :
(1) $U_1([x:y:z])=[x^2:y^2:z^2]$,
(2) $U_2([x:y:z])=[x^2+yz:y^2:z^2]$,
(3) $U_3([x:y:z])=[x^2+yz:y^2+xz:z^2]$,
(4) $U_4([x:y:z])=[x^2+\lambda xy+y^2:z^2+xy:yz]$.

Note that g_c is equivalent to U_3.

The map

$$F_c(z) = z^2 - c\bar{z}$$

has a connection with a physical model when $c = 2$. It is Chebyshev map

$$F_2(z) = z^2 - 2\bar{z}.$$

A. Lopes [1990,1992] considered the dynamics of F_2 as a special kind of Potts model and showed that triple point phase transition (three equilibrium states) exists. He conjectured that if $c > 2$, a Cantor set with expanding dynamics exists. It is known that for expanding systems equilibrium states are unique. We explain triple point phase transition. We consider the pressure

$$P(t) = \sup_{\nu \epsilon M(f)} \{h(\nu) - \frac{t}{2}\int \log|\det(Df(z))| d\nu(z)\}.$$

$M(f)$ denotes the set of invariant probabilities and $h(\nu)$ is the entropy of ν. For each t, if the measure $\mu(t)$ is the solution of the variation problem, $\mu(t)$ is called the equilibrium measure. Multiple equilibrium measures of $F_2(z) = z^2 - 2\bar{z}$ are stated as follows :

(1) if $-\frac{4}{3} < t$ then $\mu(t) = \mu(= \frac{3}{\pi^2}\frac{1}{\sqrt{-(z\bar{z})^2 + 4(z^3 + \bar{z}^3) - 18z\bar{z} + 27}}dxdy)$,

(2) if $\quad t = -\frac{4}{3}$ then there exist triple point phase transition $\mu(t)$:

μ (not magnetic), $\frac{1}{2}\delta_{p_2} + \frac{1}{2}\delta_{p_3}$ (magnetic), δ_{p_1} (anti-ferromagnetic),

(3) if $t < -\frac{4}{3}$ then there exist two equilibrium states $\mu(t)$:

$$\frac{1}{2}\delta_{p_2} + \frac{1}{2}\delta_{p_3}, \delta_{p_1}.$$

We give an affirmative answer to Lopes's conjecture. More generally, we show an analogue of the result which are well known for quadatic polynomials. In the paper we assume that c is real.

2 Dynamics of $G_c(x, y)$ and $F_c(z)$

We show the following two theorems. Let $K(g) = \{z \in \mathbf{C} \mid g^n(z) : n = 0, 1, 2, \ldots, \text{is bounded}\}$.

Theorem 1. *$K(F_c)$ is connected with the simply connected complement in $\mathbf{C}P^1$ if and only if $-4 \leq c \leq 2$.*

Theorem 2. *If $c > 2$, then*

(1) $K(F_c)$ is a Cantor set;

(2) the two-dimensional Lebesgue measure of $K(F_c)$ is 0;

(3) F_c restricted to $K(F_c)$ is topological conjugate to the shift on 4 symbols;

(4) the measure of maximal entropy of $G_c(x, y)$ is supported in the real plane $\{x = \bar{y}\}$.

We see the analogue as follows .

Let $f_c(z) = z^2 + c$ and $F_c(z) = z^2 - c\bar{z}$.

(a) $K(f_c)$ is connected with the simply connected complement if and only if $-2 \leq c \leq \frac{1}{4}$.

(A) $K(F_c)$ is connected with the simply connected complement if and only if $-4 \leq c \leq 2$.

Note that $f_c(x)$ on $[-2, \frac{1}{4}]$ and $F_c(x)$ on $[-4, 2]$ are topological conjugate.

(b) If $c < -2$ then,

(1) $K(f_c)$ is a Cantor set ;

(2) the one dimensional Lebsgue measure of $K(f_c)$ is 0;

(3) $\{K(f_c), f_c\}$ and $\{\Sigma_2, \sigma\}$ are equivalent;

(4) Julia set of f_c is included in the set $[-q, q]$.

(B) If $c > 2$ then,

(1) $K(F_c)$ is a Cantor set ;
(2) the two dimensional Lebsgue measure of $K(F_c)$ is 0;
(3) $\{K(F_c), F_c\}$ and $\{\Sigma_4, \sigma\}$ are equivalent;
(4) the smallest Julia set of G_c is included in the set $\{x = \bar{y}\}$.

To prove the assertion (1) of Theorem 2, we show the following result for non-conformal maps F_c. If $c > 2$, for any connected component $K(i_1, ..., i_n)$ in $F_c^{-n}(D)$, the diameter $[K(i_1, ..., i_n)]$ approaches 0 as $n \to \infty$. To prove this we introduce a Riemannian metric

$$\frac{1}{\mu}\{(\bar{z}^2 - 3z)dz^2 + (9 - z\bar{z})dzd\bar{z} + (z^2 - 3\bar{z})d\bar{z}^2\},$$

$$\text{where } \mu = -z^2\bar{z}^2 + 4(z^3 + \bar{z}^3) - 18z\bar{z} + 27.$$

This metric goes to ∞ on the boundary ∂S. This is a generalization of the invariant measure

$$\frac{1}{\pi\sqrt{x(1-x)}} \quad \text{for} \quad f(x) = 4x(1-x).$$

3 Proofs

We show only the proof of the assertion (4) of Theorem 2 in this paper. Proofs of the other assertions of Theorem 2 and that of Theorem 1 are stated in Uchimura [2001] and so are omitted. In this paper we use the same definitions and notations as are used in Uchimura [2001].

Lemma 1. *The number of periodic points of order n of $g_c([x : y : z]) = [x^2 - cyz : y^2 - cxz : z^2]$ with $z \neq 0$ is 4^n.*

Proof. From Corollary 3.2 of [Fornaess and Sibony, 1994], this lemma follows immediately. □

Lemma 2. *If $c > 2$, the number of periodic points of order n of the function $F_c(z) = z^2 - c\bar{z}$ is 4^n.*

Proof. From the proof of Theorem 4.1 of [Uchimura, 2001], we see that there exists a positive integer n such that

$$(F_c)^{-n}(D_c) \subset \frac{c}{2}S.$$

Let N be the smallest integer satisfying the above property. Let

$$\Gamma = (F_c)^{-N}(int(D_c)).$$

Then it can be proved that Γ is an open connected set. From Proposition 2.2 of [Uchimura, 2001], we know that there exist homeomorphisms φ_k, $(k = 0, 1, 2, 3)$, from $\frac{c^2}{4}S$ to S_k with $S_k \subset \frac{c}{2}S$ such that the composition $F_c \circ \varphi_k$ is an identity map. From Proposition 3.1 of [Uchimura, 2001], we have

$$(F_c)^{-1}(\Gamma) \subset \Gamma.$$

Hence $\bigcup_{k=0}^{3} \varphi_k(\Gamma) \quad \subset \Gamma$

and so

$$\varphi_k(\Gamma) \quad \subset \Gamma.$$

Applying Fixed Point Theorem to φ_k, we get a fixed point p_k in Γ such that $\varphi_k(p_k) = p_k$. Hence we have 4 fixed points of F_c.

By the similar argument, we can prove this lemma when $n > 1$. □

Combining Lemma 1 and Lemma 2, we have the following proposition.

Proposition 3. *If $c > 2$, then any periodic point of $G_c(x, y)$ lies in the plane $\{(x, \bar{x}) | x \in \mathbf{C}\}$.*

Let $H = \{(x, \bar{x}) | x \in \mathbf{C}\}$. We denote the Jacobian matrix of the map $G_c(x, y)$ at the point (u, v) by $DG_c(u, v)$. G_c restricted on H is the map $F_c(z)$. The map $F_c(z)$ may be viewed as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$. We denote the Jacobian matrix of the map F_c at (u_1, u_2) by $DF_c(u)$ where $u = u_1 + iu_2$, $u_1, u_2 \in \mathbf{R}$.

Lemma 4. *We consider the map $G_c(x,y)$ when c is real. Let (u,v) be a periodic point. Suppose the periodic point (u,v) lies in H. Then the set of eigenvalues of $DG_c(u,\bar{u})$ are identical with that of $DF_c(u)$.*

Proof. Clearly,

$$DG_c(x,y) = \begin{pmatrix} 2x & -c \\ -c & 2y \end{pmatrix}.$$

Then

$$DG_c(u,\bar{u}) = \begin{pmatrix} 2(u_1 + iu_2) & -c \\ -c & 2(u_1 - iu_2) \end{pmatrix}.$$

On the other hand,

$$DF_c(u) = \begin{pmatrix} 2u_1 - c & -2u_2 \\ 2u_2 & 2u_1 + c \end{pmatrix}.$$

Set

$$U = \tfrac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}.$$

Clearly U is an unitary matrix. Then we can easily prove that

$$U^{-1}DG_c(u,\bar{u})U = DF_c(u). \qquad \square$$

In Proposition 3, we show that if $c > 2$, all periodic points of $G_c(x,y)$ lie in H. Next we show they are all repelling.

Proposition 5. *If $c > 2$, then any periodic point of $G_c(x,y)$ is repelling.*

Proof. From Lemma 4, we see that to prove this proposition it suffices to show that any periodic point of $F_c(z)$ is repelling. This follows from the fact that for any connected component $K(i_1,...,i_n)$ in $F_c^{-n}(D)$, the diameter $[K(i_1,...,i_n)]$ approaches to 0 as $n \to \infty$. $\square$

Combining Proposition 5 and Corollary V.2.1. in [Briend, 1997], we can prove the assertion (4) of Theorem 2. $\square$

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