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APPLICATIONS OF MAÑÉ'S THEOREM

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ABSTRACT. In this paper, we show that the Julia set of a critically non-recurrent rational map on Riemann sphere is either shallow with respect to spherical metric or itself the whole sphere. Rigidity and ergodicity results are also obtained for such maps.

Key words and phrases. Julia set, critically non-recurrent rational map, distortion, shallow, rigidity, ergodicity.

1. INTRODUCTION

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with degree $\deg(f) \geq 2$. A point z is said to be a *periodic point* if $f^k(z) = z$ for some $k \geq 1$. The minimal of such k is called the *period* of z. For a periodic point z_0, denote the *multiplier* of z_0 by $\lambda = (f^k)'(z_0)$. The periodic point z_0 is either *attracting*, *indifferent* or *repelling* according to $|\lambda| < 1, |\lambda| = 1$ or $|\lambda| > 1$. In the indifferent case, we say z_0 is *parabolic* if λ is the root of unity.

The *Julia set*, denoted by $J(f)$, is the closure of set of repelling periodic points. Its complement is called *Fatou set*, denoted by $F(f)$. A connected component of $F(f)$ is called a *Fatou component*. D. Sullivan proved that each Fatou component U is preperiodic, i.e., there exist integers $m \geq 1, n \geq 0$ so that $f^{m+n}(U) = f^n(U)$ and every periodic Fatou component is either an attracting basin, a parabolic basin, a Siegel disk or a Herman ring. For the classical results of complex dynamics, see [B], [Mi] and [CG].

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The *post-critical set* $P(f)$ of f, i.e., the closure of the forward orbits of critical points, will play the crucial rule in the study of complex dynamics. Recall that the ω-limit set $\omega(c)$ of c is $\{z \in \overline{\mathbb{C}} |$ there exists $n_k \to \infty$ such that $z = \lim_{k\to\infty} f^{n_k}(c)\}$.

Definition 1.1. A rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called *critically non-recurrent* if each critical point $c \in J(f)$ is non-recurrent, i.e., $c \notin \omega(c)$.

We equip Riemann sphere $\overline{\mathbb{C}}$ with the standard spherical metric $ds = \frac{2}{1+|z|^2}|dz|$. If we identify the unit sphere S^2 in $\mathbb{R}^3$ with the Riemann sphere, then this metric coincides with the spherical distance and is invariant under $SO(3)$ as well as $1/z$. Restricted to any proper compact subset of Riemann sphere, this metric is equivalent to the Euclidean metric. Denote $B(x, \varepsilon)$ the ball centered at x and of radius ε in the spherical metric on $\overline{\mathbb{C}}$ and $D(x, \epsilon)$ the disk centered at x and of radius ϵ in the Euclidean metric on $\mathbb{C}$.

The following definition is given by C. McMullen[Mc].

Definition 1.2. A compact subset $X \subset \overline{\mathbb{C}}$ is *shallow* if there exists $0 < k < 1$ such that any ball $B(z, r)$, where $z \in X$ and $0 < r < 1$, contains a smaller ball $B(y, kr) \subset \overline{\mathbb{C}} \setminus X$.

It's easy to prove that if X is shallow, then its box dimension $\mathrm{BD}(X) < 2$ and its Hausdorff dimension $\mathrm{HD}(X) \leq \mathrm{BD}(X) < 2$.

Our first result on critically non-recurrent rational maps is the following.

Theorem 1.1. *For a critically non-recurrent rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, the Julia set $J(f)$ is either shallow or $J(f) = \overline{\mathbb{C}}$.*

An immediate consequence of Theorem 1.1 is:

Corollary 1.1 (Cf. [U]). *For a critically non-recurrent rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, HD$(J(f)) < 2$ or $J(f) = \overline{\mathbb{C}}$.*

Remark 1.1. C. McMullen [Mc] and F. Przytycki and S. Rohde [PR] proved the same theorems for quadratic polynomials with Siegel disk of bounded type rotation numbers and Collet-Eckmann rational maps without parabolic points.

Qusiconformality is a relaxation of conformality in complex analysis. It turns out to be very useful in the study of complex dynamics and can be defined in more general setting.

Definition 1.3. A homeomorphism $\phi : X \to X$ of metric space (X, d) is called *quasiconformal* if there exists a constant K such that

$$\limsup_{r\to 0} \frac{\max_{d(y,x)=r} d(\phi(y), \phi(x))}{\min_{d(y,x)=r} d(\phi(y), \phi(x))} \leq K$$

for all $x \in X$.

A conjugacy quasiconformal off $J(f)$ being actually globally quasiconformal reflects the rigidity of the systems. The next result is about the rigidity of critically non-recurrent rational maps.

Theorem 1.2. *Let f be a critically non-recurrent rational map. If the orientation preserving homeomorphism $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ conjugates f to a rational map g, then ϕ is isotopic relative to $J(f)$ to a quasiconformal conjugacy.*

Remark 1.2. P. Haïssinsky [H] obtained the same result for uniformly weakly hyperbolic rational maps defined therein in different way. F. Przytycki and S. Rohde [PR1] proved it for topological Collet-Eckmann rational maps.

Recall that a homeomorphism $\phi : X \to X$ is *isotopic* to $\psi : X \to X$ *relative to a subset* $A \subset X$ if there exists a continuous map $H : X \times [0,1] \to X$ such that $H(x, 0) = \phi$ and $H(x, 1) = \psi$ for all $x \in X$ and $H(x, t) = x$ for all $x \in A$ and $0 \leq t \leq 1$.

The Lebesgue measure on Riemann sphere induced by the spherical metric is denoted by mes(X) for measurable subset $X \subset \overline{\mathbb{C}}$. Therefore the full measure of the Riemann sphere is 4π.

Definition 1.4. A rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is *ergodic* if any measurable set A satisfying $f^{-1}(A) = A$ has zero or full measure in the sphere.

Now we come to our ergodicity result on critically non-recurrent rational maps.

Theorem 1.3. *For a critically non-recurrent rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, either the Julia set $J(f)$ is the whole sphere and f is ergodic or the Hausforff dimension of the Julia set $J(f)$ is strictly less than 2.*

The following notations will be used in the subsequent sections.

(1) Reserve $d(d_E$, resp.) to denote the spherical(Euclidean, resp.) distance on $\overline{\mathbb{C}}(\mathbb{C}$, resp.) and $\operatorname{diam} X(\operatorname{diam}_E X$, resp.) the corresponding diameter of the subset $X \subset \overline{\mathbb{C}}(X \subset \mathbb{C}$, resp.).

(2) The distortion $\operatorname{Dist}(X, z)(\operatorname{Dist}_E(X, z)$, resp.) of a compact subset $X \subset \overline{\mathbb{C}}(X \subset \mathbb{C}$, resp.) about $z \notin X$ is

$$\frac{\max_{y \in X} d(y, z)}{\min_{y \in X} d(y, z)} \left(\frac{\max_{y \in X} |y - z|}{\min_{y \in X} |y - z|}, \text{resp.}\right).$$

(3) Two positive numbers A and B are K-commensurable or simply commensurable if $K^{-1} \le A/B \le K$ for some $K \ge 1$ independent of A and B. $A \asymp B$ means A and B are commensurable.

2. Background materials

2.1. Distortion lemmas. The distortion theorem of univalent maps is a powerful tool in the study of complex dynamics. The following distortion theorem of the version for p-valent maps is well-known(See [CJY] or Appendix C in [ST]).

Lemma 2.1. *Let U_1 and U_2 be simply connected domains in $\overline{\mathbb{C}}$ and $g : U_1 \to U_2$ be a proper holomorphic map of deg $g \le p$. Then*

$$\{w | \rho_{U_2}(w, g(z_0)) \le C^{-1}\} \subset g(\{z | \rho_{U_1}(z, z_0) \le r\}) \subset \{w | \rho_{U_2}(w, g(z_0)) \le$$

for any $z_0 \in U_1$, where ρ_{U_1} and ρ_{U_2} are hyperbolic metrics of U_1 and U_2, and C is a constant depending only on p and r.

Denote ρ the hyperbolic metric in $D(0,1)$ and $B_\rho(0,r)$ the hyperbolic ball centered at 0 of radius r.

Lemma 2.2. *Let $h: D(0,1) \to D(0,1)$ be a proper holomorphic map of degree at most p and $h(0) = 0$. Then there exists a constant R^* depending only on p and R such that $D(0,R^*) \subset h(B_\rho(0,R))$ and the component of $h^{-1}(D(0,R^*))$ containing 0 is a subset of $B_\rho(0,R)$.*

Proof. Assume $\deg h = p' \le p$. In the case $p' = 1$, the conclusion follows immediately from Koebe one-quarter theorem of univalent maps. Now we suppose $p' \ge 2$. Let $0 = z_0, z_1, \cdots, z_{p'-1}$ be the preimages of $h(z_0) = 0$ and $h_j : D(0,1) \to D(0,1)$ be the univalent map so that $h_j(z_j) = 0, 0 \le j \le p' - 1$. Then

$$h(z) = e^{i\theta} \prod_{j=0}^{p'-1} h_j(z)$$

for some $\theta \in \mathbb{R}$.

There are at most p' points of $\{z_j | 0 \le j \le p' - 1\}$ in the closed hyperbolic ball $\overline{B_\rho(0,R)}$. We can find a hyperbolic circle γ centered at 0 in $B_\rho(0,R)$ such that

$$\rho(z, z_j) \ge \frac{R}{2(p'-1)}$$

for any $z \in \gamma$ and $0 \le j \le p' - 1$. Then for all $z \in \gamma$,

$$\rho(h_j(z), 0) = \rho(h_j(z), h_j(z_j)) = \rho(z, z_j) \ge \frac{R}{2(p'-1)}.$$

Going back to the Euclidean metric, we have

$$d_E(h_j(z), 0) \ge \frac{e^{\frac{R}{2(p'-1)}} - 1}{e^{\frac{R}{2(p'-1)}} + 1} = C_1.$$

Then

$$d_E(h(z), 0) = \prod_{j=0}^{p'-1} d_E(h_j(z), 0) \ge {C_1}^{p'} = R^*(p', R).$$

and $R^* = \min\{R^*(p', R) : 1 \le p' \le p\}$ is the constant needed. □

From Lemma 2.2, we have a useful corollary in spherical metric.

Corollary 2.1. *Fix the positive integer p and $\varepsilon_0 > 0$. Let U be a simply connected subset of $\overline{\mathbb{C}}$ missing a spherical disk of radius ε_0 and $g : U \to B(w_0, 2\delta)$ be a proper holomorphic map of $\deg g \le p, w_0 = g(z_0)$, and $\delta < \frac{\pi - \varepsilon_0}{2}$. If g maps $B(z_0, r) \subset U$ into $B(w_0, \delta)$, then there exist constants K depending only on p and ε_0, and $r^* \ge \operatorname{diam} g(B(z_0, r))/K$ so that $B(w_0, r^*) \subset g(B(z_0, r))$ and the component of $g^{-1}(B(w_0, r^*))$ containing z_0 is a subset of $B(z_0, r)$.*

Proof. By rotating U and $B(w_0, 2\delta)$ respectively to avoid the spherical disk at infinity and of radius ε_0, we prove the statement in Euclidean metric first.

Assume $(\partial g(D(z_0, r))) \cap (\partial D(w_0, \delta)) \ne \emptyset$ at first. Let U' be the component of $g^{-1}(D(w_0, \delta))$ containing z_0. Then mod $(U \setminus U') \ge \frac{\log 2}{2\pi p}$. Therefore, $\operatorname{diam}_{\rho_U} D(z_0, r) \asymp 1$ and, from distortion lemmas for univalent maps, $\rho_U(x, z_0) \asymp 1$ for any point $x \in \partial D(z_0, r)$, for a constant $C(p)$ depending only on p. Take $R = \min_{x \in \partial D(z_0, r)} \rho_U(x, z_0)$. By Lemma 2.2, there exists $r^* = 2\delta R^*$ such that $D(w_0, r^*) \subset g(D(z_0, r))$ and the component of $g^{-1}(D(w_0, r^*))$ containing z_0 is a subset of $D(z_0, r)$. Denote $K = 1/R^*$. Then $r^* \ge \operatorname{diam} g(D(z_0, r))/K$.

In the case $\partial g(D(z_0, r)) \cap \partial D(w_0, \delta) = \emptyset$, take $\delta' = d(w_0, \partial g(D(z_0, r))) \le \delta$. We replace U by the component of $g^{-1}(D(w_0, 2\delta'))$ containing z_0. Using the same method as above, we get the constant K.

Coming back to the spherical metric, we know the constant K depends also on ε_0. □

2.2. **Mañé's theorem.** In [Ma](see also [ST]), R. Mañé proved a beautiful theorem. It is the key lemma in the proof of our main results.

Lemma 2.3 (Mañé's theorem). *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree at least two and p an integer depending only on f. If a point $x \in J(f)$ is not a parabolic periodic point and is not contained in the ω-limit set of a recurrent critical point, then for all $\varepsilon > 0$ there exists*

$\delta > 0$, such that for each $n \geq 0$ and each connected component U of $f^{-n}(B(x,\delta))$,

(1) *the spherical diameter of U is $< \varepsilon$ and $deg(f^n : U \to B(x,\delta)) \leq p$, and*
(2) *for all $\varepsilon_1 > 0$ there exists $n_0 > 0$ such that if $n \geq n_0$, the spherical diameter of component U of $f^{-n}(B(x,\delta))$ is $\leq \varepsilon_1$.*

Note that for a critically non-recurrent rational map, a common δ can be taken in above lemma for all points in a compact subset of Julia set containing no parabolic points.

2.3. **Dynamics near parabolic points.** Let $f_0 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with a non-degenerate parabolic fixed point 0, i.e., $f_0'(0) = 1$ and $f_0''(0) \neq 0$. Choose a neighborhood U_0 of 0 so that f_0 maps U_0 homeomorphically onto a neighborhood U_0'.

Lemma 2.4 (Fatou coordinates). *Take f_0, U_0 and U_0' as above. Then there exist simply connected domains $D_\pm$ compactly contained in $U_0 \cap U_0'$, whose union forms a punctured neighborhood of 0, satisfying*

$$f_0^{\pm}(\overline{D}_\pm) \subset D_\pm \cup \{0\} \text{ and } \cap_{n\geq 0} f_0^{\pm n}(\overline{D}_\pm) = \{0\}.$$

Moreover, there exist univalent maps $\Phi_\pm : D_\pm \to \mathbb{C}$ such that

(1) $\Phi_\pm(f_0(z)) = \Phi_\pm(z) + 1,$
(2) *$Range(\Phi_+) \supset \{\zeta | -\frac{3}{4}\pi < \arg(\zeta - R_0) < \frac{3}{4}\pi\}$ and $Range(\Phi_-) \supset \{\zeta | \frac{1}{4}\pi < \arg(\zeta + R_0) < \frac{7}{4}\pi\}$ for some $R_0 > 0$.*

The domains D_+ and D_- in Lemma 2.4 are called attracting petal and repelling petal respectively. We will make use of this coordinates to deal with the parabolic points in critically non-recurrent rational maps.

Set $\tilde{V}_0^* = \{\zeta | \text{Re}\zeta < -R_0\}, \tilde{V}_0 = \{\zeta | \text{Re}\zeta < -R_0, |\text{Im}\zeta| > R_0\}, \tilde{\Omega}_0 = \{\zeta | -R_0 - 1 < \text{Re}\zeta < -R_0, |\text{Im}\zeta| < R_0\}, \tilde{\Omega}_j = \tilde{\Omega}_0 - j, j \geq 1$ and $V_0^* = \Phi_-^{-1}(\tilde{V}_0^*), V_0 = \Phi_-^{-1}(\tilde{V}_0), \Omega_j = \Phi_-^{-1}(\tilde{\Omega}_j), j \geq 0$. From Lemma 2.4,

V_0 is contained in the Fatou set $F(f_0) = \overline{\mathbb{C}} \setminus J(f_0)$ and $f_0 : \Omega_{j+1} \to \Omega_j$ is a homeomorphism for $j \geq 0$.

2.4. **Definition of quasiconformality.** In [KK], S. Kallunki and P. Koskela improved the remarkable result about the definition of quasiconformal map in Euclidean space in [HK]. We state it in Riemman sphere case with the original inequality replaced by an equivalent condition. This is because that the original inequality appeared in [KK] holds when $r \to 0$. Then we may use this inequality in spherical metric. Moreover, we can replace the basis of disks and its image by that of uniformly bounded distortion.

By definition, a subset $E \subset \overline{\mathbb{C}}$ of *σ-finite length* is the countable union of subsets of finite one dimensional Hausdorff measure in spherical metric(Cf. [G]).

Lemma 2.5. *Let $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homeomorphism. If there are a set E of σ-finite length and a constant K such that there exists a basis of neighborhoods $\{U(x)\}$ of x with Dist$(U(x), x) < K$ and Dist$(\phi(U(x)), \phi(x)) < K$ for each $x \in \overline{\mathbb{C}} \setminus E$, then ϕ is quasiconformal.*

3. Proofs of main results

3.1. **Proof of Theorem 1.1.** Suppose that $J(f) \neq \overline{\mathbb{C}}$. One can find a spherical disk in $\overline{\mathbb{C}}$ of radius ε_0 disjoint from $J(f)$.

For $x \in J(f)$, $r > 0$, define

$$h(x, r) = \sup\{\xi | \text{ there is } B(y, \xi) \subset F(f) \cap B(x, r)\}.$$

Then $h : J(f) \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.

Since $\text{int}(J(f)) = \emptyset$,

$$h(r) = \inf\{h(x, r) \,|\, x \in J(f)\} > 0$$

for any $r > 0$.

First of all, we suppose f is a critically non-recurrent rational map having no parabolic periodic points. From Lemma 2.3, there are $0 <$

$\delta_0 < \frac{\pi-\varepsilon_0}{2}$ and $p < \infty$ so that for any $x \in J(f)$, $n > 0$ and any component V of $f^{-n}(B(x, 2\delta_0))$, V is simply connected domain missing a spherical disk of radius of ε_0, and

$$\deg(f^n : V \to B(x, 2\delta_0)) \leq p.$$

For any $z_0 \in J(f)$ and $0 < r \leq \delta_0$, look at the forward images $B_m = f^m(B(z_0, r))$, $z_m = f^m(z_0)$, $m \geq 0$.

Let n_0 be the smallest integer so that $\text{diam} B_{n_0+1} > \delta_0$. Then $l_0\delta_0 \leq \text{diam} B_{n_0} \leq \delta_0$ for some $0 < l_0 < 1$. By Corollary 2.1, there are a constant K depending only on p and ε, and a disk $B(z_{n_0}, r_0) \subset B_{n_0}$ with $r_0 \geq \text{diam}\, B_{n_0}/K \geq l_0\delta_0$ such that the component of $f^{-n_0}(B(z_{n_0}, r_0))$ containing z_0 is a subset of $B(z_0, r)$. There exists $B(y_0, \frac{1}{2}h(r_0)) \subset B(z_{n_0}, r_0) \cap F(f)$. Let D_0 be a component of $f^{-n_0}(B(y_0, \frac{1}{2}h(r_0)))$ contained in $B(z_0, r), y \in D_0 \cap f^{-n_0}(y_0)$. Then $\text{Dist}(\partial D_0, y) \leq M$ for some M depending only on p and $\text{diam}\, D_0 \asymp r$. Therefore, there exists $0 < k' < 1$ which does not depend on z_0 and r so that $B(y, k'r) \subset D_0 \subset B(z_0, r) \cap F(f)$.

Now let f be a non-recurrent rational map having parabolic periodic points. For simplicity, we suppose f has only one non-degenerate parabolic fixed point 0.

Take $\Omega_j (j \geq 0)$ as in Section 2.3. The set $X_0 = J(f) \setminus (\cup_{j \geq 1} \Omega_j \cup \{0\})$ is a compact subset of $J(f)$. By Lemma 2.3, there exist $\delta_1 > 0$ and $p_1 < \infty$ so that for all $x \in X_0$, $n > 0$, $0 < \delta \leq \delta_1$ and any component V of $f^{-n}(B(x, 2\delta))$, V is simply connected and

$$\deg(f^n : V \to B(x, 2\delta)) \leq p_1.$$

Let K_1 be the constant depending only on p_1 and ε_0 in Corollary 2.1. Suppose both $\text{diam}\, D$ and $\text{diam}(f(D))$ are less than δ. If D contains a disk $B(z, \epsilon)$ with $\epsilon \geq \text{diam}\, D/K_1$, then $f(D)$ contains a disk $B(f(z), \epsilon^*)$ such that the component of $f^{-1}(B(f(z), \epsilon^*))$ containing z is a subset of $B(z, \epsilon)$, where $\epsilon^* \geq \text{diam}\, f(D)/K_0$ for a constant K_0 depending only on K_1 and ε_0.

Choose N large enough so that $N \geq 128K_0R_0$ and $d(0, \partial\Omega_n) > 4K_0\text{diam}\Omega_n$ for $n \geq N$. Take $\delta_2 = \eta\text{diam}\,\Omega_N$ for a suitable η so that

$$B(x, 2\delta_2) \subset V_0^* = V_0 \cup (\cup_{j\geq 0}\Omega_j)$$

for any $x \in \cup_{j=1}^{N}\Omega_j$.

Now we prove that the Julia set $J(f)$ is shallow.

For any $z_0 \in J(f)$ and $0 < r \leq \delta_0 = \min(\delta_1, \delta_2)$, denote $B_m = f^m(B(z_0, r))$, $z_m = f^m(z_0)$, $m \geq 0$. Let n_0 be the smallest integer such that $\text{diam}B_{n_0+1} > \delta_0$. Then $\text{diam}B_m \leq \delta_0$ for $m \leq n_0$ and $\text{diam}B_{n_0} \asymp \delta_0$.

If $z_{n_0} \in X_0$, then the same argument as in no parabolic periodic points case shows there exists $B(y, k_1r) \subset B(z_0, r) \cap F(f)$ for some $0 < k_1 < 1$.

Suppose $z_{n_0} \in (\cup_{j\geq 1}\Omega_j) \cup \{0\}$ and $m_0 \geq 0$ is the smallest integer such that $z_j \in (\cup_{j\geq 1}\Omega_j) \cup \{0\}$ for all $m_0 \leq j \leq n_0$. Then B_{m_0} contains a disk $W = B(z_{m_0}, \epsilon^*)$ with $\epsilon^* \geq \text{diam}B_{m_0}/K_0$ such that the component of $f^{-m_0}(W)$ containing z_0 is a subset of $B(z_0, r)$.

We claim that there exists a disk $B(y_0, r_0) \subset B_{m_0} \cap F(f)$ such that $f^{-m_0}(B(y_0, r_0))$ has a component contained in $B(z_0, r)$ for some $r_0 \asymp \text{diam}\, B_{m_0}$.

Case 1. If $z_{m_0} = 0$, take $r_0 = \frac{1}{2}\epsilon^*$.

Case 2. If $z_{n_0} \in \Omega_{j_0}$ for $1 \leq j_0 \leq N$, then $B_{n_0} \subset B(z_{n_0}, \delta_0)$ and $B(z_{n_0}, 2\delta_0) \subset V_0^*$. Let B' be a component of $f^{-(n_0-m_0)}(B(z_{n_0}, 2\delta_0))$ containing B_{m_0}. Then $f^{n_0-m_0} : B' \to B(z_{n_0}, 2\delta_0)$ is univalent and $\text{mod}\,(B' \setminus B_{m_0}) \geq \frac{1}{2\pi}\log 2$. By distortion theorem of univalent map, $\text{Dist}(\partial W_1, z_{n_0}) \asymp 1$ and there is $B(y_0^*, r_0^*) \subset W_1 \cap F(f)$ for some $r_0^* \asymp \delta_0$, where $W_1 = f^{n_0-m_0}(W)$. Hence there is $B(y_0, r_0) \subset W \cap F(f)$ for some $r_0 \asymp \text{diam}B_{m_0}$.

Case 3. If $z_{n_0} \in \Omega_{j_0}$ for $j_0 > N$, $z_{m_0} \in \Omega_{j_0} + (n_0 - m_0) = \Omega_{i_0}$.

(3a). When $\text{diam}\, B_{m_0} > 2K_0\text{diam}\,\Omega_{i_0}$, $d(z_{m_0}, \partial W) > 2\text{diam}\,\Omega_{i_0}$, take $r_0 = \frac{1}{4}d(z_{m_0}, \partial W)$.

(3b). When $\operatorname{diam} B_{m_0} \leq 2K_0 \operatorname{diam} \Omega_{i_0}$, $D = D(z_{m_0}, 2\operatorname{diam} B_{m_0}) \subset V^*$. Let $\tilde{D} = \Phi_-(D), \tilde{W} = \Phi_-(W), \tilde{W}_1 = \Phi_-(W_1)$ and $\tilde{z}_{m_0} = \Phi_-(z_{m_0})$. Then $\operatorname{Dist}_E(\partial\tilde{W}, \tilde{z}_{m_0}) \leq 16 \operatorname{Dist}(\partial W, z_{m_0}) \leq 16K_0$.

If $d_E(\tilde{z}_{m_0}, \partial\tilde{W}) > 4R_0$, then there is a disk $\tilde{B} = D(\tilde{y}_0, \frac{1}{2}d_E(\tilde{z}_{m_0}, \partial\tilde{W})) \subset \tilde{V}_0$ such that $\Phi_-^{-1}(\tilde{B}) = B$ contains a disk $B(y_0, r_0) \subset W \cap V_0 \subset W \cap F(f)$ for $r_0 \asymp \operatorname{diam} B_{m_0}$.

If $d_E(\tilde{z}_{m_0}, \partial\tilde{W}) \leq 4R_0$, then $\tilde{W} \subset D(\tilde{z}_{m_0}, 64K_0R_0)$ and $\tilde{W}_1 = \tilde{W} + (n_0 - m_0) \subset D(\tilde{z}_{n_0}, 64K_0R_0)$, $\tilde{z}_{n_0} = \tilde{z}_{m_0} + (n_0 - m_0) = \Phi_-(z_{n_0})$. Since $\tilde{z}_{n_0} \in \tilde{\Omega}_{j_0}$, $j_0 > N \geq 128K_0R_0$, we have $\tilde{D}' = D(\tilde{z}_{n_0}, 128K_0R_0) \subset \tilde{V}_0^*$, $D' = \Phi_-^{-1}(\tilde{D}') \subset V^*$, $B_{n_0} \subset D'$ and $\operatorname{mod}(D' \setminus B_{n_0}) \geq \frac{1}{2\pi}\log 2$. Hence $\operatorname{Dist}(\partial W_1, z_{n_0}) \leq 16\operatorname{Dist}_E(\partial\tilde{W}_1, \tilde{z}_{n_0}) = 16\operatorname{Dist}_E(\partial\tilde{W}, \tilde{z}_{m_0}) \leq 16^2K_0$.

Let D'' be the component of $f^{-(n_0-m_0)}(D')$ containing B_{m_0}. Then $f^{n_0-m_0} : D'' \to D'$ is univalent. By distortion theorem, there is a disk $B(y_0, r_0) \subset W \cap F(f)$ for some $r_0 \asymp \operatorname{diam} B_{m_0}$.

The proof of our claim is completed.

Now we come back to the proof of the main theorem.

If $m_0 = 0$, we are done. If $m_0 \geq 1$, then $B_{m_0-1} \cap f^{-1}(B(y_0, r_0))$ contains a disk with radius $r_0' \asymp \operatorname{diam} B_{m_0-1}$. The same argument as in no parabolic points case shows that there is $B(y, k_2r) \subset B(z_0, r) \cap F(f)$ for some $0 < k_2 < 1$. Take $k' = \min(k_1, k_2)$. Then for any $z_0 \in J(f)$, $r > 0$, $B(z_0, r)$ contains a disk $B(y, k'r) \subset F(f)$ and hence $J(f)$ is shallow.

3.2. **Proof of Theorem 1.2.** Suppose that f and g are two critically non-recurrent rational maps which are conjugated by a homeomorphism ψ. By a routine argument on Fatou set, ψ is isotopic relative to $J(f)$ to a new conjugacy ϕ which is quasiconformal on Fatou set $F(f)$. We shall prove that ϕ is actually a globally quasiconformal map.

Let $X_0(f) \subset J(f)$(resp. $X_0(g) \subset J(g)$) be the compact sets defined in the proof of Theorem 1.1. Let $\delta_f(\delta_g$, resp.)$< \frac{\pi}{2}$(We will apply

Corollary 2.1 for $\varepsilon_0 = \frac{\pi}{2}$) and integers $p(f)(p(g)$, resp.) be the numbers guaranteed by Mañé's theorem for $X_0(f)$($X_0(g)$, resp.) and for $\varepsilon = \frac{\pi}{2}$. In cases $J(f) = \overline{\mathbb{C}}$ and $J(f)$ containing no parabolic points, let $X_0(f) = J(f)$.

Take a small $0 < r_0 < \frac{1}{2}\delta_f$ such that for all $x \in J(f)$, $\phi(B(x, r_0)) \subset B(\phi(x), \frac{1}{2}\delta_g)$.

Let $x \in J(f)$ and $U_n(x)$ be a component of $f^{-n}(B(f^n(x), r_0))$ containing x. There is a constant K_1 does not depend on y such that $\mathrm{Dist}(\phi(B(y, r_0)), \phi(y)) \leq K_1$. If x is not eventually mapped to the parabolic periodic points, it visits $X_0(f)$ infinitely many times. It follows from the distortion theorem for p-valent version that there are subsequence $\{n_k\}$ and a constant K independent of x such that

$$\mathrm{Dist}(U_{n_k}(x), x) \leq K$$

and

$$\mathrm{Dist}(\phi(U_{n_k}(x)), \phi(x)) \leq K.$$

It then follows from Lemma 2.5 that $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is quasiconformal. The proof is complete.

3.3. **Proof of Theorem 1.3.** By Corollary 1.1, we assume the Julia set of f is the whole sphere.

Let $X \subset \overline{\mathbb{C}}$ be a completely invariant subset. Suppose that $\mathrm{mes}(X) > 0$. We shall prove that X has the full measure.

Let $x_0 \in X$ be a density point of X. Let $x_n = f^n(x)$, $n > 0$. We can choose a subsequence $\{x_{n_k}\}$ converging to x^*. Let $A = \{x_{n_k}\} \cup \{x^*\}$. Then A is a proper compact subset of $J(f)$. Let δ and p be the numbers given for ε in Mañé's theorem with ε-neighborhood of A missing a spherical disk of radius ε_0 and U_{n_k} be the component of $f^{-n_k}(B(x_{n_k}, \frac{1}{2}\delta))$ containing x_0. Then U_{n_k} shrinks to x_0 nicely since $\mathrm{Dist}(U_{n_k}, x_0) \leq K$, K depending only on p and ε_0. Therefore,

$$\lim_{k\to\infty} \frac{\mathrm{mes}(U_{n_k} \cap X)}{\mathrm{mes}(U_{n_k})} = 1,$$

which is equivalent to

$$\lim_{k\to\infty} \frac{\mathrm{mes}(U_{n_k} \cap (\overline{\mathbb{C}} \setminus X))}{\mathrm{mes}(U_{n_k})} = 0.$$

It follows from distortion theorem that,

$$\lim_{k\to\infty} \frac{\mathrm{mes}(B(x_{n_k}, \frac{1}{2}\delta) \cap (\overline{\mathbb{C}} \setminus X))}{\mathrm{mes}(B(x_{n_k}, \frac{1}{2}\delta))} = 0,$$

which is equivalent to

$$\lim_{k\to\infty} \frac{\mathrm{mes}(B(x_{n_k}, \frac{1}{2}\delta) \cap X)}{\mathrm{mes}(B(x_{n_k}, \frac{1}{2}\delta))} = 1,$$

and hence

$$\frac{\mathrm{mes}(B(x^*, \frac{1}{2}\delta) \cap X)}{\mathrm{mes}(B(x^*, \frac{1}{2}\delta))} = 1.$$

This implies that $B(x^*, \frac{1}{2}\delta) \subset X$ a.e.

Since $B(x^*, \frac{1}{2}\delta) \subset \overline{\mathbb{C}} = J(f)$, there exists an integer k such that $f^k(D(x^*, \frac{1}{2}\delta)) = \overline{\mathbb{C}}$. Hence $\overline{\mathbb{C}} \subset X$ a.e. This means that X has full measure and completes the proof.

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