<table>
<thead>
<tr>
<th>Title</th>
<th>Dynamics of Polynomial Automorphisms of $\mathbb{C}^2$: Herman ring (Complex dynamics and related fields)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Jin, Teisuke</td>
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Kyoto University
Dynamics of Polynomial Automorphisms of $\mathbb{C}^2$: Herman ring

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Abstract

Herman ring is a periodic set which is biholomorphic to an annulus and rotates irrationally by iteration. Though the structure is known well its existence is unknown. We will show that there are no Herman rings under some conditions in the dynamics of the title.

1 Introduction

In this paper we denote $z = (x, y) \in \mathbb{C}^2$. Take an appropriate $m \in \mathbb{N}$. Let $p_j(y)$ be polynomials such that degree $d_j > 1$ for $j = 1, \ldots, m$. We call $f_j(x, y) = (y, p_j(y) - \delta_j x)$ generalized Hénon mappings, where $\delta_j \neq 0$. Moreover we define

$$f = f_m \circ \cdots \circ f_1, \quad \delta = \delta_1 \cdots \delta_m, \quad d = d_1 \cdots d_m.$$ 

Note that $\delta$ is the Jacobian determinant of $f$, and that $f_j^{-1}$ are also generalized Hénon maps.

In [FM] Friedland and Milnor classified the polynomial automorphisms of $\mathbb{C}^2$ into three types:

- an affine mapping: $(x, y) \mapsto (\mu_{11}x + \mu_{12}y + \lambda_1, \mu_{21}x + \mu_{22}y + \lambda_2)$,

- an elementary mapping: $(x, y) \mapsto (\mu_1 x + \lambda, \mu_2 y + p(x))$,

- a composite of generalized Hénon mappings: $(x, y) \mapsto f(x, y)$.

Since the dynamical structures of the former two mappings are simple, they were investigated sufficiently in [FM]. So we study the last one.

We define $K^{\pm} = \{z \in \mathbb{C}^2 \mid \{f^{\pm n}(z) \mid n \in \mathbb{N}\} \text{ is bounded}\}$, $J^{\pm} = \partial K^{\pm}, K = K^+ \cap K^-, J = J^+ \cap J^-$. They are closed invariant sets and are important objects in dynamical systems. Moreover we define $I^\pm = \mathbb{C}^2 \setminus K^{\pm}$. For $R > 0$, we define $V^+ = \{z \in \mathbb{C}^2 \mid |x| > \max\{|y|, R\}\}$, $V^- = \{z \in \mathbb{C}^2 \mid |y| > \max\{|x|, R\}\}$ and $V = \{z \in \mathbb{C}^2 \mid |x|, |y| \leq R\}$. It is known that $K^{\pm} \subset V \cup V^{\pm}$ for sufficiently large $R > 0$.

We define the Green functions $G^{\pm}$ as (cf. [BS1, Section 3])

$$G^{\pm}(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ \|f^{\pm n}(z)\|.$$ 

$G^{\pm}$ are non-negative continuous plurisubharmonic functions such that $G^{\pm}(z) > 0$ if and only if $z \in I^\pm$, $G^{\pm}|_{I^\pm}$ are pluriharmonic, and $G^{\pm} \circ f = d^{\pm 1} \cdot G^{\pm}$.

Before we define Herman ring, we state a classification of Fatou components. We proceed with the following volume property.

Proposition 1.1. ([FM, Lemma 3.7]) Denote by Vol() the usual Lebesgue volume in $\mathbb{C}^2$. We have

- if $|\delta| < 1$, then $\text{Vol}(K^+) = \infty$ or 0, $\text{Vol}(K^-) = 0$,

- if $|\delta| = 1$, then $\text{Vol}(K^+) = \text{Vol}(K^-) < \infty$,

- if $|\delta| > 1$, then $\text{Vol}(K^+) = 0$, $\text{Vol}(K^-) = \infty$ or 0.
In this paper we assume $|\delta| < 1$, i.e. dissipative. Then only $K^+$ can have non-empty internals by the above proposition. We call each component of int $K^+$ Fatou component. Its classification theorem is as follows.

**Theorem 1.2.** ([BS2, Section 5]) Each connected component of int $K^+$ is classified as follows.

\[
\begin{array}{c|c|c}
\text{wandering domain} & \text{non-recurrent domain} & \text{recurrent domain} \\
\text{periodic domain} & \text{basin of a sink} & \text{Siegel cylinder} \\
& \text{Herman cylinder} & \\
\end{array}
\]

Before we define the names, we mention about existence and non-existence of the above domains.

As far as the author knows, it is unknown whether wandering domains exist or not. Non-recurrent domains exist and have been investigated only a little ([Hak, U1, U2, W]). Of course there are basins of sinks. Fornerss and Sibony investigated in [FS, Section 2] that there are Siegel cylinders. It is unknown whether Herman cylinders exist or not.

The only known fact with respect to non-existence of Herman cylinder is as follows: if $f$ is uniformly hyperbolic on $J$, then Fatou components consist of basins of finite sinks (cf. [BS1, Theorem 5.6]).

Let us return to the definition of the names. Fatou component $U$ is wandering if $f^n(U) \cap U = \emptyset$ for any $n \in \mathbb{N}$, periodic if $f^p(U) = U$ for some $p \in \mathbb{N}$. We call $p$ period for the minimum $p$. We say $U$ is recurrent if there are compact $C \subseteq U$ and $z \in U$ such that $f^n(z) \in C$ for infinitely many $n \in \mathbb{N}$.

For $E \subseteq \mathbb{C}^2$, we define

\[
W^s(E) = \{ z \in \mathbb{C}^2 \mid d(f^n(z), f^n(E)) \to 0 \ (n \to \infty) \}, \\
W^u(E) = \{ z \in \mathbb{C}^2 \mid d(f^n(z), f^n(E)) \to 0 \ (n \to -\infty) \}, \\
W_0^s(E) = \bigcup_{C \subset E: \text{compact}} W^s(C), \\
W_0^u(E) = \bigcup_{C \subset E: \text{compact}} W^u(C).
\]

Let $z_1$ be a periodic point with period $p$. We call $z_1$ sink if both eigenvalues of $Df^p(z_1)$ are lower than 1 in modulus, source if greater, saddle point if one lower and another greater. We say $U$ is a basin of a sink if $U = W^s(z_1)$ for some sink $z_1$.

We call $D \subseteq \mathbb{C}^2$ Siegel disk if $D$ satisfies the following properties: $D$ is a periodic set with period $p$ and there is a bijective holomorphic map $\varphi : \Delta \to D$ such that $f^p(\varphi(\zeta)) = \varphi(b\zeta)$ for $\zeta \in \Delta$, where $\Delta$ is a unit disk on a complex plane and $b$ is an irrational rotation, i.e. $b = e^{i\pi\theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Here, we have to take the maximum $D$ with respect to inclusion. We call $U$ Siegel cylinder if $U = W^s_0(D)$ for some Siegel disk $D$.

We call $H \subseteq \mathbb{C}^2$ Herman ring if $H$ satisfies the following properties: $H$ is a periodic set with period $p$ and there is a bijective holomorphic map $\varphi : A \to H$ such that $f^p(\varphi(\zeta)) = \varphi(b\zeta)$ for $\zeta \in A$, where $A = \{ \zeta \in \mathbb{C} \mid r_1 < |\zeta| < r_2 \}$ is an annulus and $b$ is an irrational rotation, i.e. $b = e^{i\pi\theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Here, we have to take the maximum $H$ with respect to inclusion. We call $U$ Herman cylinder if $U = W^s_0(H)$ for some Herman ring $H$.

We have arrived at a good position to describe our question.

**Problem.** ([BFGK, Problem 10.2.2(i)]) In case of dissipative, does a Hénon map admit a Herman ring (Herman cylinder)?

In section 2, we will investigate several properties of a Herman cylinder. In particular, Proposition 2.9 will give a classification. In Theorem 3.1 of section 3, we will show that one of the types in the classification is impossible. Perhaps it might be a clue either to prove there are no Herman rings or to construct a Herman ring.

# 2 Fundamental properties of Herman cylinder

## 2.1 Functional properties

Note that we assume $|\delta| < 1$ in this paper. The following is a known fact.
Proposition 2.1. ([BS2, section 5]) Define \( L(\zeta, \eta) = (b\zeta, b^{p}\eta) \). Then there exists a biholomorphic map \( \Phi : A \times C \rightarrow W_{0}^{\sigma}(H) \) such that \( \Phi(A \times \{0\}) = H \) and \( f^{n} \circ \Phi = \Phi \circ L \).

Proposition 2.2. Let \( U \) be an arbitrary Fatou component and \( M \) simply connected one dimensional complex manifold in \( C^{2} \). Then \( M \cap U \) is simply connected.

Proof. Assume that \( M \cap U \) is not simply connected. Then there is a point \( z_{1} \in M \setminus U \) which is surrounded by \( M \cap U \) on \( M \). By perturbing \( M \) to \( M' \), we can take \( z_{2} \in M' \setminus K^{+} \) which is surrounded by \( M' \cap U \) on \( M' \). We recall the Green function \( G^{+} \), which vanishes on \( M' \cap U \) and is positive on \( M' \setminus K^{+} \). It contradicts the maximum principle. \( \square \)

We define \( u_{a}(\eta) = u(a, \eta) = G^{-} \circ \Phi(a, \eta) \) for \( a \in A, \eta \in C \). Then \( u_{a} \) is a subharmonic function on \( C \).

In general, let \( v \) be a non-negative subharmonic function. We define the order of \( v \) by

\[
\text{ord } v = \limsup_{r \to \infty} \frac{\log \max_{|\eta| = r} v(\eta)}{\log r},
\]

Let \( \rho \) be the order of \( v \), then we say \( v \) is of mean type of order \( \rho \) if

\[
0 < \limsup_{r \to \infty} \frac{\max_{|\eta| = r} v(\eta)}{r^{\rho}} < \infty.
\]

Proposition 2.3. For \( a \in A, u_{a} \) is of mean type of order:

\[
\rho = \text{ord } u_{a} = \frac{\log d}{\log(1/|\delta|)}.
\]

Proof. It is sufficient to show that \( u_{a} \) is of mean type under the assumption that \( \rho = \frac{\log d}{\log(1/|\delta|)} \).

\[
\begin{align*}
sup_{|\zeta| = a} \limsup_{r \to \infty} \frac{\max_{|\eta| = r} u_{\zeta}(\eta)}{r^{\rho}} & \leq \limsup_{r \to \infty} \max_{|\zeta| = a} \frac{\max_{|\eta| = r} u_{\zeta}(\eta)}{r^{\rho}}. \\

\end{align*}
\]

For \( r > 1 \), we take \( n \in \mathbb{Z} \) such that \( 1/|\delta|^{pn} \leq r < 1/|\delta|^{p(n+1)} \).

\[
\begin{align*}
\max_{|\zeta| = a} \max_{|\eta| = a} \frac{u_{\zeta}(\eta)}{r^{\rho}} & \leq \max_{|\zeta| = a} \max_{|\eta| = 1/|\delta|^{p(n+1)}} G^{-} \circ \Phi(\zeta, \eta) \\
& \leq \max_{|\zeta| = a} \max_{|\eta| = 1/|\delta|^{p(n+1)}} G^{-} \circ f^{-p(n+1)} \circ \Phi \circ L^{n+1}(\zeta, \eta) \\
& = \max_{|\zeta| = a} \max_{|\eta| = 1} \frac{d^{p(n+1)} \cdot G^{-} \circ \Phi \circ L^{n+1}(\zeta, \eta)}{d^{pn}} \\
& = d^{p} \max_{|\zeta| = a} \max_{|\eta| = 1} u_{\zeta}(\eta).
\end{align*}
\]

Therefore we have

\[
\limsup_{r \to \infty} \frac{\max_{|\eta| = r} u_{a}(\eta)}{r^{\rho}} < \infty.
\]

Similarly we can compute as follows.

\[
\begin{align*}
\inf_{|\zeta| = a} \limsup_{r \to \infty} \frac{\max_{|\eta| = r} u_{\zeta}(\eta)}{r^{\rho}} & \geq \limsup_{r \to \infty} \min_{|\zeta| = a} \frac{\max_{|\eta| = r} u_{\zeta}(\eta)}{r^{\rho}}. \\

\end{align*}
\]

For \( r > 1 \), we take \( n \in \mathbb{Z} \) such that \( 1/|\delta|^{pn} \leq r < 1/|\delta|^{p(n+1)} \).

\[
\begin{align*}
\min_{|\zeta| = a} \max_{|\eta| = a} \frac{u_{\zeta}(\eta)}{r^{\rho}} & \geq \min_{|\zeta| = a} \max_{|\eta| = 1/|\delta|^{p(n+1)}} G^{-} \circ \Phi(\zeta, \eta) \\
& \leq \min_{|\zeta| = a} \max_{|\eta| = 1/|\delta|^{p(n+1)}} G^{-} \circ f^{-p(n+1)} \circ \Phi \circ L^{n}(\zeta, \eta) \\
& = \min_{|\zeta| = a} \max_{|\eta| = 1} \frac{d^{p(n+1)} \cdot G^{-} \circ \Phi \circ L^{n}(\zeta, \eta)}{d^{pn}} \\
& = d^{-p} \min_{|\zeta| = a} \max_{|\eta| = 1} u_{\zeta}(\eta).
\end{align*}
\]
Let us show that the last side is positive. Assume for some \( a' \) with \( |a'| = |a|, \ u_{a'}|_{\eta} \leq 1 \equiv 0 \). Then
\[
u((b^{-n}a') \times \{ \eta \in C \mid |\eta| \leq 1/|\delta|^{pn} \})
= u(L^{-n}(\{ a' \} \times \{ \eta \in C \mid |\eta| \leq 1 \}))
= d^{n}u(\{ a' \} \times \{ \eta \in C \mid |\eta| \leq 1 \}) = 0
\]
Since \( \bigcup_{n=0}^{\infty} (b^{-n}a') \times \{ \eta \in C \mid |\eta| \leq 1/|\delta|^{pn} \} \) is dense in \( \{ \zeta \in C \mid |\zeta| = |a| \} \times C, \ u_{a} \equiv 0 \).
On the other hand, \( G^{-1}_{V_+} > 0 \) and the range of the non-constant holomorphic map \( \Phi_{a} \) is contained in \( V \cup V^{+} \).
So \( u_{a} \not\equiv 0 \). It is a contradiction.
Therefore we have
\[
\limsup_{r \to \infty} \frac{\max_{|\eta| = r} u_{a}(\eta)}{r^{\rho}} > 0.
\]
\( \square \)

2.2 Formal classification

Let \( C = \{ c_{1}, \ldots, c_{n} \} \) be a finite ordered subset of a metric space with a metric \( d \). We call \( C \) \( \epsilon \)-chain if \( d(c_{j}, c_{j+1}) < \epsilon \) for any \( 1 \leq j < n \).

**Lemma 2.4.** Let \( \{ \epsilon_{j} \}_{j \in \mathbb{N}} \) be a positive decreasing sequence converging to 0. Take \( \epsilon_{j} \)-chain \( C_{j} = \{ c_{j1}, \ldots, c_{jn} \} \).
Assume \( \{ c_{j1} \}_{j \in \mathbb{N}} \) converges and \( \bigcup_{j=1}^{\infty} C_{j} \) is compact. Then the \( \omega \)-limit set:
\[
\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} C_{j}
\]
is a connected compact set.

**Proof.** Assume \( E = \bigcap_{j(k-1)}^{\infty} C_{j} \) is not connected. Then there exist compact sets \( E_{1} \) and \( E_{2} \) such that \( E = E_{1} \cup E_{2} \) and \( E_{1} \cap E_{2} = \emptyset \). Observe that \( d(E_{1}, E_{2}) > 0 \). We may assume \( \{ c_{j1} \}_{j \in \mathbb{N}} \) converges in \( E_{1} \). Then there is a sequence \( \{ c_{jk} \}_{j \in \mathbb{N}} \) which accumulates on \( E_{2} \).
On the other hand, because \( \{ \epsilon_{j} \} \) decreases to 0, there is a sequence \( \{ c_{jk} \}_{j \in \mathbb{N}} \) which accumulates on \( \{ w \in \bigcup C_{j} \mid \min \{ d(w, E_{1}), d(w, E_{2}) \} \geq d(E_{1}, E_{2})/3 \} \). It is a contradiction. \( \square \)

**Lemma 2.5.** Let \( X \subset \mathbb{R}^{2} \) be a closed subset and \( Y \) a compact component of \( X \). Then there is a simple closed curve \( \Gamma \subset \mathbb{R}^{2} \setminus X \) which winds \( Y \) once.

**Proof.** At first we show that there is \( \epsilon > 0 \) such that the subset of \( X \) which can be joined to \( Y \) by \( \epsilon \)-chain on \( X \) is compact.
Assume the contrary. Take a positive decreasing sequence \( \{ \epsilon_{j} \} \) converging to 0 and \( w_{1} \in Y \) and \( r > 0 \) with \( Y \subset B(w_{1}, r) \). By the assumption, for any \( j \in \mathbb{N} \) we can take \( \epsilon_{j} \)-chain \( C_{j} \subset X \) such that the start point of \( C_{j} \) is \( w_{1} \) and \( C_{j} \setminus B(w_{1}, r) \neq \emptyset \) and \( C_{j} \subset B(w_{1}, 2r) \). By the previous lemma, we can conclude that the connected component of \( X \) containing \( w_{1} \) exceeds \( B(w_{1}, r) \). It is a contradiction.
Let \( Y' \subset X \) be the compact set which can be joined to \( Y \) by \( \epsilon \)-chain. Each point on \( X \setminus Y' \) is at least \( \epsilon \) far from \( Y' \). It is not difficult to find a simple closed curve \( \Gamma \subset C \setminus X \) which winds \( Y' \) once. \( \square \)

We proceed with investigating the structure of a Herman cylinder.
We define \( \tilde{K} = \Phi^{-1}(K^{-}), \tilde{K}_{a} = \{ \eta \in C \mid \Phi(a, \eta) \in K^{-} \} \) for \( a \in A \). Note that \( \tilde{K} = \{(\zeta, \eta) \in A \times C \mid u(\zeta, \eta) = 0\}, \tilde{K}_{a} = \{ \eta \in C \mid u_{a}(\eta) = 0\} \).

**Definition 2.6.** We say \( \tilde{K}_{a} \) is bridged if the component of \( \tilde{K}_{a} \) containing 0 is unbounded.

**Lemma 2.7.** The following are equivalent.
(1) \( \tilde{K}_{a} \) is bridged.
(2) The component of \( \tilde{K}_{a} \) containing 0 is not a point.
(3) $\tilde{K}_a$ has an unbounded component.

Proof. (2)$\Rightarrow$(1). Assume the component of $\tilde{K}_a$ containing 0 is bounded, i.e. the component is contained in $B(0, r) = \{\eta \in \mathbb{C} | |\eta| < r\}$ for some $r > 0$. Take an increasing sequence \( \{n_j\}_{j \in \mathbb{N}} \subset \mathbb{N} \) such that $b^{-n_j}a$ converges to $a$. Then

\[
\{b^{-n_j}a\} \times \tilde{K}_{b^{-n_j}a} = L^{-n_j}(\{a\} \times \tilde{K}_a) = \{b^{-n_j}a\} \times (b/\delta)^{n_j} \tilde{K}_a \subset \tilde{K}.
\]

Let \( \{\epsilon_j\}_{j \in \mathbb{N}} \) be a positive sequence decreasing to 0. We take $\epsilon_j$-chain $C_j$ in $\{b^{-n_j}a\} \times (b/\delta)^{n_j} \tilde{K}_a$ so that the starting point of $C_j$ is $(b^{-n_j}a,0)$ and $C_j \subset \{b^{-n_j}a\} \times B(0,2r)$. Moreover we can assume $C_j \not\subset \{b^{-n_j}a\} \times B(0,r)$ for any sufficiently large $j$ because of the hypothesis (2). By Lemma 2.4 we can conclude that the component of $\tilde{K}_a$ containing 0 exceeds $B(0,r)$. It is a contradiction.

(3)$\Rightarrow$(2). Take an increasing sequence \( \{n_j\}_{j \in \mathbb{N}} \) such that $b^{n_j}a$ converges to $a$. Then

\[
\{b^{n_j}a\} \times \tilde{K}_{b^{n_j}a} = \{b^{n_j}a\} \times (\delta/\beta)^{n_j} \tilde{K}_a.
\]

Let $E$ be an unbounded component of $\tilde{K}_a$. We can take $r > 0$ such that $B(0,r) \cap E \neq \emptyset$. Let $\{\epsilon_j\}_{j \in \mathbb{N}}$ be a positive sequence decreasing to 0. We take $\epsilon_j$-chain $C_j$ in $\{b^{n_j}a\} \times (\delta/\beta)^{n_j} E$ so that the starting point of $C_j$ converges $(a,0)$ and $C_j \subset \{b^{n_j}a\} \times B(0,2r)$ and $C_j \not\subset \{b^{n_j}a\} \times B(0,r)$. By Lemma 2.4, we can conclude that the component of $\tilde{K}_a$ containing 0 is not a point.

\[\square\]

Lemma 2.8. For $a \in A$ the following hold.

(1) If $\tilde{K}_a$ has no compact components, then so is $\tilde{K}_\zeta$ for any $|\zeta| = |a|$.

(2) If $\tilde{K}_a$ is bridged, then so is $\tilde{K}_\zeta$ for any $|\zeta| = |a|$.

Proof. The proof of (2) is similar to the previous lemma. So we give only the proof of (1).

To prove (1), we show that if $\tilde{K}_a$ has a compact component then so is $\tilde{K}_a'$, for $|a'| = |a|$. By Lemma 2.5, there is a curve $\Gamma$ which surrounds the component and never intersects $\tilde{K}_a$. Take $\eta_1 \in \tilde{K}_a$ surrounded by $\Gamma$. Then there exists $\epsilon > 0$ such that $\Gamma_\epsilon = \{\zeta \in A | |\zeta - a| \leq \epsilon\} \times \Gamma$ never intersects $\tilde{K}$.

Consider $u_\zeta$. Note that $u_\zeta(\eta) = 0$ if and only if $\eta \in \tilde{K}_\zeta$. We define $c = \min_{(\zeta,\eta) \in \Gamma} u_\zeta(\eta) > 0$. When $\epsilon > 0$ is sufficiently small, $u_\zeta(\eta_1) < c$ for any $\zeta$ with $|\zeta - a| \leq \epsilon$. Recall that $u_\zeta$ is harmonic in $\mathbb{C} \setminus \tilde{K}_\zeta$ and continuous on $\mathbb{C}$. So $u_\zeta$ has zero points inside of $\Gamma$, i.e. for $\zeta$ with $|\zeta - a| \leq \epsilon$. $\tilde{K}_\zeta$ has at least one compact component inside of $\Gamma$.

Take $n \in \mathbb{N}$ such that $|b^n a' - a| \leq \epsilon$. Then

\[
\tilde{K}_{a'} = (\beta/\delta)^n \tilde{K}_{b^n a'}.
\]

Because $\tilde{K}_{b^n a'}$ has a compact component, so is $\tilde{K}_{a'}$.

\[\square\]

We obtain the following classification.

Proposition 2.9. For $a \in A$, $\tilde{K}_a$ is classified into the following three types:

(1) $\tilde{K}_a$ has no compact components,

(2) $\tilde{K}_a$ is bridged and has compact components,

(3) each component of $\tilde{K}_a$ is compact.

Moreover, for any $\zeta$ with $|\zeta| = |a|$, $\tilde{K}_a$ and $\tilde{K}_\zeta$ are classified into the same category.

2.3 Continuity about irrational rotation

In the following, we show several kinds of continuities of sets along irrational rotation.

Lemma 2.10. Take $a, a' \in A$ with $|a| = |a'|$. Let $\{n_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ be a sequence such that $ab^{n_j} \to a'$ as $j \to \infty$.

Then

\[
\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} (\delta/\beta)^{n_j} \tilde{K}_a \subset \tilde{K}_{a'}.
\]
Proof. Since

\[ \{ab^{n_j}\} \times \left( \frac{\delta^p}{b} \right)^{n_j} \overline{K}_a = L^{n_j} \{a\} \times \overline{K}_a \subset \overline{K}, \]

we have

\[
\bigcup_{j=k}^{\infty} \{ab^{n_j}\} \times \left( \frac{\delta^p}{b} \right)^{n_j} \overline{K}_a \subset \overline{K},
\]

\[
\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \{ab^{n_j}\} \times \left( \frac{\delta^p}{b} \right)^{n_j} \overline{K}_a \subset \{a'\} \times \overline{K}_{a'}. \]

On the other hand, we take

\[ \eta \in \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \left( \frac{\delta^p}{b} \right)^{n_j} \overline{K}_a. \]

This means that for any \( \epsilon > 0 \) and \( k \in \mathbb{N} \), there is \( l \geq k \) such that

\[ d \left( \eta, \left( \frac{\delta^p}{b} \right)^{n_l} \overline{K}_a \right) < \frac{\epsilon}{2}. \]

Then, there exists \( j \geq k \) such that

\[ d \left( (a', \eta), \{ab^{n_j}\} \times \left( \frac{\delta^p}{b} \right)^{n_j} \overline{K}_a \right) < \epsilon. \]

This implies

\[ (a', \eta) \in \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \{ab^{n_j}\} \times \left( \frac{\delta^p}{b} \right)^{n_j} \overline{K}_a. \]

Therefore

\[ \{a'\} \times \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \left( \frac{\delta^p}{b} \right)^{n_j} \overline{K}_a \subset \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \{ab^{n_j}\} \times \left( \frac{\delta^p}{b} \right)^{n_j} \overline{K}_a. \]

We obtain the assertion. \( \square \)

We define \( \overline{I} = A \times \mathbb{C} \setminus \overline{K} \) and \( \overline{I}_a = \mathbb{C} \setminus \overline{K}_a \) for \( a \in A \). Under the hypothesis of the above lemma, we have

\[ \bigcup_{k=1}^{\infty} \text{int} \bigcap_{j=k}^{\infty} \left( \frac{\delta^p}{b} \right)^{n_j} \overline{I}_a \supset \overline{I}_{a'}. \]

More precisely we obtain the following.

**Proposition 2.11.** Take \( a, a' \in A \) with \( |a| = |a'| \). Let \( \{n_j\}_{j \in \mathbb{N}} \subset \mathbb{N} \) be a sequence such that \( ab^{n_j} \to a' \) as \( j \to \infty \). Then each component of

\[ \bigcup_{k=1}^{\infty} \text{int} \bigcap_{j=k}^{\infty} \left( \frac{\delta^p}{b} \right)^{n_j} \overline{I}_a \]

is either in agreement with a component of \( \overline{I}_{a'} \) or contained in \( \overline{K}_{a'} \).
Proof. Take an arbitrary component $I_1$ from
\[
\bigcup_{k=1}^{\infty} \text{int} \bigcap_{j=k}^{\infty} \left( \frac{\delta^p}{b} \right)^{n_j} \tilde{I}_a.
\]
We may assume $I_1$ contains some component of $\tilde{I}_{a'}$. Take a compactly contained open set $V \subset \tilde{I}_{a'}$. Then there is $k \in \mathbb{N}$ such that for any $j \geq k$,
\[
\left( \frac{\delta^p}{b} \right)^{n_j} \tilde{I}_a \supset V, \text{ i.e. } \tilde{I}_{ab^{n_j}} \supset V.
\]
Since $u$ is continuous on $A \times \mathbb{C}$,
\[
u_{ab^{n_j}}|_V \to u_{a'}|_V
\]
uniformly as $j \to \infty$. On the other hand, since $u_{ab^{n_j}}$ is harmonic on $\tilde{I}_{ab^{n_j}}$,
\[
u_{ab^{n_j}} \in \mathcal{H}(V)
\]
for any $j \geq k$. Because $V \Subset I_1$ is arbitrary,
\[
u_{a'} \in \mathcal{H}(I_1).
\]
Recall that $I_1$ contains some component of $\tilde{I}_{a'}$, on which $u_{a'}$ is a positive harmonic function, and $u_{a'}$ vanishes on $\mathbb{C} \setminus \tilde{I}_{a'}$. Because $I_1$ is connected, $I_1$ coincides with some component of $\tilde{I}_{a'}$. □

For $c > 0$, we define
\[
\tilde{I}_{a'} = \{ \eta \in \mathbb{C} \mid u_a(\eta) > c \},
\]
which is a subset of $\tilde{I}_a$. The next lemma tells that $\tilde{I}_{a'}$ plays a role similar to $\tilde{I}_a$.

Lemma 2.12. Take $a, a' \in A$ with $|a| = |a'|$. Let $\{n_j\}_{n \in \mathbb{N}} \subset \mathbb{N}$ be an increasing sequence such that $ab^{n_j} \to a'$ as $j \to \infty$. Then
\[
\tilde{I}_{a'} \subset \bigcup_{k=1}^{\infty} \text{int} \bigcap_{j=k}^{\infty} \left( \frac{\delta^p}{b} \right)^{n_j} \tilde{I}_a \subset \bigcup_{k=1}^{\infty} \text{int} \bigcap_{j=k}^{\infty} \left( \frac{\delta^p}{b} \right)^{n_j} \tilde{I}_a.
\]
Moreover, each component of the middle side is either in agreement with some component of $\tilde{I}_{a'}$ or contained in $K_{a'}$.

Proof. It is sufficient to show the left inclusion. \( \eta \in \tilde{I}_{a'} \) if and only if $u_a(\eta) > c$. Therefore
\[
\eta \in \left( \frac{\delta^p}{b} \right)^{n_j} \tilde{I}_a \iff (ab^{n_j}, \eta) \in \{ab^{n_j}\} \times \left( \frac{\delta^p}{b} \right)^{n_j} \tilde{I}_a
\]
\[\iff (ab^{n_j}, \eta) \in L^{n_j}(\{a\} \times \tilde{I}_a)
\]
\[\iff L^{-n_j}(ab^{n_j}, \eta) \in \{a\} \times \tilde{I}_a
\]
\[\iff u(L^{-n_j}(ab^{n_j}, \eta)) > c
\]
\[\iff d^{n_j} u(ab^{n_j}, \eta) > c.
\]
Take $\eta_1 \in \tilde{I}_{a'}$. Then there is $\varepsilon_1 > 0$ with $B(\eta_1, 3\varepsilon_1) \subset \tilde{I}_{a'}$. Note that
\[
u_{a'}|_{B(\eta_1, 3\varepsilon_1)} > 0.
\]
Since $u$ is continuous, there is $k_1 \in \mathbb{N}$ such that for any $j \geq k_1$,
\[
u_{ab^{n_j}}|_{B(\eta_1, 2\varepsilon_1)} > 0.
\]
Moreover there exists $k_2 \in \mathbb{N}$ for arbitrary $j \geq k_2$,

$$u_{ab^n_j}|_{B(\eta_1, \varepsilon_1)} > \frac{c}{d^{n_j}},$$

i.e. $B(\eta_1, \varepsilon_1) \subset \left(\frac{\delta^p}{b}\right)^{n_j} \overline{I'}_a$.

Therefore we have

$$\eta_1 \in \text{int} \bigcap_{j=k_2}^{\infty} \left(\frac{\delta^p}{b}\right)^{n_j} \overline{I'}_a.$$  

This implies the left inclusion in the assertion. \hfill \Box

## 3 In case that $\overline{K}_a$ has no compact components

We have the following non-existence of Herman rings.

**Theorem 3.1.** The case (1) in Proposition 2.9 is impossible.

**Corollary 3.2.** The case that $\overline{K}_a$ is connected is impossible.

To prove the theorem we use a Böttcher function $\varphi^-$ (cf. [MNTU, Section 7.3]). $\varphi^-$ is holomorphic on $V^+$ for sufficiently large $R > 0$, and satisfies $\varphi^{-} \circ f^{-1}(z) = (\varphi^{-}(z))^d$ and $\log|\varphi^{-}(z)| = G^{-}(z)$ for $z \in V^+$. There is $M > 1$ such that $1/M \leq |\varphi^{-}(x, y)|/|x| \leq M$ for $(x, y) \in V^+$. When $|w|$ $(w \in \mathbb{C})$ is sufficiently large, $\{z \in V^+ \mid \varphi^{-}(z) = w\}$ is a simply connected one dimensional complex manifold in $V^+$.

We use a notation $\psi_{c}(\eta) = \psi(\zeta, \eta) = \varphi^{-}\circ \Phi(\zeta, \eta)$. Note that $\log|\psi_{a}| = u_a$.

**Lemma 3.3.** If $\overline{K}_a$ has no compact components, then

$$\nabla u_a(\eta) \neq (0, 0)$$

for any $\eta \in \overline{I}_a$.

**Proof.** Assume the contrary i.e. there is $\eta_0 \in \overline{I}_a$ such that $\frac{\partial u_a}{\partial \eta}(\eta_0) = 0$.

By Proposition 2.3, for any $c > 0$ the number of components of $\{\eta \in \mathbb{C} \mid u_a(\eta) > c\}$ is at most max$\{1, 2p\}$ (cf. [Hay, Theorem 8.9]). Since the number is monotone increasing along $c > 0$, we can take $c > 0$ so that the number attains its maximum.

Then $\frac{\partial u_a}{\partial \eta}$ has no zero points in $\overline{I}_a = \{\eta \in \mathbb{C} \mid u_a(\eta) > c\}$. In fact, let us assume the contrary, i.e. there is $\eta_1 \in \overline{I}_a$ with $\nabla u_a(\eta_1) = (0, 0)$. Define $c' = u_a(\eta_0)$. There are $n \geq 2, 0 \leq \theta < 2\pi$ and $\varepsilon_1 > 0$ such that

$$u_a(\eta_0 + t \exp(i(\theta + 2\pi j/n))) > c',
$$

$$u_a(\eta_0 + t \exp(i(\theta + 2\pi (j + 1/2)/n))) < c',$$

for any $0 \leq j < n$ and $0 < t < 2\varepsilon_1$. Define

$$I_1 = \{\eta \in \mathbb{C} \mid u_a(\eta) > c', \eta \text{ is in the component of } \overline{I}_a \text{ containing } \eta_0\}.$$

Because of the definition of $c$, $I_1$ is a connected open set. Moreover, $I_1$ is simply connected because so is $\overline{I}_a$. Therefore there is an arc $\Gamma \subset I_1$ which joins

$$\eta_0 + \varepsilon_1 \exp(i\theta) \text{ and } \eta_0 + \varepsilon_1 \exp(i(\theta + 2\pi/n)).$$

We can extend $\Gamma$ in a neighborhood of $\eta_0$ and obtain a closed curve $\Gamma'$ so that $u_a \geq c'$ on $\Gamma'$. But there is a point inside of $\Gamma'$ on which $u_a < c'$. It is a contradiction.

Let $\{\eta_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ be a sequence such that $ab^n_j \to a$ as $j \to \infty$. Then for $\eta \in (\delta^p/b)^{n_j} \overline{I}_a$,

$$u_{ab^n_j}(\eta) = u(ab^n_j, \eta)
= u(L^{n_j}(a, (b/\delta^p)^{n_j}\eta))
= d^{-n_j}u(a, (b/\delta^p)^{n_j}\eta)
= d^{-n_j}u_a((b/\delta^p)^{n_j}\eta).$$
\[
\frac{\partial u_{ab^{n_{j}}}}{\partial \eta}(\eta) = d^{-n_{j}} \left( \frac{b}{\delta^{p}} \right)^{n_{j}} \frac{\partial u_{a}}{\partial \eta}((b/\delta^{p})^{n_{j}}\eta) \neq 0.
\]

On the other hand, by Lemma 2.12, there are \(\epsilon_{0} > 0\) and \(k \in \mathbb{N}\) such that for any \(j \geq k\),
\[
B(\eta_{0}, \epsilon_{0}) \subset \left( \frac{\delta^{p}}{b} \right)^{n_{j}} \tilde{I}_{a}.
\]

Because \(u\) is continuous, \(u_{ab^{n_{j}}}\) converge to \(u_{a}\) uniformly in \(B(\eta_{0}, \epsilon_{0})\) as \(j \to \infty\).

When harmonic functions \(u_{ab^{n_{j}}}\) converge to a non-constant harmonic function \(u_{a}\) uniformly on \(B(\eta_{0}, \epsilon_{0})\), then antiholomorphic functions \(\overline{\partial} u_{ab^{n_{j}}}\) converge to \(\overline{\partial} u_{a}\) uniformly. By Hurwitz theorem, each zero point of \(\overline{\partial} u_{a}\) is an accumulation point of zero points of \(\overline{\partial} u_{ab^{n_{j}}}\). But \(\overline{\partial} u_{ab^{n_{j}}}\) has no zero points in \(B(\eta_{0}, \epsilon_{0})\) for any \(j \geq k\). It is a contradiction. \(\square\)

We use a notation \(H = \{\xi \in \mathbb{C} \mid \text{Re}\xi > 0\}\).

**Lemma 3.4.** Assume \(\tilde{K}_{a}\) has no compact components. Then for any component \(I_{0}\) of \(\tilde{I}_{a}\) and for any \(c > 0\) the number of components of
\[
\{\eta \in I_{0} \mid u_{a}(\eta) > c\}
\]
is exactly one.

**Proof.** Since \(I_{0}\) is simply connected, there is \(g \in O(I_{0})\) such that \(\text{Re} g = u_{a}\). Then
\[
g : I_{0} \to H.
\]

By the previous lemma, for each \(c > 0\) its level set
\[
\{\eta \in I_{0} \mid u_{a}(\eta) = c\}
\]
is a set of smooth simple arcs, whose all ends go to infinity. Therefore \(g : I_{0} \to H\) is locally biholomorphic and proper. This implies \(g\) is bijective. Hence the above each level set consists of single arc. We obtain the required result. \(\square\)

**Lemma 3.5.** Assume \(\tilde{K}_{a}\) has no compact components. Let \(I_{0}\) be an arbitrary component of \(\tilde{I}_{a}\). Then it is possible to define
\[
\log \psi_{a} : I_{0} \to H
\]
bym analytic continuation. Moreover it is biholomorphic.

**Proof.** For sufficiently large \(c > 0\), \(\psi_{a}\) is defined on
\[
I'_{0} = \{\eta \in I_{0} \mid u_{a}(\eta) > c\},
\]
because the set is contained in \(\Phi^{-1}(V^{+})\). Since \(I'_{0}\) is simply connected, \(\log \psi_{a}\) is well-defined on \(I'_{0}\).

We take \(g\) used in the previous proof. Because \(\text{Re} \log \psi_{a} = u_{a}\), \(\log \psi_{a} - g\) is a purely imaginary constant. Then \(\log \psi_{a}\) can be analytic continued to \(I_{0}\) because \(I'_{0}\) is connected. Since \(g : I_{0} \to H\) is biholomorphic, so is \(\log \psi_{a}\). \(\square\)

Let us investigate the structure of \(\tilde{I}\).

**Proposition 3.6.** For any \(a \in A\) there are \(N \in \mathbb{N}\) and a closed curve
\[
\gamma : [0, N] \to \tilde{I}
\]
such that
\[
\pi_{A} \circ \gamma(t) = a \exp(2\pi it),
\]
where \(\pi_{A} : A \times \mathbb{C} \to A\) is a natural projection.
Note that we take the minimum $N \geq 1$ when we use the proposition.

**Proof.** Assume $\tilde{a}$ has $q'$. We know $q' \leq \max\{1, 2\rho\} < \infty$. Take $\eta_1, \ldots, \eta_{q'} \in \tilde{a}$ so that any two of them belong to distinct components of $\tilde{I}_a$. There is small $\epsilon_0 > 0$ such that

$$T = \bigcup_{j=1}^{q'} \{(ae^{it}, \eta_j) \mid 0 \leq t \leq 2\epsilon_0 \} \subset \tilde{I}.$$

We define a sequence $\{n_j\} \subset \mathbb{Z}$ as follows. We take $n_1 \in \mathbb{Z}$ such that

$$\epsilon_0 \leq \arg ab^{n_1} - \arg a \leq 2\epsilon_0 \pmod{2\pi}.$$

Then we take $n_2 \in \mathbb{Z}$ such that

$$\epsilon_0 \leq \arg ab^{n_2} - \arg ab^{n_1} \leq 2\epsilon_0 \pmod{2\pi}.$$

By repeating the procedure, we return to the starting point, i.e. there is $k \in \mathbb{N}$ such that

$$0 < \arg a - \arg ab^{nk} \leq 2\epsilon_0 \pmod{2\pi}.$$

Then we can draw arcs in $\tilde{I}$ as follows. For $\eta_{j_0}$, draw an arc from $(a, \eta_{j_0})$ to $(ab^{nk}, \eta_{j_0})$ along $T$. Choose $(\delta^p/b)^{n_1} \eta_j$, so that $\eta_{j_0}$ and $(\delta^p/b)^{n_1} \eta_j$ are in the same component of $\tilde{I}_{ab^{nk}}$, then draw arcs joining the two points in the component. In the sequel draw an arc from $(ab^{nk}, \eta_{j_1})$ to $(ab^{nk}, \eta_{j_1})$ along $L^{nk}(T)$. By repeating the procedure, we can draw an arc from each $\eta_1, \ldots, \eta_{q'}$ to $\tilde{I}_a$.

If for some $\eta_j$ the arc returns to the component of $\tilde{I}_a$ containing the same $\eta_j$, we can draw an arc in the component joining the start point and the end point, and obtain a closed curve. Otherwise repeat $N$ times the above procedure and at last some end point arrives at the same component of its start point. So we can draw a closed curve.

Finally by perturbing the closed curve we obtain $\gamma$ as required.

**Lemma 3.7.** If $\tilde{K}_a$ has no compact components, $\gamma$ in Proposition 3.6 is unique in the following sense. Let $\gamma': [0, N'] \to \tilde{I}$ be another closed curve satisfying the same condition, and $\gamma(0)$ and $\gamma'(0)$ are in the same component of $\tilde{I}_a$. Then $N = N'$ and for any $t \in [0, N]$, $\gamma(t)$ and $\gamma'(t)$ are in the same component of $\tilde{I}_{a \exp(2\pi it)}$.

**Proof.** We may suppose $N \leq N'$. Let us assume for some $t \in [0, N]$, $\gamma(t)$ and $\gamma'(t)$ are in distinct components of $\tilde{I}_{a \exp(2\pi it)}$, and derive a contradiction.

By iteration of $L^{-1}$, we may assume $\gamma, \gamma' \subset \Phi^{-1}(V^+)$. Then $\psi$ is defined in a neighborhood of $\gamma$ and $\gamma'$.

The set

$$\{t \in [0, N] \mid \gamma(t) \text{ and } \gamma'(t) \text{ are in the same component of } \tilde{I}_{a \exp(2\pi it)}\}$$

is open. In fact, take $t_1$ from the set. Then there is an arc $C \subset \tilde{I}_{a \exp(2\pi it_1)}$ joining $\gamma(t_1)$ and $\gamma'(t_1)$. Because

$$\bigcup_{|\zeta| = |a|} \tilde{I}_\zeta$$

is open in $\{\zeta \in A \mid |\zeta| = |a|\} \times \mathbb{C}$, a neighborhood of $C$ is also contained in the above open set. So in a neighborhood of $t_1$, $\gamma(t)$ and $\gamma'(t)$ are joined by an arc in $\tilde{I}_{a \exp(2\pi it)}$.

Define

$$t_2 = \min\{t \in [0, N] \mid \gamma(t) \text{ and } \gamma'(t) \text{ are in distinct components of } \tilde{I}_{a \exp(2\pi it)}\},$$

$(a_2, \eta_2) = (\gamma(t_2))$ and $(a_2, \eta'_2) = (\gamma'(t_2))$. Take $\epsilon_1, \epsilon_2 > 0$ such that

$$\{a_2 \exp(2\pi it) \mid -\epsilon_1 < t \leq 0 \} \times B(\eta_2, \epsilon_2) \subset \Phi^{-1}(V^+) \subset \tilde{I},$$

$$\{a_2 \exp(2\pi it) \mid -\epsilon_1 < t \leq 0 \} \times B(\eta'_2, \epsilon_2) \subset \Phi^{-1}(V^+) \subset \tilde{I},$$

$$\{(a_2 \exp(2\pi it) \mid -\epsilon_1 < t \leq 0 \} \times \partial B(\eta_2, \epsilon_2)) \cap \gamma = \emptyset,$$

$$\{(a_2 \exp(2\pi it) \mid -\epsilon_1 < t \leq 0 \} \times \partial B(\eta'_2, \epsilon_2) \cap \gamma' = \emptyset.$$
By Lemma 3.5, for each $\zeta \in \{a_2 \exp(2\pi it) | -\epsilon_1 < t < 0\}$, $\log \psi_\zeta$ is well-defined in the component of $\tilde{I}_\zeta$ containing $\eta_2$ and $n_2$. We choose the branches of the logarithms so that $\log \psi_\zeta(\eta_2)$ varies continuously with respect to $\zeta$. Then
$$
\log \psi_\zeta(\eta_2)
$$
vary continuously on $\zeta \in \{a_2 \exp(2\pi it) | -\epsilon_1 < t < 0\}$. Moreover since $\psi_\zeta$ converges to $\psi_{a_2}$ as $\zeta \to a_2$ uniformly in a neighborhood of $\eta_2$, there is $\xi_2$ such that
$$
\log \psi_\zeta(\eta_2) \to \xi_2
$$
as $\zeta \to a_2$.

On the other hand, there is $\eta_3$ in the component of $\tilde{I}_{a_2}$ containing $\eta_2$ such that
$$
\log \psi_{a_2}(\eta_3) = \xi_2,
$$
where $\log \psi_{a_2}$ is defined so that $\log \psi_\zeta(\eta_2) \to \log \psi_{a_2}(\eta_2)$ as $\zeta \to a_2$. Observe that
$$
\log \psi_\zeta(\eta_3) \to \xi_2
$$
as $\zeta \to a_2$, because $\psi$ is continuous in a neighborhood of $(a_2, \eta_3)$.

Therefore, both $\log \psi_\zeta(\eta_2)$ and $\log \psi(\eta_3)$ converge to $\xi_2$. It contradicts with the injectivity of $\log \psi_\zeta$.

Hence $\gamma(N)$ and $\gamma'(N)$ are in the same component of $\tilde{I}_a$. We can draw a curve in the component from $\gamma'(N)$ to $\gamma(0)$, i.e. $N = N'$.

**Lemma 3.8.** Assume $\tilde{I}_a$ has no compact components. Take an arbitrary closed curve $\gamma$ as in Proposition 3.6. Then $\Phi(\gamma)$ is trivial in $\pi_1(I^-)$.

**Proof.** Since $\tilde{I}_a$ has finite components, there is $q \in \mathbb{N}$ such that both $\gamma$ and $L^q(\gamma)$ intersect a common component of $\tilde{I}_a$. We know by Lemma 3.7 that for each $t \in [0, N]$
$$
\gamma(t) \text{ and } L^q(\gamma(t + t_0)) \pmod{N}
$$
are in the same component of $\tilde{I}_{a \exp(2\pi it)}$ for some $t_0 \in \mathbb{R}$.

We can draw a curve in $\tilde{I}_{a \exp(2\pi it)}$ between $\gamma(t)$ and $L^q(\gamma'(t + t_0))$. We can extend the curve along $t$ to a strip, i.e. $\gamma(t)$ and $L^q(\gamma'(t + t_0))$ are locally homotopic. since each component of $\tilde{I}_\zeta$ is simply connected, we can join the homotopies and have that $\gamma$ and $L^q(\gamma)$ are homotopic in $\tilde{I}$.

On the other hand, there is an isomorphism $\alpha : \pi_1(I^-) \to \mathbb{Z}[2]$ such that
$$
\alpha(f(C)) = \frac{1}{d} \alpha(C)
$$
for any $C \in \pi_1(I^-)$ (cf. [MNTU, Section 7.3]).

Since $\gamma$ and $L^q(\gamma)$ are homotopic in $\tilde{I} = \Phi^{-1}(I^-)$, $\Phi(\gamma)$ and $\Phi(L^q(\gamma))$ are homotopic in $I^-$. We obtain
$$
\frac{1}{d^{pq}} \alpha(\Phi(\gamma)) = \alpha(f^p(\Phi(\gamma))) = \alpha(\Phi(L^q(\gamma))) = \alpha(\Phi(\gamma)).
$$
Therefore $\alpha(\Phi(\gamma)) = 0$.

**Proof of Theorem 3.1.** Take $\gamma$ as in Proposition 3.6. By iteration of $L^{-1}$, we may assume $\psi$ is defined on the curve. Let $\pi_C : A \times C \to C$ be a natural projection. For each $t \in [0, N]$, let $I_t$ be the component of $\tilde{I}_{a \exp(2\pi it)}$ containing $\pi_C \circ \gamma(t)$. By Lemma 3.5, we have
$$
\log \psi_{a \exp(2\pi it)} : I_t \to \mathbb{H}.
$$
We choose the branch of the logarithm so that $\log \psi_{a \exp(2\pi it)}(\pi_C \circ \gamma(t))$ varies continuously. Here, we regard $\psi_{a \exp(2\pi it)}$ and $\psi_{a \exp(2\pi it+1)}$ as different functions.
Then in general, $\log \psi_{a \exp(2\pi i \cdot 0)}$ and $\log \psi_{a \exp(2\pi i \cdot N)}$ do not have to be coincide. But since $\Phi(\gamma)$ is trivial in $\pi_1(I^-)$, they coincide. In fact, there is a 1-form $\omega$ in $I^-$ such that

$$
\pi_1(I^-) \ni C \mapsto \int_C \omega = \alpha(C) \in \mathbb{Z}\left[ \frac{1}{d} \right]
$$

and $\int \omega = \log \varphi^-$ (indefinite integral) (cf. [MNTU, Section 7.3]). Therefore

$$
\log \psi_{a \exp(2\pi i \cdot N)}(\pi_C \circ \gamma(0)) = \log \psi_{a \exp(2\pi i \cdot 0)}(\pi_C \circ \gamma(0)) + \int_\gamma \Phi^* \omega
$$

Take an appropriate $\xi \in \mathbb{H}$ so that $\text{Re} \, \xi$ is sufficiently large. Then for each $t \in [0, N]$, there is a unique point $\eta_t \in I_t$ such that

$$
\log \psi_{a \exp(2\pi it)}(\eta_t) = \xi.
$$

Then

$$
[0, N] \ni t \mapsto (a \exp(2\pi it), \eta_t) \in \bar{I}
$$

is a closed curve and satisfies

$$
\psi(a \exp(2\pi it), \eta_t) = e^\xi.
$$

for any $t \in [0, N]$.

Therefore

$$
[0, N] \ni t \mapsto \Phi(a \exp(2\pi it), \eta_t) \in W_0^s(\mathcal{H})
$$

is a non-contractible closed curve, and satisfies

$$
\{ \Phi(a \exp(2\pi it), \eta_t) \mid t \in [0, N] \} \subset (\varphi^-)^{-1}(e^\xi).
$$

This contradicts with Proposition 2.2. \qed

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References


